58. *On Quasi-Evaluations of Compact Rings.*

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Recently H. Anzai has proved that every compact associative ring containing no left-(or right-)total nul-divisor is totally disconnected and is a limit ring of finite rings[1]. In the present paper the author proposes to prove that every compact ring containing no total nul-divisor has "sufficiently many quasi-evaluations." These evaluations serve to clarify the uniform structure of compact rings, and the author has in view to apply this result, in a forthcoming paper[2], to determine the structure of locally compact totally disconnected fields.

To prepare the introduction of the quasi-evaluations in § 2, we shall, in § 1, consider the compact open ideals of the ring determined by finite sets of characters of the additive group of the ring, by means of which Anzai's result is easily deduced in a more precise form.

§ 1. Let $\mathfrak{R}$ be a compact associative topological ring, that is, an associative ring forming a compact topological group with respect to addition, where the product xy is continuous in two variables x and y.

Denote by $\mathfrak{R}^*$ the character group (in the sense of Pontrjagin-van Kampen[3]) of the compact additive group $\mathfrak{R}$, then $\mathfrak{R}^*$ is a discrete group, and

(1) $V(0;\xi_1,\xi_2,\dots,\xi_r;1/m)=\{x\,|\,|(x,\xi_i)|<1/m,\ i=1,2,\dots,r\}$,

where $\xi_1,\xi_2,\dots,\xi_r\in\mathfrak{R}^*$, $m=1,2,\dots$, $r=1,2,\dots$, form a complete system of neighbourhoods of zero in $\mathfrak{R}$[4]. Define further

$$[a]'=a\mathfrak{R}+\mathfrak{R}a+\mathfrak{R}a\mathfrak{R}$$

for $a\in\mathfrak{R}$, and denote by $[a]$ the additive subgroup $a\mathfrak{R}+\mathfrak{R}a+\Sigma\mathfrak{R}a\mathfrak{R}$ generated by $[a]'$, similarly for $A\subseteq\mathfrak{R}$ by $[A]$ the subgroup generated by all $[a]'$ with $a\in A$. Then we have

(2) $x\in[a]'$ implies $tx=x+x+\cdots+x$ (t times) $\in[a]'$

for any integer $t(>0)$, since $tx=t(ax_1+x_2a+x_3ax_4)=a(tx_1)+(tx_2)a+(tx_3)ax_4\in[a]'$; and

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1) H. Anzai, On compact topological rings, Proc. **19** (1943), 616.
2) Y. Otobe, On locally compact fields.
3) E. R. van Kampen, Locally bicompact abelian groups and their character groups, Ann. of Math., **36** (1935).
4) We denote by (x,ξ) the value (in real numbers mod 1) of the continuous additive character ξ at $x\in\mathfrak{R}$. For subsets $A,B\leq\mathfrak{R}$ and $C^*\leq\mathfrak{R}^*$, $(A,C^*)=0$ means that $(x,\xi)=0$ for all $x\in A$, $\xi\in C^*$. We put $A+B=\{x+y\,|\,x\in A,\ y\in B\}$, $AB=\{xy\,|\,x\in A,\ y\in B\}$, $\Sigma A=\{x_1+x_2+\cdots+x_r\,|\,x_i\in A\ (i=1,\dots,r),\ r=1,2,\dots\}$, etc. The annihilator $\{\mathfrak{R},C^*\}=\{x\,|\,(x,C^*)=0\}$ or $\{\mathfrak{R}^*,A\}=\{\xi\,|\,(A,\xi)=0\}$ is a closed additive subgroup of $\mathfrak{R}$ or $\mathfrak{R}^*$ respectively.

$$x, y \in [a] \quad \text{and} \quad z \in \Re \quad \text{implies} \quad x-y, \quad xz \quad \text{and} \quad zx \in [a],$$

since, say, $xz \in a(\Re z) + \Re az + \Sigma \Re x(\Re z) \subseteq a\Re + \Sigma \Re a \Re \subseteq [a]$.

Thus $[a]$ and $[A]$ are two-sided ideals in $\Re$. (If $\Re$ has the unit, $[a]$ (or $[A]$) is the ideal generated by a (or A).)

We now define for any finite set $\xi_1, \xi_2, \dots, \xi_r$ of $\Re^*$

$$(3) \qquad \begin{aligned} \Im_{\xi_1, \xi_2, \dots, \xi_r} &= \{a \mid [a] \subseteq \{\Re, (\xi_1, \xi_2, \dots, \xi_r)\}\} \\ &= \{a \mid [a]' \subseteq \{\Re, (\xi_1, \xi_2, \dots, \xi_r)\}\} \\ &= \{a \mid ([a]', \xi_i) = 0, \ i = 1, 2, \dots, r\}, \end{aligned}$$

where $(\xi_1, \xi_2, \dots, \xi_r)$ is the subgroup of $\Re^*$ generated by $\xi_1, \xi_2, \dots, \xi_r$, and $\{\Re, (\xi_1, \xi_2, \dots, \xi_r)\}$ is its annihilator[4].

Lemma 1. $\Im_{\xi_1, \xi_2, \dots, \xi_r}$ *is a compact, open, two-sided ideal in* $\Re$.

Proof. $\Im_{\xi_1, \xi_2, \dots, \xi_r}$ is evidently a two-sided ideal in the compact ring $\Re$. In order to prove that it is compact and open, it is therefore sufficient to show that it contains a neighbourhood U of zero[5]. Take the neighbourhood $V = V(0; \xi_1, \xi_2, \dots, \xi_r; 1/3)$ of zero ((1)), then there exist neighbourhoods V_1, U_1 and U of zero such that $V_1 + V_1 + V_1 \subseteq V$, $U_1\Re \subseteq V_1$, $\Re U \subseteq U_1$ and $U \subseteq U_1 \subseteq V_1$, since the multiplication is continuous and $\Re$ is compact. If $a \in U$, then $[a]' = a\Re + \Re a + \Re a \Re \subseteq V_1 + V_1 + V_1 \subseteq V$, i.e. $|([a]', \xi_i)| < 1/3 \pmod 1$ $(i = 1, 2, \dots, r)$. Then by (2) $x \in [a]'$ implies $t|(x, \xi_i)| = |(tx, \xi_i)| < 1/3 \pmod 1$ for all integers $t > 0$, hence necessarily $(x, \xi_i) = 0$ $(i = 1, 2, \dots, r)$. Thus $U \subseteq \Im_{\xi_1, \xi_2, \dots, \xi_r}$,

q. e. d.

Lemma 2. In order that all $\Im_{\xi_1, \xi_2, \dots, \xi_r}$ $(\xi_i \in \Re^*)$ *form a complete system of neighbourhoods of zero in* $\Re$, *it is necessary and sufficient that* $\Re$ *has no total nul-divisor (except zero), i. e.*

$$a\Re = \Re a = 0 \quad \textit{implies} \quad a = 0.$$

Proof. The following four conditions are clearly equivalent to one another: 1) $a \in \bigcap \Im_{\xi_1, \xi_2, \dots, \xi_r}$ for $\xi_i \in \Re^*$, $r = 1, 2, \dots$, 2) $([a]', \xi) = 0$ for all $\xi \in \Re^*$, 3) $[a]' = 0$, and (4) a is a total nul-divisor. Hence $\Re$ has no total nul-divisor if and only if $\bigcap \Im_{\xi_1, \xi_2, \dots, \xi_r} = \{0\}$. By Lem. 1 the last condition means that all $\Im_{\xi_1, \xi_2, \dots, \xi_r}$ form a complete system of neighbourhoods of zero.

All residue rings $\Re / \Im_{\xi_1, \xi_2, \dots, \xi_r}$ are finite rings because of the compactness of $\Re$ and Lem. 1, hence we can easily, by appealing Lem. 2, deduce a more precise form of the result obtained by Anzai:

Theorem 1. Every compact associative ring containing no total nul-divisor is totally disconnected and is the limit ring of a directed system of finite associative rings.

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5) Cf. Lem. *B* in the author's forthcoming paper cited in 2).

§ 2. In this section we assume that $\mathfrak{R}$ has no total nul-divisor and that $\mathfrak{R}$ is not finite. Not to interrupt the current of the proof we first remark:

$$\mathfrak{J}_{\xi_1,\dots,\xi_r,\xi}=\mathfrak{J}_{\xi_1,\dots,\xi_r} \quad \textit{if and only if} \quad \xi\in\{\mathfrak{R}^*,[\mathfrak{J}_{\xi_1,\dots,\xi_r}]\}.$$

Especially (*for* $r=0$) $\mathfrak{J}_\xi=\mathfrak{R}$ *if and only if* $\xi\in\{\mathfrak{R}^*,[\mathfrak{R}]\}$.

For, since always $\mathfrak{J}_{\xi_1,\dots,\xi_r,\xi}\leq\mathfrak{J}_{\xi_1,\dots,\xi_r}$, the equality holds if and only if $([\mathfrak{J}_{\xi_1,\dots,\xi_r}],\xi)=0$ from (3).

We now take a well-ordered (minimal) system $\{\xi^\iota\,|\,\iota<\Omega_1\}$ of generators ξ^ι of the discrete additive group $\mathfrak{R}^*$, such that $\xi^\iota\notin\{\mathfrak{R}^*,[\mathfrak{R}]\}$. We then rearrange $\{\xi^\iota\,|\,\iota<\Omega_1\}$ into $\{\xi^\alpha_n\,|\,\alpha<\Omega_0,\ n=1,2,\dots\}$ as follows: Put $\xi^0_1=\xi^0$. If $\xi^\alpha_1,\xi^\alpha_2,\dots,\xi^\alpha_n=\xi^{\iota(\alpha,n)}$ are defined, we put $\xi^\alpha_{n+1}=\xi^{\iota'}$, where ι' is an arbitrary (say, the minimal) number such that

$$(*)\qquad \xi^\alpha_{n+1}=\xi^{\iota'}\notin\{\mathfrak{R}^*,[\mathfrak{J}_{\xi^\alpha_1,\xi^\alpha_2,\dots,\xi^\alpha_n}]\}.$$

If ξ^α_n for all $\alpha<\beta$ and $n=1,2,\dots$ are defined, we put $\xi^\beta_1=\xi^{\iota''}$ where ι'' is the minimal number such that $\iota''\neq\iota(\alpha,n)$ for all $\alpha<\beta$, $n=1,2,\dots$. Then evidently

$$(4)\qquad \{\xi^\alpha_n\,|\,\alpha<\Omega_0,\ n=1,2,\dots\}=\{\xi^\iota\,|\,\iota<\Omega_1\}.$$

Since $\mathfrak{R}^*$ is infinite and torsional[6], we may assume that Ω_1 is an 'Anfangszahl' $\geq\omega$. Hence Ω_0 can be made $=1$ in the case $\overline{\Omega}_1=\aleph_0$, but $\Omega_0=\Omega_1$ if $\overline{\Omega}_1>\aleph_0$. Denote

$$(5)\qquad U^\alpha_n=\mathfrak{J}_{\xi^\alpha_1,\xi^\alpha_2,\dots,\xi^\alpha_n}\quad(\alpha<\Omega_0,\ n=1,2,\dots),\quad U^\alpha_0=\mathfrak{R},$$

then from (*) and the above-mentioned remark we have

$$(6)\qquad U^\alpha_n\gneqq U^\alpha_{n+1}\qquad(\alpha<\Omega_0,\ n=0,1,2,\dots).$$

We now define for each fixed $\alpha<\Omega_0$ a functional $|x|_\alpha$ of $x\in\mathfrak{R}$:

$$(7)\qquad \begin{aligned}|x|_\alpha&=2^{-n}\qquad\text{if } x\in U^\alpha_n \text{ and } x\notin U^\alpha_{n+1},\\ &=0\qquad\text{if } x=0.\end{aligned}$$

Then $|x|_\alpha$ is continuous on $\mathfrak{R}$, since U^α_n are closed and open (Lem. 1) and satisfy (6). We have further

$$(8)\qquad 0\leq|x|_\alpha=|-x|_\alpha\leq1,\qquad |0|_\alpha=0,$$

$$(9)\qquad |x+y|_\alpha\leq\max\{|x|_\alpha,|y|_\alpha\},$$

and

$$(10)\qquad |xy|_\alpha\leq\min\{|x|_\alpha,|y|_\alpha\},$$

since U^α_n are two-sided ideals (Lem. 1). We shall call every functional satisfying (8). (9), and (10) a "quasi-evaluation."

6) The discrete character group $\mathfrak{R}^*$ of the totally disconnected compact group $\mathfrak{R}$ is torsional (i. e. contains no element of infinite order). Cf. van Kampen, loc. cit., § 6, g.

Let U be an arbitrary neighbourhood of zero (in $\mathfrak{R}$), then by Lem. 2 there exists a finite set $\xi_1, \xi_2, \dots, \xi_r \in \{\mathfrak{R}^*, [\mathfrak{R}]\}$ such that $\mathfrak{J}_{\xi_1, \xi_2, \dots, \xi_r} \leq U$. As $\mathfrak{R}^*$ is directe, $(\xi_1, \xi_2, \dots, \xi_r)$ can be generated by a finite subset of $\{\xi^\iota \mid \iota < \Omega_1\}$, hence by (4) there are $\alpha_1, \alpha_2, \dots, \alpha_s$; $n_1, n_2, \dots, n_s$ such that $(\xi_1, \dots, \xi_r) \leq (\xi_1^{\alpha_1}, \dots, \xi_{n_1}^{\alpha_2}, \xi_1^{\alpha_1}, \dots, \xi_1^{\alpha_s}, \dots \xi_{n_s}^{\alpha_s})$. Therefore $U \geq \mathfrak{J}_{\xi_1, \dots, \xi_r} \geq \bigcap_{j=1}^{s} U_{n_j}^{\alpha_j}$, hence

$$(11) \quad U(0; \alpha_1, \alpha_2, \dots, \alpha_s; 2^{-n}) = \{x \mid |x|_{\alpha_j} < 2^{-n}, \quad j=1, 2, \dots, s\} \leq U,$$

where $n = \max_{j=1}^{s} n_j$. Since U was arbitrary, $U(0; \alpha_1, \alpha_2, \dots, \alpha_s; 2^{-n})$ for all finite sets $\alpha_1, \alpha_2, \dots, \alpha_s$ of α and $n=1, 2, \dots$ form a complete system of neighbourhoods of zero in $\mathfrak{R}$. Especially

$$(12) \qquad x=0 \quad \text{if and only if} \quad |x|_\alpha = 0 \quad \text{for all} \quad \alpha < \Omega_0.$$

Hence, the totality of all finite sets of additively invariant metrics $\rho_\alpha(x, y) = |x-y|_\alpha$, $\alpha < \Omega_0$ determines a complete system of neighbourhoods in $\mathfrak{R}$. Thus we have

Theorem 2. Any (infinite) compact associative topological ring $\mathfrak{R}$, *which has no total nul-divisor*[7], *has sufficiently many real-valued discrete*[8] *quasi-evaluations: i. e. there is a set of functionals* $\{|x|_\alpha \mid \alpha < \Omega_0\}$ *satisfying* (8), (9), (10) *and* (12); *the uniform structure of* $\mathfrak{R}$ *is being given by* $V_{\alpha_1, \alpha_2, \dots, \alpha_s; n} = \{(x, y) \mid |x-y|_{\alpha_j} < 2^{-n}, j=1, 2, \dots, s\}$ *for all finite sets* $\alpha_1, \alpha_2, \dots, \alpha_s$ $(< \Omega_0)$ *and* $n=1, 2, \dots$[9].

Corollary. $\mathfrak{R}$ *satisfies the first countability axiom, if and only if* $\mathfrak{R}$ *has a separated quasi-evaluation* $|x|$, *which satisfies* (8), (9), (10) *and*

$$(12') \qquad x=0 \quad \textit{if and only if} \quad |x|=0.$$

(This means that $\mathfrak{R}$ is a "$\mathfrak{b}_\nu$-adic ring" in the sense of van Dantzig[10].)

Remark. The condition (10) can not, in general, be replaced by the stronger one:

$$(10') \qquad |xy|_\alpha \leq |x|_\alpha \cdot |y|_\alpha.$$

In fact, if (10′) and $\Omega_0 = 1$ hold[11], each open ideal $\mathfrak{J}$ in $\mathfrak{R}$ contains a power of the ideal $\mathfrak{J}_0 = \{x \mid |x| < 1\}$. For, we have $\mathfrak{J}_0 = \{x \mid |x| \leq \gamma_0\}$ $(0 < \gamma_0 < 1)$ since $|x|$ is then necessarily discrete, and $\mathfrak{J}$ contains a neighbourhood of zero $\{x \mid |x| < \varepsilon\}$ for some $\varepsilon > 0$; hence $\mathfrak{J} \geq \mathfrak{J}_0^t$ for

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7) This assumption is satisfied if $\mathfrak{R}$ has a left- (or right-) unit e (ex (or xe)$=x$ for all $x \in \mathfrak{R}$).

8) Any quasi-evaluation $|x|$ of a compact ring is discrete. For, $\mathfrak{J} = \{x \mid |x| < \gamma_0\}$ is an open ideal, hence it is compact. Therefore $|x|$ attains the maximum γ_1 $(< \gamma_0)$ in $\mathfrak{J}$, and $\mathfrak{J} = \{x \mid |x| \leq \gamma_1\}$. Thus $|x|$ takes only values $\gamma_0 > \gamma_1 > \gamma_2 > \cdots$, which has no limit point except 0.

9) Cf. A. Weil, Sur les espaces à structure uniforme, Act. Sci. Ind. **551** (1937).

10) D. van Dantzig, Zur topologischen Algebra II, Comp. Math. **2** (1935).

11) In this case $|x|$ is a "Pseudobewertung" of K. Mahler, Über Pseudobewertungen I, Act. Math. **66** (1936).

$r_0^l < \varepsilon$. Now let $\mathfrak{R}$ be the ring of "ideale Zahlen" of Prüfer, i. e. the completion of the topological ring of all integers, in which a defining system of neighbourhoods of zero consists of all ideals $\mathfrak{J}_n = (p_1 p_2 \cdots p_n)^n$, $n = 1, 2, \ldots$, where $p_1, p_2, \cdots$ are all prime numbers[12]. Then no possible quasi-evaluation $|x|$ of $\mathfrak{R}$ satisfies (10′), because $\mathfrak{J}_{n_0} \leq \mathfrak{J}_0$ for some n_0 but $\mathfrak{J}_n \not\geq \mathfrak{J}_{n_0}^t$ $(t = 1, 2, \ldots)$ for $n > n_0$.

12) H. Prüfer, Neue Begründung der algebraischen Zahlentheorie, Math. Ann. **94** (1925). van Dantzig, loc. cit., p. 205, calls them "universelle Zahlen."