PAPERS COMMUNICATED

53. *On the Strong Summability of Series of Orthogonal Functions.*

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(Comm. by M. FUJIWARA, M.I.A., May 12, 1944.)

Let $\{\varphi_n(x)\}$ be a system of normalized orthogonal functions in (a, b) and consider the series

(1) $$\sum_{\nu=0}^{\infty} c_\nu \varphi_\nu(x)$$

such that $$\sum_{\nu=0}^{\infty} c_\nu^2 < \infty .$$

By the Riesz-Fisher theorem, the series (1) converges in the mean to a function $f(x)$ in L^2. As usual we denote by $s_n(x)$ and $\sigma_n(x)$ the partial sum and $(C, 1)$-mean of the series (1) respectively. In this paper we discuss the convergency of

(2) $$\sum_{n=1}^{\infty} | s_n(x) - f(x) |^k/n , \qquad k > 1 ,$$

and

(3) $$\sum_{n=1}^{\infty} | s_n(x) - \sigma_n(x) |^k/n , \qquad k > 1 .$$

For the case of trigonometrical system, the former is considered by Hardy and Littlewood[1] and the latter by Zygmund[2].

As an application of our theory, we shall give an alternative proof of the Rademacher[3]-Menchof[4] theorem regarding the almost everywhere convergence of the series (1).

1. *Convergency of the series* $\sum_{n=1}^{\infty} (s_n - f)^2/n$.

(1.1) *In the series* (1), *we get*

$$\int_a^b \{\sum_{n=1}^{\infty} (s_n - f)^2/n\}\, dx \leqq A \sum_{n=1}^{\infty} c_n^2 \log n^{5)} .$$

For,

$$\sum_{n=1}^{\infty} \frac{1}{n} \int_a^b (s_n - f)^2 dx = \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{\nu=n+1}^{\infty} c_\nu^2 \right) = \sum_{\nu=2}^{\infty} c_\nu^2 \sum_{n=1}^{\nu-1} \frac{1}{n} \leqq A \sum_{\nu=1}^{\infty} c_\nu^2 \log \nu ,$$

which is the required.

For the case of trigonometrical system, we have

$$\sum_{n=1}^{\infty} c_n^2 \log n \sim \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} | f(x+t) + f(x-t) - 2f(x) |^2/2t dx dt .$$

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1) G. H. Hardy and J. E. Littlewood, Duke Math. Journ., **2** (1936), pp. 354–382.
2) A. Zygmund, Fund. Math., **30** (1938), pp. 170–196.
3) H. Rademacher, Math. Ann., **87** (1922), pp. 112–138.
4) D. Menchof, Fund. Math., **4** (1923), pp. 82–105.
5) $A, B, \ldots$ are constants, not always the same from one occurrence to another.

Thus (1.1) is a special case of the result due to Hardy and Littlewood[1].

(1.2) *If* $\sum_{n=1}^{\infty} c_n^2 \log n < \infty$, *then* $\sum_{n=0}^{\infty} c_n\varphi_n$ *is* $(C, 1)$*-summable almost everywhere in* (a, b).

This is evident from (1.1) by the Kronecker theorem. This theorem is a classical result due to Weyl[2] and Hobson[3]. Moreover Borgen[4] proved that $\sum_{n=1}^{\infty} c_n^2 \{\log(\log n)\}^2 < \infty$ is sufficient for almost everywhere $(C, 1)$-summability. On the other hand Chen[5] proved that this is equivalent to the Rademacher-Menchof theorem in the next section.

2. *Proof of the Rademacher-Menchof theorem.*

(2.1) *If* $\sum_{n=1}^{\infty} c_n^2 (\log n)^2 < \infty$. *Then* $\sum_{n=0}^{\infty} c_n\varphi_n$ *is convergent almost everywhere in* (a, b).

The method of reduction of (2.1) from (1.2) has been sketched by Zygmund[6]. But for the sake of completeness we reproduce it with some simplification.

If we put

$$A_n^\alpha = \binom{n+\alpha}{\alpha} \sim \frac{n^\alpha}{\Gamma(\alpha+1)}, \quad S_n^\alpha = \sum_{\nu=0}^{n} A_{n-\nu}^\alpha c_\nu \varphi_\nu, \quad \sigma_n^\alpha = S_n^\alpha / A_n^\alpha,$$

then we have $$\sigma_n^{\alpha-1} - \sigma_n^\alpha = \left(\sum_{\nu=0}^{n} \nu A_{n-\nu}^{\alpha-1} c_\nu \varphi_\nu\right) / \alpha A_n^\alpha$$

and $$\int_a^b (\sigma_n^{\alpha-1} - \sigma_n^\alpha)^2 dx = \left\{\sum_{\nu=0}^{n} \nu^2 (A_{n-\nu}^{\alpha-1})^2 c_\nu^2\right\} / \alpha^2 (A_n^\alpha)^2.$$

Accordingly

$$\sum_{n=0}^{\infty} \int_a^b \frac{(\sigma_n^{\alpha-1} - \sigma_n^\alpha)^2}{n+1} dx = \sum_{n=0}^{\infty} \left\{\sum_{\nu=0}^{n} \nu^2 (A_{n-\nu}^{\alpha-1})^2 c_\nu^2\right\} / \alpha^2 (n+1) (A_n^\alpha)^2$$

$$\leqq B \sum_{n=0}^{\infty} \left\{\sum_{\nu=0}^{n} \nu^2 (n-\nu+1)^{2(\alpha-1)} c_\nu^2\right\} / (n+1)^{2\alpha+1}$$

$$\leqq B \sum_{\nu=0}^{\infty} \nu^2 c_\nu^2 \sum_{n=\nu}^{\infty} (n-\nu+1)^{2(\alpha-1)} / (n+1)^{2\alpha+1}$$

$$\leqq B \sum_{\nu=0}^{\infty} \nu^2 c_\nu^2 \sum_{n=\nu}^{2\nu} (n-\nu+1)^{2(\alpha-1)} / (n+1)^{2\alpha+1} + B \sum_{\nu=0}^{\infty} \nu^2 c_\nu^2 \sum_{n=2\nu+1}^{\infty} (n-\nu+1)^{2(\alpha-1)} / (n+1)^{2\alpha+1}$$

$$\leqq P + Q, \quad \text{say.}$$

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1) G. H. Hardy and J. E. Littlewood, loc. cit.
2) H. Weyl, Math. Ann., **67** (1909), pp. 225–245.
3) E. W. Hobson, Proc. London Math. Soc., **12** (1912), pp. 297–308.
4) S. Borgen, Math. Ann., **98** (1928), pp. 125–150.
5) K. Chen, Tôhoku Math. Journ., **29** (1928), pp. 125–150.
6) A. Zygmund, Fund. Math., **10** (1927), pp. 356–362.

Then $$P \leqq B\sum_{\nu=0}^{\infty} \nu^2 c_\nu^2(\nu+1)^{-2\alpha-1}\sum_{n=\nu}^{\infty}(n-\nu+1)^{2(\alpha-1)} \quad (1\geqq \alpha>1/2)$$

$$\leqq B\sum_{\nu=0}^{\infty}\nu^{-2\alpha+1}c_\nu^2\nu^{2\alpha-2+1}\leqq C_1\sum_{\nu=1}^{\infty}c_\nu^2,$$

$$Q\leqq B\sum_{\nu=1}^{\infty}\nu^{-2\alpha+1}c_\nu^2\sum_{n=2\nu+1}^{\infty}n^{2(\alpha-1)}\leqq C_2\sum_{\nu=1}^{\infty}c_\nu^2.$$

Thus we get

$$\int_a^b\sum_{n=0}^{\infty}(\sigma_n^{\alpha-1}-\sigma_n^\alpha)^2/(n+1)\,dx\leqq D\sum_{n=0}^{\infty}c_n^2, \quad \text{where} \quad 1\geqq\alpha>1/2.$$

In the analogous way, we get for $\alpha=1/2$

$$\int_a^b\sum_{n=0}^{\infty}(\sigma_n^{\alpha-1}-\sigma_n^\alpha)^2/(n+1)\,dx\leqq E\sum_{n=1}^{\infty}c_n^2\log n.$$

Thus we proved the theorem:

(2.1.1.) *If* $1\geqq\alpha>1/2$ *then we have*

$$\int_a^b\sum_{n=0}^{\infty}(\sigma_n^{\alpha-1}-\sigma_n^\alpha)^2/(n+1)\,dx\leqq A\sum_{n=0}^{\infty}c_n^2,$$

and for $\alpha=1/2$,

$$\int_a^b\sum_{n=0}^{\infty}(\sigma_n^{\alpha-1}-\sigma_n^\alpha)^2/(n+1)\,dx\leqq B\sum_{n=1}^{\infty}c_n^2\log n.$$

Further we have

(2.1.2.) *If* $\sum_{\nu=0}^{n}(\sigma_\nu^\alpha)^2/(n+1)\to 0$, *then* $\sigma_n^{\alpha+1/2+\varepsilon}\to 0$, *for* $\alpha>-1/2$, $\varepsilon>0$ *and* $s_n=o(\sqrt{\log n})$ *for* $\alpha=-1/2$.

For, $$|S_n^{\alpha+1/2+\varepsilon}|=|\sum_{\nu=0}^{n}S_\nu^\alpha A_{n-\nu}^{-1/2+\varepsilon}|=\sum_{\nu=0}^{n}|\sigma_\nu^\alpha A_\nu^\alpha A_{n-\nu}^{-1/2+\varepsilon}|$$

$$\leqq\sqrt{\sum_{\nu=0}^{n}(\sigma_\nu^\alpha)^2}\sqrt{\sum_{\nu=0}^{n}(A_\nu^\alpha A_{n-\nu}^{-1/2+\varepsilon})^2}\leqq o(\sqrt{n})O\Big(\sqrt{\sum_{\nu=0}^{n}A_\nu^\alpha A_{n-\nu}^{-1+2\varepsilon}}\Big)$$

$$=o(\sqrt{n})O(\sqrt{n^{2\alpha+2\varepsilon}})=o(n^{\alpha+1/2+\varepsilon}).$$

The remaining part is analogous.

Proof of the theorem. If $\sum_{n=1}^{\infty}c_n^2\log n<\infty$, then by (1.2), (1) is $(C,1)$-summable. From (2.1.1.) and (2.1.2.), $s_n=o(\sqrt{\log n})$. By the well known theorem, the series

$$\sum_{n=2}^{\infty}\frac{c_n}{\sqrt{\log n}}\varphi_n$$

converges almost everywhere, provided that $\sum_{n=1}^{\infty}c_n^2\log n<\infty$. Thus $\sum_{n=0}^{\infty}c_n\varphi_n$ converges almost everywhere, provided that $\sum_{n=1}^{\infty}c_n^2(\log n)^2<\infty$.

3. *Behaviour of the series* $\sum_{n=1}^{\infty}|s_n-f|^k/n$.

(3.1) *If* $|\varphi_n(x)|\leqq K$, $(n=0,1,2,\dots)$ *and* $f\sim\sum_{\nu=0}^{\infty}c_\nu\varphi_\nu$, *then*

$$\int_a^b\sum_{n=1}^{\infty}|s_n-f|^q/ndx\leqq A\sum_{n=1}^{\infty}|c_n|^q n^{q-2}\log n\,,$$

and $$\Big(\int_a^b\sum_{n=1}^{\infty}|s_n-f|^q/ndx\Big)^{1/q}\leqq B\Big(\sum_{n=1}^{\infty}|c_n|^p(\log n)^{p/q}\Big)^{1/p},$$

where $$1<p\leqq 2\leqq q<\infty,\qquad 1/p+1/q=1\,.$$

For, by Paley's theorem,

$$\sum_{n=1}^{\infty}\frac{1}{n}\int_a^b|s_n-f|^q dx\leqq A\sum_{n=1}^{\infty}\frac{1}{n}\sum_{\nu=n+1}^{\infty}|c_\nu|^q\nu^{q-2}.$$

The righthand side series is less than

$$\leqq A\sum_{\nu=2}^{\infty}|c_\nu|^q\nu^{q-2}\sum_{n=1}^{\nu-1}1/n\leqq B\sum_{\nu=1}^{\infty}|c_\nu|^q\nu^{q-2}\log\nu\,.$$

Further $$\{\sum_{\nu=1}^{\infty}|c_\nu|^q\nu^{q-2}(\log^{1/q}\nu)^q\}^{\frac{1}{q}}\leqq C\Big(\sum_{\nu=1}^{\infty}|c_\nu|^p(\log\nu)^{p/q}\Big)^{1/p}.$$

Accordingly $$\Big(\int_a^b\sum_{n=1}^{\infty}|s_n-f|^q/ndx\Big)^{1/q}\leqq C\Big(\sum_{\nu=1}^{\infty}|c_\nu|^p(\log\nu)^{p/q}\Big)^{1/p}.$$

Thus we get the theorem.

Analogously we get

(3.2) *If* $|\varphi_n(x)|\leqq K$ $(n=0,1,2,\dots)$, *then*

$$\int_a^b\sum_{n=1}^{\infty}|s_n-f|^p/ndx\geqq D\sum_{n=1}^{\infty}|c_n|^p n^{p-2}\log n$$

and $$\Big(\int_a^b\sum_{n=1}^{\infty}|s_n-f|^p/ndx\Big)^{1/p}\geqq E\Big(\sum_{n=1}^{\infty}|c_n|^q(\log n)^{q/p}\Big)^{1/q},$$

where $$1<p\leqq 2\leqq q<\infty,\qquad 1/p+1/q=1\,.$$

These results were given by Izumi and Kawata[1] under more severe conditions.

4. *Behaviour of* $\sum_{n=1}^{\infty}|s_n-\sigma_n|^k/n$.

(4.1) *If* $|\varphi_n(x)|\leqq K$ $(n=0,1,2,\dots)$, *then*

$$\int_a^b\sum_{n=1}^{\infty}|s_n-\sigma_n|^q/ndx\leqq A\sum_{n=1}^{\infty}n^{q-2}|c_n|^q,$$

and $$\Big(\int_a^b\sum_{n=1}^{\infty}|s_n-\sigma_n|^q/ndx\Big)^{1/q}\leqq B\Big(\sum_{n=1}^{\infty}|c_n|^p\Big)^{1/p},$$

where, $$1<p\leqq 2\leqq q<\infty,\qquad 1/p+1/q=1$$

For, $$s_n-\sigma_n=\Big(\sum_{\nu=1}^{n}\nu c_\nu\varphi_\nu\Big)/(n+1)\,.$$

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1) S. Izumi and T. Kawata, Tôhoku Math. Journ. **45** (1939), pp. 134–144.

By Paley's theorem,

$$\int_a^b |s_n-\sigma_n|^q dx=\int_a^b |(\sum_{\nu=1}^{n} \nu c_\nu\varphi_\nu)/(n+1)|^q dx$$
$$\leqq A\{\sum_{\nu=1}^{n} |\nu c_\nu|^q \nu^{q-2}\}/n^q .$$

Therefore $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}\int_a^b |s_n-\sigma_n|^q dx \leqq A\sum_{n=1}^{\infty}\frac{1}{n^{q+1}}\{\sum_{\nu=1}^{\infty}|\nu c_\nu|^q\nu^{q-2}\}$

$$\leqq A\sum_{\nu=1}^{\infty}\nu^{2q-2}|c_\nu|^q\sum_{n=\nu}^{\infty}1/n^{q+1}$$
$$\leqq B\sum_{\nu=1}^{\infty}\nu^{2q-2}|c_\nu|^q\cdot\nu^{-q}\leqq B\sum_{\nu=1}^{\infty}\nu^{q-2}|c_\nu|^q .$$

Thus we get the first inequality of the (4.1). The remaining is given by

$$(\sum_{\nu=1}^{\infty}\nu^{q-2}|c_\nu|^q)^{1/q}\leqq C(\sum_{\nu=1}^{\infty}|c_\nu|^p)^{1/p} .$$

Thus we complete the proof of theorem.

Analogously we get

(4.2) *If* $|\varphi_n(x)|\leqq K$, $(n=0,1,2,\ldots)$, *then*

$$\int_a^b\sum_{n=1}^{\infty}|s_n-\sigma_n|^p/n dx\geqq C\sum_{n=1}^{\infty}n^{p-2}|c_n|^p ,$$
$$\left(\int_a^b\sum_{n=1}^{\infty}|s_n-\sigma_n|^p/n dx\right)^{1/p}\geqq D(\sum_{n=1}^{\infty}|c_n|^q)^{1/q} .$$

where $\quad 1<p\leqq 2\leqq q<\infty ,\qquad 1/p+1/q=1 .$

These results were considered also by Izumi and Kawata[1] under more severe conditions.

5. *Behaviour of the sequence* $\{s_{p_\nu}\}$.

For any increasing sequence $\{p_\nu\}$, $(C,1)$-summability of $\{s_{p_\nu}\}$ is considered by Zalcwasser[2]. He opened the problem: *For any* $f(x)\in L^2$, *is the sequence* $\{s_{p_\nu}\}$ $(C,1)$-*summable for all* $\{p_\nu\}$, *where* $\varphi_n(x)$ *is trigonometrical system.* Regarding this problem we get

(5.1) *If* $\sum_{\nu=1}^{\infty}c_\nu^2$ *converges,* $(C,1)$-*summability of* $\{s_{p_\nu}\}$ *is equivalent to the convergency of* $\{s_{p_{2^\nu}}\}$.

If $c_\nu\neq 0$, then we put

$$\psi_\nu(x)=(c_{p_{\nu-1}+1}\varphi_{p_{\nu-1}+1}+\cdots+c_{p_\nu}\varphi_{p_\nu})/\gamma_\nu ,$$

where $\quad \gamma_\nu=(c^2_{p_{\nu-1}+1}+\cdots+c^2_{p_\nu})^{1/2} ,$

1) S. Izumi and T. Kawata, Tôhoku Math. Journ., **45** (1939), pp. 212-218.
2) Z. Zalcwasser, Studia Math., **6** (1936), pp. 82-88.

and consider the series $\sum_{\nu=0}^{\infty} \gamma_\nu \psi_\nu(x)$.

Since $\{\psi_n(x)\}$ is a normalized orthogonal system and $\sum_{\nu=0}^{\infty} \gamma_\nu^2 < \infty$, the $(C, 1)$-summability of $\sum_{\nu=0}^{\infty} \gamma_\nu \psi_\nu(x)$ is equivalent to the convergency of $\{t_{2^\nu}\}$ where t_ν is the ν-th partial sum of $\sum_{\nu=0}^{\infty} \gamma_\nu \psi_\nu(x)$[1]. Thus we get the theorem.

From this, Zalcwasser's problem will perhaps be negatively answered, but the author could not conclude it.

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1) S. Kaczmarz und H. Steinhaus, *Theorie der Orthogonalreihen*, (1935), p. 190.