95. *Equivalence of Two Topologies of Abelian Groups.*

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(Comm. by T. TAKAGI, M.I.A., July 12, 1944.)

Let G be a locally compact (=bicompact), separable abelian group and let X be the totality of continuous characters[1] $\chi(g)$ of G. It is well known[2] that X is also a locally compact, separable abelian group by the multiplication

$$\chi_1\chi_2(g)=\chi_1(g)\chi_2(g)$$

and by Pontrjagin's topology induced from the (closed) neighbourhood:

$$U(\chi_1)=\{\chi;\ \sup_{g\in G_0}|\chi(g)-\chi_1(g)|\leqq\varepsilon,\qquad G_0=\text{compact subset of } G\}.$$

X also constitutes a locally compact, separable topological space $\widetilde{X}$ by the topology induced from the (closed) neighbourhood:

$$\widetilde{V}(\chi_1)=\left\{\chi;\left|\int_G x_i(g)\chi(g)dg-\int_G x_i(g)\chi_1(g)dg\right|\leqq\varepsilon,\quad i=1,2,\dots,n\right\}$$

where $x_i(g)\in L_1(G)$ viz. $x_i(g)$ denote measurable functions integrable over G with respect to Haar's invariant measure dg on G. The latter topology is introduced by I. Gelfand and D. Raikov[3], and its equivalence to Pontrjagin's topology plays a fundamental rôle in the ring-theoretic treatment and extension of the classical Fourier analysis based upon the theory of normed ring[4]. However the proof of the equivalence is, so far as we know, not published by the Russian school, though stated and used by them repeatedly[5].

The purpose of the present note is i): to give it a proof and ii) to show that the character group is a topological group in Gelfand-Raikov's topology even when G is not separable. For the purpose we make use of the following

Lemma. For any χ_2, the mapping

$$\chi\longrightarrow\chi_2\chi$$

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1) A continuous character of G is a continuous homomorphic mapping of G in the topological group of rotations of a circle.

2) L. Pontrjagin: Topological group, Princeton (1939), 127.

3) C. R. URSS, **28**, 3 (1940).

4) D. Raikov: C. R. URSS, **28**, 4 (1940). M. Krein: C. R. URSS, **30**, 6 (1941). D. Raikov: C. R. URSS, **30**, 7 (1941). K. Yosida: Proc. **20** (1944), 269. The author (Yosida) wishes to withdraw the §3 of this note, since the Lemma 2 is valid for $z\in L_1(G)$ only and thus the arguments in §3 is insufficient. A complete proof and the fact that Bochner-Raikov's theorem may be derived from Plancherel's theorem will be published elsewhere.—During the proof, Y. Kawada kindly communicated that 3° may be obtained from Bochner-Raikov's theorem.

5) H. Anzai kindly communicated M. Fukamiya's unpublished proof of the equivalence, which is entirely different from ours.

of $\widetilde{X}$ on $\widetilde{X}$ is a topological one.

Proof. The inclusion $\chi\in\widetilde{U}(\chi_1)$ is equivalent to the inclusion $\chi_2\chi\in\widetilde{U}(\chi_2\chi_1)$ where

$$\begin{cases} U(\chi_2\chi_1)=\left\{\chi' ; \left|\int_G x_i'(g)\chi'(g)dg-\int_G x_i'(g)\chi_2\chi_1(g)dg\right|\leqq\varepsilon, \quad i=1,2,\dots,n\right\} \\ x_i'(g)=x_i(g)\chi_2^{-1}(g), \qquad \chi_2^{-1}(g)=\chi_2(-g) \end{cases}$$

Q. E. D.

Proof of i). It is evident that the mapping

$$X\ni\chi\rightarrow\chi\in\widetilde{X} \tag{1}$$

is continuous. Hence, by the lemma, we have only to show that for any neighbourhood $U(\chi_0)$ in X of the unit-character $\chi_0\big(\chi_0(g)\equiv 1\big)$ there exists a neighbourhood $\widetilde{V}(\chi_0)$ in $\widetilde{X}$ of χ_0 satisfying $U(\chi_0)\geqq\widetilde{V}(\chi_0)$ as subset (without topology) of X.

Let $W(\chi_0)$ be a compact and symmetric neighbourhood in X of χ_0:

$$W(\chi_0)=W(\chi_0)^{-1}=\{\chi^{-1};\ \chi\in W(\chi_0)\} \tag{2}$$

such that

$$U(\chi_0)\geqq W(\chi_0)^2=\{\chi_\alpha\chi_\beta;\ \chi_\alpha,\chi_\beta\in W(\chi_0)\}. \tag{3}$$

Since X is separable there exists an enumerable sequence $\{\chi_i\}\leqq X$ such that

$$X=\bigcup_{i=1}^{\infty}\chi_i W(\chi_0),$$

where

$$\chi_i W(\chi_0)=\{\chi;\ \chi=\chi_i\chi',\ \chi'\in W(\chi_0)\}. \tag{4}$$

By the continuity of the mapping (1), the image $\widetilde{\chi_i W(\chi_0)}$ in $\widetilde{X}$ of the compact set $\chi_i W(\chi_0)$ in X is also a compact set of $\widetilde{X}$. $\widetilde{X}$ being complete as a locally compact space, at least one compact set $\widetilde{\chi_i W(\chi_0)}$ contains a neighbourhood $\widetilde{V}_1(\chi_a)$. This results from the fact that a complete space is not of Baire's first category. Hence, by the lemma, the neighbourhood $\widetilde{\chi_a^{-1}\chi_i W(\chi_0)}$ in $\widetilde{X}$ contains a neighbourhood $\widetilde{V}_1(\chi_0)$ in $\widetilde{X}$ of χ_0. Thus there exists χ such that the neighbourhood $\chi\widetilde{W}(\chi_0)$ in $\widetilde{X}$ contains a neighbourhood in $\widetilde{X}$ of χ_0. Hence $\chi\widetilde{W}(\chi_0)\ni\chi_0$ and thus $\widetilde{W}(\chi_0)\ni\chi^{-1}$, $\widetilde{W}(\chi_0)\ni\chi$ by (2). Therefore, by (3), there exists a neighbourhood $\widetilde{V}(\chi_0)$ in $\widetilde{X}$ of χ_0 which satisfies

$$U(\chi_0)\geqq W(\chi_0)^2\geqq\widetilde{V}(\chi_0), \qquad \text{as subsets (without topology) of } X.$$

Q. E. D.

Remark. The separability of G is only used in (4). Hence i) holds good if, for example, X is compact or connected.

Proof of ii). Since $x(g)\in L_1(G)$ implies $y(g)=x(-g)\in L_1(G)$ and v. v., a subset $\widetilde{V}$ of $\widetilde{X}$ is a neighbourhood $\chi\in\widetilde{X}$ if and only if V^{-1} is a neighbourhood of χ^{-1}. Thus χ^{-1} is a continuous function of χ.

It remains to show that $\chi_1\chi_2$ is a continuous function of two variables χ_1, χ_2. By the lemma and by the commutativity of $\widetilde{X}$, it is sufficient to prove that for every neighbourhood $\widetilde{V}$ of the unit-character χ_0 there exists a neighbourhood $\widetilde{W}$ of χ_0 such that $\widetilde{W}^2 \subseteqq \widetilde{V}$.

A generic neighbourhood of χ_0 is the intersection of a finite number of sets of the form

$$\widetilde{U}(x, \varepsilon)=\left\{\chi\,;\left|\int_G x(g)\chi_0(g)dg-\int_G x(g)\chi(g)dg\right|<\varepsilon\right\},$$

where $\varepsilon > 0$ and $x(g) \in L_1(G)$. For any such ε, x there exists a step function $y(g) \in L_1(G)$ satisfying $\|x-y\| < \varepsilon/3$. Since this inequality implies that for every $\chi \in \widetilde{X}$ $\left|\int_G x(g)\chi(g)dg-\int_G y(g)\chi(g)dg\right| < \varepsilon/3$, we have $\widetilde{U}(y, \varepsilon/3) \subseteqq \widetilde{U}(x, \varepsilon)$. We may therefore take as $x(g)$ only step functions from $L_1(G)$, and so only characteristic functions x_E of measurable subsets E of G with $0 < m(E) < \infty$ (m indicates Haar's measure).

Put

$$\widetilde{U}(E, \varepsilon)=\left\{\chi\,;\left|\int_G x_E(g)\chi_0(g)dg-\int_G x_E(g)\chi(g)dg\right|<\varepsilon\right\}$$
$$=\left\{\chi\,;\left|\int_E\big(1-\chi(g)\big)dg\right|<\varepsilon\right\}.$$

Let $\varepsilon' > 0$ and put $D(\chi, E, \varepsilon')=\{g\,;\, g \in E,\ |1-\chi(g)| > \varepsilon'\}$; this is a measurable set with finite measure. It is easily seen that, if $\varepsilon' < \varepsilon$ and if ε', $\delta > 0$ are sufficiently small, the set $V(E, \varepsilon', \delta)=\left\{\chi\,;\, m\big(D(\chi, E, \varepsilon')\big) < \delta\right\}$ is contained in $\widetilde{U}(E, \varepsilon)$. Moreover, we have always $V(E, \varepsilon'/2, \delta)^2 \subseteqq V(E, \varepsilon', \delta)$. Hence it is sufficient to show that for any $\varepsilon', \delta > 0$ there exists an $\varepsilon > 0$ such that $\widetilde{U}(E, \varepsilon) \subseteqq V(E, \varepsilon', \delta')$. Such an ε may be given, as will be shown below, by $\varepsilon=\varepsilon''\delta$, where $\varepsilon''=\inf\limits_{|1-\chi(g)|>\varepsilon'} \cdot \big(1-\Re\chi(g)\big)$, $\Re$ indicating the real part of a complex number. Since $|\chi(g)|=1$, we have $1-\Re\chi(g) \geqq 0$ everywhere and $\varepsilon'' > 0$.

Suppose $\chi \in \widetilde{U}(E, \varepsilon)$, i. e. $\left|\int_E\big(1-\chi(g)\big)dg\right| < \varepsilon$. Then $D(\chi, E, \varepsilon')$ coincides with $C=\{g\,;\, g \in E,\ 1-\Re\chi(g) > \varepsilon''\}$. Since $\left|\int_E\big(1-\chi(g)dg\right|^2=\left\{\int_E\big(1-\Re\chi(g)\big)dg\right\}^2+\left\{\int_E \Im\chi(g)dg\right\}^2$, where $\Im$ indicates the imaginary part of a complex number, we have

$$\varepsilon > \int_E\big(1-\Re\chi(g)\big)dg \geqq \int_C\big(1-\Re\chi(g)\big)dg \geqq \varepsilon'' m(C)=\varepsilon'' m\big(D(\chi, E, \varepsilon')\big).$$

Hence $m\big(D(\chi, E, \varepsilon')\big) < \varepsilon/\varepsilon''=\delta$, i. e. $\chi \in V(E, \varepsilon', \delta)$. Thus we have shown $\widetilde{U}(E, \varepsilon) \subseteqq V(E, \varepsilon', \delta)$, and the proof is complete.