112. On Fourier Constants.

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G. H. Hardy[1] proved the following theorem:

(*A*) *If* $\{a_n\}$ *are the Fourier constants of a function of* L_p $(p \geqq 1)$, *then* $\{(\sum_{k=1}^{n} a_k)/n\}$ *are also the Fourier constants of a function of* L_p.

Recently T. Kawata[2] has proved a dual form of (*A*), that is:

(*B*) *If* $\{a_n\}$ *are the Fourier sine constants of a function of* L_p $(p > 1)$, *then* $\{\sum_{k=n}^{\infty} a_k/k\}$ *are also the Fourier sine constants of a function of* L_p. *Moreover if* $\{a_n\}$ *are the Fourier sine constants of a function of* L_z, *then* $\{\sum_{k=n}^{\infty} a_k/k\}$ *are the Fourier sine constants of a function of* L.

In the present note the author considers the case of cosine constants and completes (*B*) in the following form.

Theorem 1. If $\{a_n\}$ *are the Fourier constants of a function* L_p $(p > 1)$, *then* $\{\sum_{k=n}^{\infty} a_k/k\}$ *are also the Fourier constants of a function of* L_p. *Moreover if* $\{a_n\}$ *are the Fourier constants of a function of* L_z, *then* $\{\sum_{k=n}^{\infty} a_k/k\}$ *are the Fourier constants of a function of* L.

The method of proof is analogous to that of Kawata, but is somewhat delicate.

Proof of the case L_p. It is sufficient to prove the theorem for pure cosine series without constant term, that is $\int_0^\pi f(x)dx = 0$.

Let

$$(1) \qquad f(x) \sim \sum_{k=1}^{\infty} a_k \cos kx, \qquad f(x) \in L_p,$$

$$(2) \qquad g(x) \sim \sum_{k=1}^{\infty} \frac{1}{k} \cos kx,$$

then $g(x) \in L_r$ for all $r \geqq 1$ by the Hausdorff-Young theorem.

By Parseval's relation[3], we have

$$(3) \qquad \sum_{k=n}^{\infty} \frac{a_k}{k} = \frac{2}{\pi}\int_0^\pi f(x)g(x)dx - \frac{2}{\pi}\int_0^\pi f(x) \sum_{k=1}^{n-1} \frac{\cos kx}{k}\, dx.$$

The left-hand side series is summable $(C, 1)$, and further in this case it converges as $f(x) \in L_p$.

1) G. H. Hardy, Messenger of Math., **58** (1928), 50-52.
2) T. Kawata, Proc. **20** (1944), 218-222.
3) A. Zygmund, *Trigonometrical series*, (1935), 88.

Let $\int_0^x f(t)dt = F(x)$, then $F(\pi)=0$.

Since $$g(x)=\frac{1}{2}\log\frac{1}{2(1-\cos x)},$$

and $$\lim_{x\to+0}\left(\log\frac{1}{x}\right)\int_0^x f(t)dt=\lim_{x\to+0}\left(\log\frac{1}{x}\right)\cdot x\int_0^x |f(t)|^p dx=0,$$

the right-hand side of (3) becomes

$$\frac{2}{\pi}\Big[F(x)g(x)\Big]_0^\pi+\frac{2}{\pi}\int_0^\pi F(x)\frac{1}{2}\cot\frac{x}{2}dx-\frac{2}{\pi}\Big[F(x)\sum_{k=1}^{n-1}\frac{\cos kx}{k}\Big]_0^\pi$$
$$-\frac{2}{\pi}\int_0^\pi F(x)\sum_{k=1}^{n-1}\sin kx dx$$
$$=\frac{2}{\pi}\left\{\int_0^\pi F(x)\frac{1}{2}\cot\frac{x}{2}dx-\int_0^\pi F(x)\frac{\cos\frac{1}{2}x-\cos\left(n-\frac{1}{2}\right)x}{2\sin\frac{1}{2}x}dx\right\}$$
$$=\frac{2}{\pi}\int_0^\pi F(x)\frac{\cos\left(n-\frac{1}{2}\right)x}{2\sin\frac{1}{2}x}dx$$
$$=\frac{2}{\pi}\int_0^\pi F(x)\frac{1}{2}\cot\frac{1}{2}x\cos nx dx+\frac{1}{\pi}\int_0^\pi F(x)\sin nx dx.$$

Since $F(x)\frac{1}{2}\cot\frac{1}{2}x\in L_p$ and $\int_0^\pi F(x)\sin nx dx=0(n^{-1})$, we get the first part of the theorem.

Proof of the case L_z. For every $\lambda<1/e$, $e^{\lambda g}\in L$[4]. Since L_z and $L_{\exp.\lambda}$ are Young's complementary classes, (3) is still valid and convergency is assured by the Hardy-Littlewood theorem[5].

$$\lim_{x\to+0}\left(\log\frac{1}{x}\right)\int_0^x f(t)dt=0$$

follows from the inequality[6]

$$\left(\log\frac{1}{x}\right)\int_0^x|f(t)|dt\leqq\int_0^x|f(t)|\log\frac{1}{t}dt\leqq 2\int_0^x|f|\log^+|f|dt+\frac{4\sqrt{x}}{e},$$
$$(|x|<1).$$

And that $F(x)/x\in L$ is nothing but the maximal theorem of Hardy-Littlewood. Thus we complete the proof of the theorem.

Remark. There exist Fourier cosine constants of a function L such as $\sum_{k=1}^{\infty}a_k/k=\infty$. $\sum_{n=2}^{\infty}\frac{\sin nx}{(\log n)^2}$ is sine series of a function of L.

4) A. Zygmund, *T. S.*, 234.
5) A. Zygmund, *T. S.*, 138.
6) G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities*, (1934), 168–169.

As $\sum_{k=n}^{\infty}\frac{1}{k(\log k)^2}\sim\int_x^{\infty}\frac{dt}{t(\log t)^2}=\frac{1}{\log x}\sim\frac{1}{\log n}$, $\sum_{k=n}^{\infty}\frac{1}{k(\log k)^2}$ cannot be sine constants[7]. Thus our theorem is best possible in a sense.

In the Fourier integral we get analogous theorem by Titschmarsh's argument[8].

Theorem 2. If $F(x)$ *is the transform of* $f(x)\in L_p$ $(1<p\leqq 2)$, *then* $\int_x^{\infty}\frac{F(t)}{t}dt$ *is the transform of* $\frac{1}{x}\int_0^x f(t)dt$ *which belongs to* L_p.

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7) A. Zygmund, *T. S.*, 112.

8) E. C. Titchmarsh, *Introduction to the theory of Fourier integrals*, (1937), 93.