105. On the Reducibility of the Differential Equations in the n-Body Problem.

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It is known that the system of differential equations for the motion of n bodies can be reduced to a system of differential equations of order $6n-12$ from that of order $6n$ by the aid of the Eulerian integrals of the eliminations of the node and of the time. Lie's theory on the contact transformation groups and the function-groups has been applied for carrying out the effective reduction of the order of this system of differential equations[1]. Among others É. Cartan's procedure is the most elegant in employing the theory of integral invariants[2]. In the present note I propose to modify the procedure by avoiding the explicit appearance of time in the treatment[3] and also to discuss the n-body problem in the planar case.

Let, according to Poincaré[4], $x_{3j-2}, x_{3j-1}, x_{3j}$ be the Cartesian coordinates of the j-th body with mass $m_{3j-2}=m_{3j-1}=m_{3j}$, $(j=1,2,\dots,n)$, and $y_{3j-2}, y_{3j-1}, y_{3j}$ be the Cartesian components of the momentum of the j-th body. Then the motion of the n bodies is represented by the following canonical system of differential equations.

$$\frac{dx_i}{dt}=\frac{\partial H}{\partial y_i}, \qquad \frac{dy_i}{dt}=-\frac{\partial H}{\partial x_i}, \qquad (i=1,2,\dots,3n-1,3n),$$

where

$$H=T-U,$$

$$T=\sum_{k=1}^{3n}\frac{1}{2m_k}y_k^2, \qquad U=\sum_{i\neq j}\frac{m_{3i}m_{3j}}{\Delta_{i,j}},$$

$$\Delta_{i,j}^2=(x_{3i-2}-x_{3j-2})^2+(x_{3i-1}-x_{3j-1})^2+(x_{3i}-x_{3j})^2.$$

This system of differential equations admit the infinitesimal transformations:

$$A_0f=\frac{\partial f}{\partial t}, \quad A_1f=\sum_{j=1}^{n}\frac{\partial f}{\partial x_{3j-2}}, \quad A_2f=\sum_{j=1}^{n}\frac{\partial f}{\partial x_{3j-1}}, \quad A_3f=\sum_{j=1}^{n}\frac{\partial f}{\partial x_{3j}},$$

$$A_4f=\sum_{j=1}^{n}\left(-x_{3j}\frac{\partial f}{\partial x_{3j-1}}+x_{3j-1}\frac{\partial f}{\partial x_{3j}}\right), \quad A_5f=\sum_{j=1}^{n}\left(-x_{3j-2}\frac{\partial f}{\partial x_{3j}}+x_{3j}\frac{\partial f}{\partial x_{3j-2}}\right),$$

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1) S. Lie, Math. Ann., **8** (1874), 215; Gesammelte Abhandlung, **4** (1929), 1; Goursat, Leçons sur l'intégration des équations différentielles aux dérivées partielles du premier ordre, 1921; Engel-Faber, Die Lie'sche Theorie der partiellen Differentialgleichungen erster Ordnung, 1932; Englund, Sur les méthodes d'intégration de Lie et le problème de la mécanique céleste, Thèse, Uppsala, 1916; Engel, Göttinger Nachrichten, Math.-Phys. Kl., 1916, 270; 1917, 189.

2) E. Cartan, Leçons sur les invariants intégraux, 1922.

3) Y. Hagihara, Comptes Rendus Acad. Sc. Paris, **207** (1938), 390.

4) H. Poincaré, Bulletin Astr., **14** (1897), 53; Acta Mathematica, **21** (1897), 83 Leçons de mécanique céleste, **1** (1905). Chap. I.

$$A_6f=\sum_{j=1}^{n}\left(-x_{3j-1}\frac{\partial f}{\partial x_{3j-2}}+x_{3j-2}\frac{\partial f}{\partial x_{3j-1}}\right),$$

$$A_7f=\sum_{j=1}^{n}\left(m_{3j-2}\frac{\partial f}{\partial y_{3j-2}}+t\frac{\partial f}{\partial x_{3j-2}}\right),\quad A_8f=\sum_{j=1}^{n}\left(m_{3j-1}\frac{\partial f}{\partial y_{3j-1}}+t\frac{\partial f}{\partial x_{3j-1}}\right),$$

$$A_9f=\sum_{j=1}^{n}\left(m_{3j}\frac{\partial f}{\partial y_{3j}}+t\frac{\partial f}{\partial x_{3j}}\right).$$

Corresponding to each of these transformations we have ten Eulerian integrals:

$$H_0=H,\quad H_1=-\sum_{j=1}^{n}y_{3j-2},\quad H_2=-\sum_{j=1}^{n}y_{3j-1},\quad H_3=-\sum_{j=1}^{n}y_{3j},$$

$$H_4=\sum_{j=1}^{n}(x_{3j}y_{3j-1}-x_{3j-1}y_{3j}),\quad H_5=\sum_{j=1}^{n}(x_{3j-2}y_{3j}-x_{3j}y_{3j-2}),$$

$$H_6=\sum_{j=1}^{n}(x_{3j-1}y_{3j-2}-x_{3j-2}y_{3j-1}),\quad H_7=\sum_{j=1}^{n}(m_{3j-2}x_{3j-2}-y_{3j-2}t),$$

$$H_8=\sum_{j=1}^{n}(m_{3j-1}x_{3j-1}-y_{3j-1}t),\quad H_9=\sum_{j=1}^{n}(m_{3j}y_{3j}-y_{3j}t).$$

In order to eliminate t among these integrals we write

$$H_7'=H_3H_8-H_2H_9,\quad H_8'=H_1H_9-H_3H_7,\quad H_9'=H_2H_7-H_1H_8,$$

where we have the identity:

$$H_1H_7'+H_2H_8'+H_3H_9'\equiv 0.$$

Hence the nine functions among the ten $H_0, H_1, H_2, H_3, H_4, H_5, H_6, H_7'$, H_8', H_9' form the function-group in the sense of Lie. The schema for Poisson's brackets is:

	H_0	H_1	H_2	H_3	H_4	H_5	H_6	H_7'	H_8'	H_9'
H_0	0	0	0	0	0	0	0	0	0	0
H_1	0	0	0	0	0	H_3	$-H_2$	0	MH_3	$-MH_2$
H_2	0	0	0	0	$-H_3$	0	H_1	$-MH_3$	0	MH_1
H_3	0	0	0	0	H_2	$-H_1$	0	MH_2	$-MH_1$	0
H_4	0	0	H_3	$-H_2$	0	H_6	$-H_5$	0	H_9'	$-H_8'$
H_5	0	$-H_3$	0	H_1	$-H_6$	0	H_4	$-H_9'$	0	H_7'
H_6	0	H_2	$-H_1$	0	H_5	$-H_4$	0	H_8'	$-H_7'$	0
H_7'	0	0	MH_3	$-MH_2$	0	H_9'	$-H_8'$	0	MH_9'	$-MH_8'$
H_8'	0	$-MH_3$	0	MH_1	$-H_9'$	0	H_7'	$-MH_9'$	0	MH_7'
H_9'	0	MH_2	$-MH_3$	0	H_8'	$-H_7'$	0	MH_8'	$-MH_7'$	0

where M denotes the total mass of the n bodies.

Let Δ be the determinant formed of the above matrix of order 10. The rank of the determinant Δ is 3. Hence the number of distinguished functions is 3.

Now, according to Lie, if the order of the function-group is r and the number of the distinguished functions is m, then the number of independent functions which are mutually in involution is $(r+m)/2$. In our case $r=9$, $m=3$. Thus the number of independent functions mutually in involution is 6. In the following I obtain these three distinguished functions and the six independent functions mutually in involution.

H_0 is in involution with any of the nine functions $H_1, H_2, H_3, H_4, H_5, H_6, H_7', H_8', H_9'$, as is evident from the above schema.

The functions H_1, H_2, H_3 are also mutually in involution, because Poisson's brackets formed of these three functions are all zero. The functions in involution with the six functions $H_1, H_2, H_3, H_4, H_5, H_6$ must be functions of these six functions. Let it be denoted by $\Phi(H_1, H_2, H_3, H_4, H_5, H_6)$, then it must satisfy

$$(H_1, \Phi)=0\,, \quad (H_2, \Phi)=0\,, \quad (H_3, \Phi)=0\,.$$

However, as there is an identity

$$H_1(H_1, \Phi)+H_2(H_2, \Phi)+H_3(H_3, \Phi)\equiv 0\,,$$

only two of these three equations are independent. From

$$H_3\frac{\partial\Phi}{\partial H_5}-H_2\frac{\partial\Phi}{\partial H_6}=0\,, \qquad -H_3\frac{\partial\Phi}{\partial H_4}+H_1\frac{\partial\Phi}{\partial H_6}=0$$

we get
$$\Phi=H_1H_4+H_2H_5+H_3H_6\,.$$

Thus H_0, H_1, H_2, H_3, Φ are mutually in involution.

Let Π be a function of the ten functions H_i, $(i=0, 1, \dots, 6)$, and H_j', $(j=7, 8, 9)$, and be in involution with H_0, H_1, H_2, H_3, Φ. Then it must satisfy

$$(H_0, \Pi)=(H_1, \Pi)=(H_2, \Pi)=(H_3, \Pi)=(\Phi, \Pi)=0\,.$$

From these equations we get

$$\Pi=(MH_4-H_7')^2+(MH_5-H_8')^2+(MH_6-H_9')^2\,.$$

Thus the six functions mutually in involution are

$$H_0, H_1, H_2, H_3, \Phi, \Pi\,.$$

The distinguished functions f are in involution with any other function in the function-group. f must be a function of H_1, H_2, H_3, Φ, Π and such that

$$(H_4,f)=(H_5,f)=(H_6,f)=(H_7',f)=(H_8',f)=(H_9',f)=0\,.$$

The solutions of this system of differential equations are

$$f=\Psi\equiv\frac{1}{2}(H_1^2+H_2^2+H_3^2)\,,$$

and
$$f=\Pi\,.$$

Hence H_0, Ψ and Π are the required distinguished functions.

Finally Lie's theorem states that the order 2ν of a system of differential equations is reduced to $2(\nu-\mu)$, when there exist μ independent integrals mutually in involution. We have six independent functions mutually in involution. Thus $\mu=6$. Hence the order $6n$ of our system of differential equations of the n-body problem is reduced to $6n-16$.

In the planar problem of n bodies the order of the system of differential equations is $4n$. Take $x_{3j-2}=y_{3j-2}=0$, $j=1,2,\dots,n$. The known integrals are

$$H_0=T-U\,,\quad H_1=-\sum_{j=1}^{n} y_{3j-1}\,,\quad H_2=-\sum_{j=1}^{n} y_{3j}\,,$$

$$H_6=\sum_{j=1}^{n}(x_{3j}y_{3j-1}-x_{3j-1}y_{3j})\,,$$

$$H_7=\sum_{j=1}^{n}(m_{3j-1}x_{3j-1}-y_{3j-1}t)\,,\qquad H_8=\sum_{j=1}^{n}(m_{3j}x_{3j}-y_{3j}t)\,.$$

The schema of the Poisson brackets is:

	H_0	H_1	H_2	H_6	H_7	H_8
H_0	0	0	0	0	0	0
H_1	0	0	0	$-H_2$	$-M$	0
H_2	0	0	0	H_1	0	$-M$
H_6	0	H_2	$-H_1$	0	H_8	$-H_7$
H_7	0	M	0	$-H_8$	0	0
H_8	0	0	M	H_7	0	0

The rank of the determinant formed of this matrix is 4. Hence $m=2$. $r=6$ on the other hand. Thus the number of independent functions mutually in involution is 4 and the number of distinguished functions is 2. The four independent functions mutually in involution are H_0, H_1, H_2, and $X=MH_6-H_2H_7+H_1H_8$. The two distinguished functions are H_0 and X. Hence the order of the system of differential equations is reduced to $4n-8$.