130. On Hopf's Ergodic Theorem.

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1. Let E be a measurable set of points in $|z|<1$. We define its hyperbolic measure $\sigma(E)$ by $\sigma(E)=\iint_E \frac{rdrd\theta}{(1-r^2)^2}$ $(z=re^{i\theta})$. Similarly the hyperbolic length $\lambda(C)$ of a rectifiable curve C is defined by $\lambda(C)=\int_C \frac{|dz|}{1-|z|^2}$.

Let G be a Fuchsian group of linear transformations, which make $|z|<1$ invariant and D_0 be its fundamental domain, which contains $z_0=0$ and is bounded by at most enumerably infinite number of orthogonal circles to $|z|=1$, z_n be equivalents of $z_0=0$ and $n(r)$ be the number of z_n in $|z|\leqq r$. For any z in $|z|<1$, we denote its equivalent in D_0 by (z). Let $E(\theta)$ be the set of points $(re^{i\theta})$ in D_0, which are equivalent to points on a radius $z=re^{i\theta}$ $(0\leqq r<1)$. In my former paper[1], I have proved:

Theorem 1. (i) *If* $\sum_{n=0}^{\infty}(1-|z_n|)=\infty$, *then* $E(\theta)$ *is everywhere dense in* D_0 *for almost all* $e^{i\theta}$ *on* $|z|=1$, (ii) *If* $\sum_{n=0}^{\infty}(1-|z_n|)<\infty$, *then* $\lim_{r\to 1}|(re^{i\theta})|=1$ *for almost all* $e^{i\theta}$ *on* $|z|=1$.

Theorem 2. The necessary and sufficient condition that there exists a set e *on* $|z|=1$, *which is invariant by* G *and* $0<me<2\pi$, *is that* $\sum_{n=0}^{\infty}(1-|z_n|)<\infty$.

Theorem 1 (i) is an extension of Myrberg's theorem[2], who assumed that D_0 lies with its boundary entirely in $|z|<1$, in which case, it is easily proved that $\sum_{n=0}^{\infty}(1-|z_n|)=\infty$.

2. Let $\eta_1=e^{i\theta}$, $\eta_2=e^{i\varphi}$ be two points on $|z|=1$, $|w|=1$ respectively. Then the pair (η_1,η_2) can be considered as a point on a torus $\Omega(0\leqq\theta\leqq 2\pi,\ 0\leqq\varphi\leqq 2\pi)$. For any measurable set E on Ω, we define its measure mE by $mE=\iint_E d\theta d\varphi$, so that $m\Omega=4\pi^2$.

Let S be any substitution of G and T: $\eta_1'=S(\eta_1)$, $\eta_2'=S(\eta_2)$, then the totality of T constitutes a group $\mathfrak{G}$, which is isomorphic to G. Hopf proved the theorem[3]:

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1) M. Tsuji: Theory of conformal mapping of a multiply connected domain, III. Jap. Journ. Math. **19** (1944).

2) Myrberg: Ein Satz über die Fuchsschen Gruppen und seine Anwendungen in der Funktionentheorie. Annales Academie Sci. Fennicae. **32** (1929).

3) E. Hopf: Fuchsian groups and ergodic theory. Trans. Amer. Math. Soc. **39** (1936). Ergodentheorie. Berlin (1937).

Theorem 3 (*Hopf*). *If $\sigma(D_0) < \infty$, then there does not exist a set E on Ω, which is invariant by $\mathfrak{G}$ and $0 < mE < 4\pi^2$.*

From Hopf's lemma 1, it is easily proved that if $\sigma(D_0) < \infty$, then $n(r) \geqq \frac{\text{const.}}{1-r}$, $(0 \leqq r < 1)$. We will prove the following extension of Hopf's theorem.

Theorem 4 (*Main theorem*). *If $\overline{\lim_{r\to 1}}\, n(r)(1-r) > 0$, then there does not exist a set E on Ω, which is invariant by $\mathfrak{G}$ and $0 < mE < 4\pi^2$.*

3. We will use some lemmas in the proof.

Lemma 1. Let E be a measurable set on Ω and $f(\theta, \varphi)$ be its chracteristic function and

$$u(z, w) = u(re^{i\theta}, \rho e^{i\varphi})$$
$$= \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} \frac{f(\theta', \varphi')(1-r^2)(1-\rho^2)d\theta' d\varphi'}{\left(1-2r\cos(\theta'-\theta)+r^2\right)\left(1-2\rho\cos(\varphi'-\varphi)+\rho^2\right)}$$
$$(0 \leqq r < 1,\ 0 \leqq \rho < 1).$$

Then $u(z, w) \to f(\theta, \varphi)$ almost everywhere on Ω, when $z \to e^{i\theta}$, $w \to e^{i\varphi}$ non-tangentially to $|z|=1$, $|w|=1$ respectively.

Proof. By the strong density theorem,

$$\frac{1}{\delta\delta'}\int_{\theta_0-\delta}^{\theta_0+\delta}\int_{\varphi_0-\delta'}^{\varphi_0+\delta'} |f(\theta, \varphi) - f(\theta_0, \varphi_0)|\, d\theta d\varphi \to 0, \quad \text{as } \delta \to 0, \quad \delta' \to 0 \qquad (1)$$

almost everywhere on Ω. It can be proved that if (1) holds at (θ_0, φ_0), then $u(z, w) \to f(\theta_0, \varphi_0)$, when $z \to e^{i\theta_0}$, $w \to e^{i\varphi_0}$ non-tangentially to $|z|=1$, $|w|=1$ respectively.

Lemma 2. If $\overline{\lim_{r\to 1}}\, n(r)(1-r) > 0$, then there does not exist a set e on $|z|=1$, which is invariant by G and $0 < me < 2\pi$.

Proof. Under the hypothesis, it is easily proved that $\int_0^1 n(r)dr = \infty$, or $\sum_{n=0}^{\infty}(1-|z_n|) = \infty$, so that the lemma follows from Theorem 2.

Lemma 3. Let $K_0 : |z| \leqq r_0$ be a disc contained in D_0 and K_n be its equivalents and $rL(r)$ be the measure of the part of $|z|=r$ contained in $\sum_{n=0}^{\infty} K_n$. If $\overline{\lim_{r\to 1}}\, n(r)(1-r) > 0$, there exists $\rho_\nu \to 1$, such that $L(\rho_\nu) \geqq a > 0$ $(\nu = 1, 2, \dots)$.

Lemma 4. Let $K_0 : |z| = r_0$ and $K : \left|\frac{z-a}{1-\bar{a}z}\right| = r_0$ $(|a| < 1)$ be two circles in $|z| < 1$. We transform K into K_0 by S:

$$S:\ z' = e^{i\theta}\cdot\frac{z-a}{1-\bar{a}z}, \text{ such that } S(K) = K_0,\ S(0) = \rho_0 e^{i\varphi_0}.$$

Then $S(K_0) = \bar{K}$ is obtained from K by a .rotation about $z=0$.

Let e be a set on $|z|=1$ contained in an arc $C:\pi \geqq |\arg z - \arg a| \geqq \eta > 0$. Then $S(e)$ is contained in an arc $\bar{C}$ on $|z|=1$, whose center is at $e^{i\varphi_0}$, such that $m\bar{C}=\varkappa R$ (R=radius of K) and

$$\frac{1}{2\pi}me > \frac{mS(e)}{m\bar{C}} > \lambda me,$$

where
$$\varkappa = \frac{2\pi}{r_0 \sin^2 \eta}, \qquad \lambda = \frac{\sin^2 \eta (1-r_0^2)}{8\pi}.$$

4. *Proof of Theorem 4.* Suppose that there exists a set E on Ω, which is invariant by $\mathfrak{G}$ and $0 < mE < 4\pi^2$. Let $f(\theta, \varphi)$ be its characteristic function and we construct $u(z, w)$ as Lemma 1. Then $u(z, w) \to f(\theta, \varphi)$ almost everywhere on Ω, when $z \to e^{i\theta}$, $w \to e^{i\varphi}$ non-tangentially to $|z|=1$, $|w|=1$ respectively. For any substitution S of G,

$$u\big(S(z), S(w)\big) = u(z, w). \tag{1}$$

Let $E(\theta_0)$, $\bar{E}(\varphi_0)$ be the sub-sets of E, which lie on the line $\theta=\text{const.}=\theta_0$ and $\varphi=\text{const.}=\varphi_0$ respectively, then

$$mE = \int_0^{2\pi} mE(\theta)d\theta = \int_0^{2\pi} m\bar{E}(\varphi)d\varphi. \tag{2}$$

Now

$$u(0, w) = \frac{1}{2\pi}\int_0^{2\pi} \frac{F(\varphi')(1-\rho^2)d\varphi'}{1-2\rho\cos(\varphi'-\varphi)+\rho^2},$$

where
$$F(\varphi) = \frac{1}{2\pi}\int_0^{2\pi} f(\theta, \varphi)\, d\theta = \frac{1}{2\pi} m\bar{E}(\varphi). \tag{3}$$

Let $E(\theta)=0$ on a set e of positive measure, then since such a set is invariant by G, we have $me=2\pi$ by Lemma 2. Hence $mE=0$ by (2), which contradicts the hypothesis, so that $mE(\theta) \neq 0$ for almost all θ. Hence if η is small, then there exists a set e of measure $me > 2\pi - \varepsilon$ ($\varepsilon < \pi$), such that

$$mE(\theta) \geqq 4\eta \text{ for any } \theta \in e.$$

Let E_1 be a sub-set of E consisting of points (θ, φ), such that

$$E_1:\ \theta \in e, \qquad |\varphi - \theta| \geqq \eta. \tag{4}$$

If $E_1(\theta)$ is defined as $E(\theta)$ with respect to E_1, then

$$mE_1(\theta) \geqq mE(\theta) - 2\eta \geqq 4\eta - 2\eta = 2\eta, \tag{5}$$

so that
$$mE = \int_e mE(\theta)d\theta \geqq 2\eta\, me \geqq 2\eta\,(2\pi - \varepsilon) \geqq 2\pi\eta.$$

By Egoroff's theorem, there exists a closed sub-set E_0 of E_1 of positive measure consisting of points (θ, φ), such that

E_0: (i) $\theta \in e_0$, $|\varphi-\theta| \geqq \eta$, where e_0 is a closed sub-set of e, such that $me_0 > 2\pi-\varepsilon$, (6)

(ii) $mE_0(\theta) \geqq \eta$, where $E_0(\theta)$ is defined as $E(\theta)$ with respect to E_0, (7)

(iii) $u(z, w) \to 1$ uniformly, when $z \to e^{i\theta}$, $w \to e^{i\varphi}$ from the inside of angular domains: $\Delta(\theta)$: $|\arg(1-ze^{-i\theta})| \leqq \omega$, $\Delta(\varphi)$: $|\arg(1-we^{-i\varphi})| \leqq \omega$, where ω is so chosen that, if an equivalent K_j of K_0: $|z| \leqq r_0$ intersects a radius $z=re^{i\theta}$ $(0 \leqq r < 1)$, then its non-euclidean center z_j is contained in $\Delta(\theta)$.

Hence if $|z_j-e^{i\theta}| < \delta(\varepsilon)$, $|w-e^{i\varphi}| < \delta(\varepsilon)$, $w \in \Delta(\varphi)$, then

$$1-\varepsilon \leqq u(z_j, w) \leqq 1, \tag{8}$$

where $\delta(\varepsilon)$ depends on ε only and is independent of (θ, φ) on E_0.

Let $L(r)$ be defined as Lemma 3. Then there exists $\rho_\nu > 1$, such that $L(\rho_\nu) \geqq a$ $(\nu=1, 2, \dots)$. Now the part of $|z|=\rho_\nu$ contained in $\sum_{n=0}^{\infty} K_n$ consists of a set of arcs. If we project these arcs from $z=0$ on $|z|=1$, we have a set of arcs on $|z|=1$. We divide these arcs into two classes: $\sum_j \alpha_j(\rho_\nu) + \sum_j \beta_j(\rho_\nu)$, where $\alpha_j(\rho_\nu)$ contains at least one point $\theta_j \in e_0$ and $\beta_j(\rho_\nu)$ does not contain such points. If we denote the arc length of an arc α on $|z|=1$ by $|\alpha|$, then

$$L(\rho_\nu) = \sum_j |\alpha_j(\rho_\nu)| + \sum_j |\beta_j(\rho_\nu)| \geqq a.$$

Let e_0' be the complementary set of e_0, then $me_0' < \varepsilon$, so that if we take $\varepsilon \leqq \frac{a}{2}$, then $\sum_j |\beta_j(\rho_\nu)| \leqq me_0' < \varepsilon \leqq \frac{a}{2}$. Hence

$$L'(\rho_\nu) = \sum_{j=1}^{m=m(\nu)} |\alpha_j(\rho_\nu)| \geqq \frac{a}{2}, \tag{9}$$

where $\alpha_j(\rho_\nu)$ is the projection of an arc on $|z|=\rho_\nu$ contained in K_j, which intersects a radius $z=re^{i\theta_j}$ $(0 \leqq r < 1)$, such that $\theta_j \in e_0$.

Let $z_j = r_j e^{i\psi_j}$ be its non-euclidean center and put $U_j(w) = u(z_j, w)$. Then $U_j(w)$ is a bounded harmonic function in $|w| < 1$, so that by Fatou's theorem, $\lim U_j(w)$ exists almost everywhere on $|w|=1$, when w tends to $|w|=1$ non-tangentially. We write this limiting value by $u(z_j, e^{i\varphi})$. Hence there exists a sub-set $E_0'(\theta_j)$ of $E_0(\theta_j)$, such that $mE_0'(\theta_j) = mE_0(\theta_j)$ and for any $\varphi \in E_0'(\theta_j)$, the limiting value $u(z_j, e^{i\varphi})$ exists.

Let $\theta_j \in e_0$, $\varphi \in E_0'(\theta_j)$, then by (7),

$$mE_0'(\theta_j) \geqq \eta, \tag{10}$$

and from (6), if $\nu \geqq \nu_0$, $E_0'(\theta_j)$ is contained in an arc C_j on $|z|=1$, such that

$$C_j: \pi \geqq |\arg z - \psi_j| \geqq \frac{\eta}{2}. \tag{11}$$

Let S_j be the substitution of G, such that $S_j(K_j)=K_0$, $S_j(z_j)=0$ and put $S_j(K_0)=\bar{K}_j$. Then by Lemma 4, $\bar{K}_j$ is obtained from K_j by a rotation about $z=0$. Let $|z|=\rho_\nu$ intersect $\bar{K}_j$ in an arc, whose projection from $z=0$ on $|z|=1$ be $\bar{\alpha}_j(\rho_\nu)$. We put

$$A_\nu=\sum_{j=1}^{m=m(\nu)}\bar{\alpha}_j(\rho_\nu)\,,\qquad A=\overline{\lim_{\nu\to\infty}}A_\nu=(A_1+A_2+\cdots)(A_2+A_3+\cdots)\cdots\,,$$

then, since $|\bar{\alpha}_j(\rho_\nu)|=|\alpha_j(\rho_\nu)|$, $mA_\nu=L'(\rho_\nu)=\sum_{j=1}^{m}|\alpha_j(\rho_\nu)|\geqq\frac{a}{2}$, so that

$$mA\geqq\frac{a}{2}\,. \tag{12}$$

By (10), (11) and Lemma 4, $S_j\big(E_0'(\theta_j)\big)$ is contained in an arc $\bar{C}_j$ on $|z|=1$, concentric with $\bar{\alpha}_j(\rho_\nu)$ and $m\bar{C}_j=\varkappa R_j$ (R_j=radius of K_j), such that

$$\frac{mS_j\big(E_0'(\theta_j)\big)}{m\bar{C}_j}>\lambda mE_0'(\theta_j)\geqq\lambda\eta\,, \tag{13}$$

where $\varkappa,\lambda$ depend on η anly. Since $\varkappa>2\pi$, $\bar{C}_j$ contains $\bar{\alpha}_j(\rho_\nu)$.

We put

$$M_\nu=S_1\big(E_0'(\theta_1)\big)+S_2\big(E_0'(\theta_2)\big)+\cdots+S_m\big((E_0'(\theta_m)\big),\ \big(m=m(\nu)\big)\,,$$
$$M^{(\nu)}=M_\nu+M_{\nu+1}+\cdots,\ M=\overline{\lim_{\nu\to\infty}}M_\nu=(M_1+M_2+\cdots)(M_2+M_3+\cdots)\cdots\,.$$

Let $\varphi\in A$, then $\varphi\in A_{\nu_n}$ $(n=1,2,\ldots)$, so that

$$\varphi\in\bar{\alpha}_{j_n}(\rho_{\nu_n})\subset\bar{C}_{j_n}\big(1\leqq j_n\leqq m(\nu_n)\big)\,.$$

Hence if $\nu\leqq\nu_n$, then by (13),

$$m(M^{(\nu)}\cdot\bar{C}_{j_n})\geqq m(M_{\nu_n}\cdot\bar{C}_{j_n})\geqq m\big(S_{j_n}\big(E_0'(\theta_{j_n})\big)\cdot\bar{C}_{j_n}\big)$$
$$=m\big(S_{j_n}\big(E_0'(\theta_{j_n})\big)\geqq\lambda\eta m\bar{C}_{j_n}\,.$$

Since $\bar{C}_{j_n}\to\varphi$ for $n\to\infty$, the lower density of $M^{(\nu)}$ at φ is $\geqq\lambda\eta$, so that $mM^{(\nu)}\geqq mA\geqq\frac{a}{2}$. Hence

$$mM=\lim_{\nu\to\infty}mM^{(\nu)}\geqq\frac{a}{2}\,.$$

Let $\varphi\in M$, then $\varphi\in M_{\nu_n}$ $(n=1,2,\ldots)$, so that

$$\varphi\in S_{j_n}\big(E_0'(\theta_{j_n})\big)\big(1\leqq j_n\leqq m(\nu_n)\big)\,.$$

Let K_{j_n} be the disc, such that $S_{j_n}(K_{j_n})=K_0$ intersecting a radius $z=re^{i\theta_{j_n}}$ $(0\leqq r\leqq 1,\ \theta_{j_n}\in e_0)$, whose non-euclidean center is z_{j_n}.

Then

$$S_{j_n}^{-1}(\varphi)=\varphi_{j_n}\in E_0'(\theta_{j_n})\,,\qquad S_{j_n}(z_{j_n})=0\,.$$

By making $w\to e^{i\varphi_{j_n}}$ in (8), we have by (1),

$$1-\varepsilon \leqq u(z_{j_n}, e^{i\varphi_{j_n}})=u(0, e^{i\varphi}) \leqq 1\ ,$$

if $|z_{j_n}-e^{i\theta_{j_n}}|<\delta(\varepsilon)$. Since $|z_{j_n}-e^{i\theta_{j_n}}|\to 0$ for $n\to\infty$, we have $u(0, e^{i\varphi})=1$. Hence $u(0, e^{i\varphi})=1$ at every point φ on M. Since the set on $|w|=1$, such that $u(0, w)=1$ is invariant by G and $mM>0$, we have by Lemma 2, $mM=2\pi$, so that $u(0, w)=1$ almost everywhere on $|w|=1$. Since by (3), $u(0, e^{i\varphi})=\frac{1}{2\pi}m\bar{E}(\varphi)$ almost everywhere on $|w|=1$, we have $m\bar{E}(\varphi)=2\pi$ almost everywhere on $|w|=1$, so that by (2), $mE=4\pi^2$, which contradicts the hypothesis, which proves the Theorem.

Theorem 5. If $\sum_{n=0}^{\infty}(1-|z_n|)<\infty$, *then there exists a set* E *on* Ω, *which is invariant by* $\mathfrak{G}$ *and* $0<mE<4\pi^2$.

Proof. By theorem 2, there exists a set e on $|z|=1$, which is invariant by G and $0<me<2\pi$. Then the product set $E=e\times e$ is invariant by $\mathfrak{G}$ and $0<mE=(me)^2<4\pi^2$. We can also prove directly as follows.

Let $Q: \left|\theta-\frac{\pi}{2}\right|\leqq\varepsilon,\ |\varphi-\pi|\leqq\varepsilon\left(\varepsilon<\frac{\pi}{16}\right)$ be a square contained in Ω and $\alpha, \bar{\alpha}, \beta, \bar{\beta}$ be arcs on $|z|=1$, such that

$$\alpha:\ \left|\arg z-\frac{\pi}{2}\right|\leqq\varepsilon, \qquad \bar{\alpha}:\ \left|\arg z-\frac{\pi}{2}\right|\leqq\frac{\pi}{8},$$

$$\beta:\ |\arg z-\pi|\leqq\varepsilon, \qquad \bar{\beta}:\ |\arg z-\pi|\leqq\frac{\pi}{8},$$

and $\bar{\omega}$ be the complementary set of $\bar{\alpha}+\bar{\beta}$ on $|z|=1$. Let $K_0: |z|\leqq r_0$ be a disc contained in the fundamental domain of G and K_n, $z_n=r_ne^{i\theta_n}$ be equivalents of K_0, $z_0=0$ by G respectively, such that $S_n(K_n)=K_0$, $S_n(z_n)=0$, $z_n\in K_n$ and ρ_n be its radius.

(i) If $\theta_n\in\bar{\omega}$, then, since $\varepsilon<\frac{\pi}{16}$, for any z on α, β, $|\arg z-\theta_n|\geqq\frac{\pi}{16}$. Hence by Lemma 4,

$$mS_n(\alpha)<\frac{\chi\rho_n}{2\pi}m\alpha=\frac{\varepsilon\chi}{\pi}\rho_n\ , \qquad mS_n(\beta)<\frac{\varepsilon\chi}{\pi}\rho_n\ ,$$

so that

$$mS_n(Q)<\frac{\varepsilon^2\chi^2}{\pi^2}\rho_n^2<2\varepsilon\chi\rho_n\ .$$

(ii) If $\theta_n\in\bar{\alpha}$, then for any z on β, $|\arg z-\theta_n|\geqq\frac{\pi}{16}$, so that $mS_n(\beta)<\frac{\varepsilon\chi}{\pi}\rho_n$. Since $mS_n(\alpha)\leqq 2\pi$, we have

$$mS_n(Q)<2\varepsilon\chi\rho_n\ .$$

We have the same inequality, if $\theta_n\in\bar{\beta}$.

Since as easily be proved, $\rho_n = \frac{r_0(1-|z_n|^2)}{1-r_0^2} < \frac{2r_0}{1-r_0^2}(1-|z_n|)$, we have $\sum_{n=0}^{\infty}\rho_n < \infty$. If we take ε so small, that

$$\sum_{n=0}^{\infty} mS_n(Q) < 2\varepsilon\chi \sum_{n=0}^{\infty}\rho_n < 4\pi^2,$$

then $E = \sum_{n=0}^{\infty} S_n(Q)$ is invariant by $\mathfrak{G}$ and $0 < mE \leqq \sum_{n=0}^{\infty} mS_n(Q) < 4\pi^2$, q.e.d.

Remark. The condition $\sum_{n=0}^{\infty}(1-|z_n|) < \infty$ is equivalent to $\int_0^1 n(r)dr < \infty$. Hence we have three cases:

(I) $\int_0^1 n(r)dr = \infty$; (a) $\overline{\lim_{r\to1}}\, n(r)(1-r) > 0$, (b) $\lim_{r\to1} n(r)(1-r) = 0$,

(II) $\int_0^1 n(r)dr < \infty$.

In case (I) (a), by Theorem 4, there does not exist a set E on Ω, which is invariant by $\mathfrak{G}$ and $0 < mE < 4\pi^2$. In case (II), by Theorem 5, there exists such invariant sets. In case (I) (b) we have no informations about the existence of such invariant sets. It seems that there exist groups of class (I) (b), for which such invariant sets exist and groups, for which such invariant sets do not exist, but I have no examples for it.

5. Consider n points: $\eta_1 = e^{i\theta_1}, \dots, \eta_n = e^{i\theta_n}$ on $|z_1| = 1, \dots, |z_n| = 1$ respectively. Then the pair $(\eta_1, \dots, \eta_n)$ can be considered as a point on an n-dimensional torus Ω_n $(0 \leqq \theta_j \leqq 2\pi,\ j = 1, 2, \dots, n)$ and the measure of a set E on Ω_n is defined by $mE = \int_E \cdots \int d\theta_1 \dots d\theta_n$, so that $m\Omega_n = (2\pi)^n$. Let S be any substitution of G and $T: \eta_1' = S(\eta_1), \dots, \eta_n' = S(\eta_n)$. Then the totality of T constitues a group $\mathfrak{G}_n$, which is isomorphic to G. We will prove:

Theorem 6. If $n \geqq 3$, then there exists always a set E on Ω_n, which is invariant by $\mathfrak{G}_n$ and $0 < mE < (2\pi)^n$.

Proof. We assume $n = 3$, the other case can be proved similarly.

Let $Q: \left|\theta - \frac{\pi}{2}\right| \leqq \varepsilon$, $|\varphi - \pi| \leqq \varepsilon$, $\left|\psi - \frac{3\pi}{2}\right| \leqq \varepsilon\left(\varepsilon < \frac{\pi}{16}\right)$ be a cube on Ω_3 in (θ, φ, ψ)-space and $\alpha, \bar{\alpha}, \beta, \bar{\beta}, \gamma, \bar{\gamma}$ be arcs on $|z| = 1$, such that

$$\alpha: \left|\arg z - \frac{\pi}{2}\right| \leqq \varepsilon, \qquad \bar{\alpha}: \left|\arg z - \frac{\pi}{2}\right| \leqq \frac{\pi}{8},$$

$$\beta: |\arg z - \pi| \leqq \varepsilon, \qquad \bar{\beta}: |\arg z - \pi| \leqq \frac{\pi}{8},$$

$$\gamma: \left|\arg z - \frac{3\pi}{2}\right| \leqq \varepsilon, \qquad \bar{\gamma}: \left|\arg z - \frac{3\pi}{2}\right| \leqq \frac{\pi}{8}$$

and $\bar{\omega}$ be the complementary set of $\bar{\alpha} + \bar{\beta} + \bar{\gamma}$ on $|z| = 1$.

Let $K_0: |z| \leqq r_0$ be a disc contained in the fundamental domain of G and K_n, $z_n = re^{i\theta_n}$ be equivalents of K_0, $z_0=0$ by G respectively such that $S_n(K_n)=K_0$, $S_n(z_n)=0$, $z_n \in K_n$ and ρ_n be its radius, then since K_n are non-overlapping, $\sum_{n=0}^{\infty} \rho_n^2 < 1$.

(i) If $\theta_n \in \bar{\omega}$, then, since $\varepsilon < \frac{\pi}{16}$, for any z on α, β, γ, $|\arg z - \theta_n| \geqq \frac{\pi}{16}$, so that by Lemma 4, $mS_n(\alpha) \leqq \frac{\chi \rho_n}{2\pi} me = \frac{\varepsilon\chi}{\pi}\rho_n$, $mS_n(\beta) \leqq \frac{\varepsilon\chi}{\pi}\rho_n$, $mS_n(\gamma) \leqq \frac{\varepsilon\chi}{\pi}\rho_n$. Hence $mS_n(Q) \leqq \left(\frac{\varepsilon\chi}{\pi}\right)^3 \rho_n{}^3 < \varepsilon^2\chi^2\rho_n^2$.

(ii) If $\theta_n \in \bar{\alpha}$, then for any z on β, γ, $|\arg z - \theta_n| \geqq \frac{\pi}{16}$, so that $mS_n(\beta) \leqq \frac{\varepsilon\chi}{\pi}\rho_n$, $mS_n(\gamma) \leqq \frac{\varepsilon\chi}{\pi}\rho_n$. Since $mS_n(\alpha) \leqq 2\pi$, we have $mS_n(Q) \leqq \frac{2\varepsilon^2\chi^2}{\pi}\rho_n^2 < \varepsilon^2\chi^2\rho_n^2$.

We have the same inequality, if $\theta_n \in \bar{\beta}$ or $\theta_n \in \bar{\gamma}$.

If we take ε so small, that $\sum_{n=0}^{\infty} mS_n(Q) < \varepsilon^2\chi^2 \sum_{n=0}^{\infty}\rho_n^2 = \varepsilon^2\chi^2 < 8\pi^3$, then $E = \sum_{n=0}^{\infty} S_n(Q)$ is invariant by $\mathfrak{G}_3$ and $0 < mE \leqq \sum_{n=0}^{\infty} mS_n(Q) < 8\pi^3$, q.e.d.