128. *Mean Concentration Function and the Law of Large Numbers.*

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1. Let $F(x)$ be a probability distribution. Put

(1) $$Q_F(l)=\sup_{-\infty<x<\infty}\{F(x+l+0)-F(x-0)\}, \qquad l>0.$$

$Q_F(l)$ is called the *maximal concentration function* of $F(x)$ and plays a fundamental rôle in the theory of P. Lévy on the series of independent random variables[1]. Recently, T. Kawata[2] introduced a *mean concentration function*:

(2) $$C_F(l)=\frac{1}{2l}\int_{-\infty}^{\infty}\{F(x+l+0)-F(x-l-0)\}^2dx, \qquad l>0,$$

and has shown that most of P. Levy's results can be obtained in an analytical way by appealing to this function. It is easy to see, by Plancherel's theorem and Lévy's inversion formula, that $C_F(l)$ can be expressed by Fejér integral:

(3) $$C_F(l)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\sin lt}{lt}\right)^2|f(t)|^2dt=2\int_0^{2l}\left(1-\frac{x}{2l}\right)d\widetilde{F}(x),$$

where $f(t)$ is the characteristic function of $F(x)$, and $\widetilde{F}(x)=\big(1-F(-x)\big)*F(x)$ is the symmetrized distribution of $F(x)$. $\big(F(x)*G(x)$ denotes the convolution of two distributions $F(x)$ and $G(x)$. Thus $|f(t)|^2$ is the characteristic function of $\widetilde{F}(x)\big)$. In the present paper we propose to adopt a new mean concentration function:

(4) $$\psi_F(l)=l\int_0^{\infty}e^{-lt}|f(t)|^2dt=\int_0^{\infty}e^{-t}\left|f\left(\frac{t}{l}\right)\right|^2dt$$
$$=2\int_0^{\infty}\frac{l^2}{l^2+x^2}d\widetilde{F}(x), \qquad l>0.$$

This function, based on Poisson integral, well turn out to be a useful tool in some problems in the theory of independent random variables.

2. It is easy to see that $\psi_F(l)$ possesses similar properties as $Q_F(l)$ and $C_F(l)$. $\psi_F(l)$ is non-negative, monotone non-decreasing in l, and

(5) $$\lim_{l\to\infty}\psi_F(l)=1,$$

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1) P. Lévy, L'addition des variables aléatoires, Paris, 1937.

2) T. Kawata, The function of the mean concentration function of a chance variable, Duke Math. Journ., **9** (1941). T. Kawata, Tokyo Buturi Gakko Zassi, **50** (1942), 11.

$$(6)\qquad \lim_{l\to 0}\psi_F(l)=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|f(t)|^2dt$$
$$=\sum_{n=0}^{\infty}\{F(a_n+0)-F(a_n-0)\}^2,$$

where $\{a_n \mid n=1,2,\ldots\}$ is the set of all discontinuities of $F(x)$. (5) is clear from the definition (4), and (6) follows from Wiener's formula[1]. Further, as a functional of $F(x)$, $\psi_F(l)$ diminishes by convolution, i. e. $F_{12}(x)=F_1(x)*F_2(x)$ implies $\psi_{F_{12}}(l)\leqq\psi_{F_1}(l)$ and $\psi_{F_{12}}(l)\leqq\psi_{F_2}(l)$. Finally, $1-\frac{x}{2l}\leqq\frac{l^2}{l^2+x_2}$ for $0\leqq x\leqq 2l$ implies

$$(7)\qquad C_F(l)=2\int_0^{2l}\Big(1-\frac{x}{2l}\Big)d\widetilde{F}(x)\leqq\psi_F(l)=2\int_0^{\infty}\frac{l^2}{l^2+x^2}d\widetilde{F}(x)$$
$$\leqq 2\sum_{m=0}^{\infty}\frac{l^2}{l^2+m^2l^2}\Big\{\widetilde{F}\big((m+1)l\big)-\widetilde{F}(ml)\Big\}$$
$$\leqq 2Q_{\widetilde{F}}(l)\sum_{m=0}^{\infty}\frac{1}{1+m^2}\leqq(2+\pi)Q_{\widetilde{F}}(l)\leqq(2+\pi)Q_F(l),$$

which together with the inequality

$$(8)\qquad \frac{1}{2}\big(Q_F(l)\big)^2\leqq C_F(l)$$

will give a relation among the magnitude of $Q_F(l)$, $C_F(l)$ and $\psi_F(l)$.

Lemma. Let $\{X_n \mid n=1,2,\ldots\}$ be a sequence of random variables. For each n, denote by $F_n(x)$ the distribution function of X_n. In order that there exists a sequence of real numbers $\{A_n \mid n=1,2,\ldots\}$ such that $X_n-A_n\to 0$ in probability, it is necessary and sufficient that $\psi_{Fn}(l)\to 1$ for some $l>0$.

3. Let $\{X_n \mid n=1,2,\ldots\}$ be a sequence of independent random variables, and let $\{B_n \mid n=1,2,\ldots\}$ be a sequence of positive numbers. We say that the *law of large numbers* holds for these sequences, if there exists a sequence of real numbers $\{A_n \mid n=1,2,\ldots\}$ such that

$$(9)\qquad \frac{1}{B_n}\sum_{k=1}^{n}X_k-A_n\to 0 \qquad \text{in probability.}$$

Theorem. In order that the law of large numbers holds for the sequences $\{X_n \mid n=1,2,\ldots\}$ and $\{B_n \mid 1,2,\ldots\}$, it is necessary and sufficient that

$$(10)\qquad 2\sum_{k=1}^{n}\int_0^{\infty}\frac{x^2}{B_n^2+x^2}d\widetilde{F}_k(x)\equiv\sum_{k=1}^{n}\{1-\psi_{F_k}(B_n)\}\to 0,$$

where $F_n(x)$ and $\widetilde{F}_n(x)$ are the distribution function of X_n and its symmetrized distribution, respectively.

Remark. This condition is clearly equivalent with the combination of the following two:

$$(11)\qquad \sum_{k=1}^{n}\int_{B_n}^{\infty}d\widetilde{F}_k(x)\to 0,$$

1) S. Bochner, Vorlesungen über Fouriersche Integrale, Leipzig, 1923, p. 30.

(12) $$\frac{1}{B_n^2}\sum_{k=1}^n\int_0^{B_n} x^2 d\tilde{F}_k(x) \to 0\,.$$

On the other hand, it was proved by W. Feller[1) that

(13) $$\sum_{k=1}^n\int_{|x|>B_n} dF_k(x) \to 0\,,$$

(14) $$\frac{1}{B_n^2}\sum_{k=1}^n\int_{|x|\leqq B_n} x^2 dF_k(x) \to 0$$

are sufficient for the law of large numbers and that these are necessary if 0 is a median of each $F_n(x)$. It may be shown without any difficulty that our results are essentially equivalent with his.

Proof of the theorem. In order that (9) holds for a suitable sequence $\{A_n \mid n=1,2,\dots\}$, it is necessary and sufficient that

(15) $$1-\psi_{G_n}(1)=1-\psi_{H_n}(B_n)=\int_0^\infty e^{-t}\left(1-\Pi_{k=1}^n\left|f_k\left(\frac{t}{B_n}\right)\right|^2\right)dt \to 0\,,$$

where $G_n(x)$ and $H_n(x)$ are the distribution functions of $\frac{1}{B_n}\sum_{k=1}^n X_k$ and $\sum_{k=1}^n X_k$ respectively. This follows from the Lemma stated above. Thus our problem is to prove the equivalence of (15) with (10) or

(16) $$\sum_{k=1}^n\{1-\psi_{F_k}(B_n)\}=\int_0^\infty e^{-t}\left\{\sum_{k=1}^n\left(1-\left|f_k\left(\frac{t}{B_n}\right)\right|^2\right)\right\}dt \to 0\,.$$

It is clear that (16) implies (15), since

(17) $$1-\Pi_{k=1}^n u_k \leqq \sum_{k=1}^n(1-u_k)$$

if $0\leqq u_k \leqq 1$, $k=1,\dots,n$. Conversely, (15) implies

(18) $$\int_0^T e^{-t}\left(1-\Pi_{k=1}^n\left|f_k\left(\frac{t}{B_n}\right)\right|^2\right)dt \to 0$$

for any finite T. Since

(19) $$\Pi_{k=1}^n\left|f_k\left(\frac{t}{B_n}\right)\right|^2 \to 1 \quad \text{uniformly in} \quad |t|\leqq T$$

and since

(19) $$\sum_{k=1}^n(1-u_k)\leqq 2(1-\Pi_{k=1}^n u_k)$$

if $0\leqq u_k\leqq 1$, $k=1,\dots,n$ and $\Pi_{k=1}^n u_k \geqq \frac{1}{2}$, so we see that (18) implies

(20) $$\int_0^T e^{-t}\left\{\sum_{k=1}^n\left(1-\left|f_k\left(\frac{t}{B_n}\right)\right|^2\right)\right\}dt \to 0\,.$$

(16) follows from this easily if we observe that each $f_k(t)$ satisfies the inequality[2)

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1) W. Feller, Über das Gesetz der grossen Zahlen, Acta Szeged, 8 (1936-37).

2) A. Khintchine, Contribution à l'arithmétique des lois de distribution, Bull. de l'univ. d'état à Moscou, Série intern., Sect. A, vol. 1, fasc. 1 (1937).

$$1-|f(2t)|^2 \leqq 4(1-|f(t)|^2) \tag{21}$$

for any t, which is an immediate consequence of the inequality: $1-\cos 2t \leqq 4(1-\cos t)$. This completes the proof of our theorem.

4. By using the same idea we can discuss the three series theorem of Khintchine-Kolmogoroff[1)] and a theorem of Lévy-Doeblin[2)] concerning the uniform diminution of the maximal concentration function. The detail of the argument is left to a forthcoming paper.

In concluding the paper, the author wishes to express his hearty thanks to Drs. T. Kawata and S. Kakutani for their valuable advices.

1) A. Khintchine and A. Kolmogoroff, Über Konvergenz von Reihen deren Glieder durch den Zufall bestimmt werden, Rec. Math. **32** (1925).

2) P. Lévy, loc. cit. (1), p. 148.