127. On the Osculating Representation for a Dynamical System with Slow Variation.

Second Note.

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In a preceding note[1] the author has enunciated one of the results of his study on the osculating representation for a dynamical system with slow variation, in which the associated curtailed system of differential equations for the motion of the dynamical system is osculatingly represented by quasi-periodic functions of Bohl's class. In the present note I give one of the results under the additional condition that the Hamiltonian function H is, besides being analytic with regard to the first pairs of variables x_i and y_i in a domain $|x_i|, |y_i| < D$, $(i=1, 2, \dots, m)$, and periodic in t with period 2π as in the preceding note, also analytic with regard to the second pairs of variables ξ_j and η_j in a domain $|\xi_j - A_j|$, $|\eta_j - B_j| < \varDelta$ in the immediate neighbourhood of the initial point $\xi_j = A_j$, $\eta_j = B_j$, $(j=1, 2, \dots, n)$, where A_j and B_j are constants, in anticipating the possibility of attacking the problems as to the foundations of the theory of long period variations in celestial mechanics and of the theories of degenerate systems and of adiabatic invariants in quantum mechanics.

The differential equations of the problem have been reduced in the preceding note[3] to the normalised form

$$(1)\quad \begin{cases} \dfrac{d\bar{x}_i}{dt} = \left\{-\sqrt{-1}\,\lambda_i + \left(\dfrac{\partial K^{(s)}}{\partial c_i}\right)\right\}\cdot \bar{x}_i\,, \\ \qquad \dfrac{d\bar{y}_i}{dt} = -\left\{-\sqrt{-1}\,\lambda_i + \left(\dfrac{\partial K^{(s)}}{\partial c_i}\right)\right\}\cdot \bar{y}_i\,, \quad (i=1, 2, \dots, m)\,, \\ \dfrac{d\xi_j}{dt} = \dfrac{\sqrt{-1}}{2}\left(\dfrac{\partial K^{(s)}}{\partial \eta_j}\right), \quad \dfrac{d\eta_j}{dt} = -\dfrac{\sqrt{-1}}{2}\left(\dfrac{\partial K^{(s)}}{\partial \xi_j}\right), \\ \qquad\qquad (j=1, 2, \dots, n)\,, \end{cases}$$

in which $K^{(s)}$ is a finite power series arranged in ascending powers *of the constants* $c_1, c_2, \dots, c_m$, *beginning with the terms of the second* degree, the coefficients of the various powers of c_i 's being in the present case analytic with respect to ξ_j, η_j and t in the immediate neighbourhood of $\xi_j = A_j$, $\eta_j = B_j$, $(j=1, 2, \dots, n)$, and for all values of t, and is periodic in t with period 2π.

By the change of variables

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1) Y. Hagihara, Proc. **20** (1944), 617.

2) A part of the results has been communicated to the American Mathematical Society in December 1928. *Cf.*, Bull. Amer. Math. Soc., **35** (1929), 178.

3) Y. Hagihara, *loc. cit.*, Equation (6).

$$(2)\quad \begin{cases} c_i=\mu\sigma_i\,, & \tau-\tau_0=\mu^2(t-t_0)\,, \quad (i=1,2,\dots,m)\,, \\ \alpha_j=\xi_j-A_j\,, & \beta_j=\eta_j-B_j\,, \quad (j=1,2,\dots,n)\,, \end{cases}$$

the associated curtailed system of (1) is further transformed into

$$(3)\quad \frac{d\alpha_j}{d\tau}=\frac{\partial\Phi^{(s)\prime}}{\partial\beta_j}\,,\qquad \frac{d\beta_j}{d\tau}=-\frac{\partial\Phi^{(s)\prime}}{\partial\alpha_j}\,,\qquad (j=1,2,\dots,n)\,,$$

with the condition

$$\alpha_j=\beta_j=0\,,\qquad \text{for}\qquad \tau=\tau_0\,,\quad (j=1,2,\dots,n)\,,$$

where $\mu^2\Phi^{(s)\prime}$ is the transform of $\frac{\sqrt{-1}}{2}K^{(s)}$ after the change of variables (2), such that

$$\Phi^{(s)\prime}=\Phi_0^{(s)\prime}+\mu\Phi_1^{(s)\prime}+\mu^2\Phi_2^{(s)\prime}+\cdots+\mu^v\Phi_v^{(s)\prime}\,,$$

with

$$(4)\quad \begin{cases} v=\dfrac{s}{2}-2\,, & \text{if } s \text{ is even,} \\[2ex] v=\dfrac{s-1}{2}-2\,, & \text{if } s \text{ is odd,} \end{cases}$$

and $\Phi_r^{(s)\prime}$ is a homogeneous polynomial of degree $r+2$ in $\sigma_1, \sigma_2, \dots, \sigma_m$, $(r=0,1,2,\dots,v)$, and analytic with regard to ξ_j, η_j and τ, $(j=1,2,\dots n)$.

Now arrange $\Phi^{(s)\prime}$ in ascending powers of α_j and β_j, $(j=1,2,\dots,n)$, in the form

$$(5)\quad \Phi^{(s)\prime}=\Psi_0^{(s)}+\Psi_1^{(s)}+\cdots+\Psi_r^{(s)}+\cdots,$$

where $\Psi_r^{(s)}$ is a homogeneous polynomial of the r-th degree in α_j and β_j, $(j=1,2,\dots,n\,;\ r=1,2,\dots,$ *ad inf.*). Suppose that

$$(6)\quad \frac{\partial\Phi^{(s)\prime}}{\partial A_j}=\frac{\partial\Phi^{(s)\prime}}{\partial B_j}=0\,,\qquad (j=1,2,\dots,n)\,.$$

According to our assumption the original system of differential equations is satisfied by $x_i=y_i=0$, $\xi_j=A_j$, $\eta_j=B_j$, $(i=1,2,\dots,m\,;\ j=1,2,\dots,n)$. If $c_1=\cdots=c_m=0$, then (6) is satisfied up to the order of magnitude μ^{v+2} with v given by (4). Hence, if (6) is satisfied up to this order μ^{v+2}, we have, after a linear transformation of the variables α_j and β_j to α_j' and β_j',

$$(7)\quad \begin{cases} \dfrac{d\alpha_j'}{d\tau}=\dfrac{\partial Q^{(s)}}{\partial\beta_j'}\,,\qquad \dfrac{d\beta_j'}{d\tau}=-\dfrac{\partial Q^{(s)}}{\partial\alpha_j'}\,,\quad (j=1,2,\dots,n)\,, \\ Q^{(s)}=-2\sqrt{-1}\,\Phi^{(s)\prime} \\ \qquad=-\sqrt{-1}\displaystyle\sum_{l=1}^{n}\nu_l\alpha_l'\beta_l'+Q_3^{(s)}+Q_4^{(s)}+\cdots, \end{cases}$$

in the domain $|\alpha_j'|, |\beta_j'|<\Delta$, $(j=1,2,\dots,n)$. Here it is assumed that there are n real, distinct, non-zero pairs of characteristic numbers $\frac{1}{2}\nu_l$, $(l=1,2,\dots,n)$, for the matrix formed of the coefficients of the quadratic terms of α_j and β_j in the expansion of $\Phi^{(s)\prime}$, and further that there is no linear homogeneous relation with rational coefficients among these $\mu^2\nu_l$'s, λ_k 's and 1.

Repeat $u-2$ times the contact transformation

$$(8)\quad \begin{cases} \beta'_j=\dfrac{\partial\Gamma}{\partial\alpha'_j}, \quad \bar{\alpha}_j=\dfrac{\partial\Gamma}{\partial\bar{\beta}_j}, \quad (j=1,2,\dots,n), \\ \Gamma=\sum\limits_{l=1}^{n}\alpha'_l\bar{\beta}_l+\Gamma_3+\Gamma_4+\cdots+\Gamma_u, \end{cases}$$

where $\Gamma_3, \Gamma_4, \dots, \Gamma_u$ denote respectively homogeneous polynomials in α'_j and $\bar{\beta}_j$ of the degree indicated by the suffixes, as in the preceding note[1] for the variables x'_i and $\bar{y}_i$. Then under the above assumptions the given system of differential equations is transformed up to the degree s with regard to x_i and y_i and up to the degree u with regard to α_j and β_j to the following remarkable form

$$(9)\quad \begin{cases} \dfrac{d\bar{x}_i}{dt}=\dfrac{\partial\sum^{(s,u)}}{\partial c_i}\cdot\bar{x}_i, \quad \dfrac{d\bar{y}_i}{dt}=-\dfrac{\partial\sum^{(s,u)}}{\partial c_i}\cdot\bar{y}_i, \quad (i=1,2,\dots,m), \\ \dfrac{d\bar{\alpha}_j}{dt}=\dfrac{\partial\sum^{(s,u)}}{\partial\gamma_j}\cdot\bar{\alpha}_j, \quad \dfrac{d\bar{\beta}_j}{dt}=-\dfrac{\partial\sum^{(s,u)}}{\partial\gamma_j}\cdot\bar{\beta}_j, \quad (j=1,2,\dots,n), \end{cases}$$

with

$$\bar{x}_i\bar{y}_i=c_i, \qquad \bar{\alpha}_j\bar{\beta}_j=\gamma_j,$$

$$\begin{aligned} \textstyle\sum^{(s,u)} &= -\sqrt{-1}\cdot\sum_{k=1}^{m}\lambda_k c_k+S^{(s,u)} \\ &= -\sqrt{-1}\cdot\sum_{k=1}^{m}\lambda_k c_k-\sqrt{-1}\,\mu^2\cdot\sum_{l=1}^{n}\nu_l\gamma_l \\ &\quad+\sum \varkappa_{\alpha_1\alpha_2\dots\alpha_m\beta_1\beta_2\dots\beta_n}c_1^{\alpha_1}c_2^{\alpha_2}\cdots c_m^{\alpha_m}\gamma_1^{\beta_1}\gamma_2^{\beta_2}\cdots\gamma_n^{\beta_n}, \end{aligned}$$

where the last sum is extended for positive integral values of $\alpha_1, \alpha_2, \dots, \alpha_m, \beta_1, \beta_2, \dots, \beta_n$, including zero, satisfying

$$1<\alpha_1+\alpha_2+\cdots+\alpha_m\leqq\frac{s}{2}, \quad \text{for } s \text{ even}, \quad \frac{s-1}{2} \text{ for } s \text{ odd},$$

$$1<\beta_1+\beta_2+\cdots+\beta_n\leqq\frac{u}{2}, \quad \text{for } u \text{ even}, \quad \frac{u-1}{2} \text{ for } u \text{ odd},$$

and $\varkappa_{\alpha_1\alpha_2\dots\alpha_m\beta_1\beta_2\dots\beta_n}$'s are periodic in t with period 2π.

Put

$$(10)\quad \begin{cases} \sqrt{-1}\,r_i=\dfrac{\partial\sum^{(s,u)}}{\partial c_i}=\sqrt{-1}\,\lambda_i+\dfrac{\partial S^{(s,u)}}{\partial c_i}=\dfrac{2\pi\sqrt{-1}}{t_i}, \\ \qquad\qquad (i=1,2,\dots,m), \\ \sqrt{-1}\,\rho_j=\dfrac{\partial\sum^{(s,u)}}{\partial\gamma_j}= \qquad \dfrac{\partial S^{(s,u)}}{\partial\gamma_j}=\dfrac{2\pi\sqrt{-1}}{\tau_j}, \\ \qquad\qquad (j=1,2,\dots,n), \end{cases}$$

where r_i and ρ_j, and accordingly t_i and τ_j depend on s and u, and, according to our assumption, are all real. Thus we have the solution of (9) in the form

$$\begin{cases} \bar{x}_i = x_i^0 e^{\sqrt{-1} r_i t}, & \bar{y}_i = y_i^0 e^{-\sqrt{-1} r_i t}, \qquad (i=1,2,\dots,m), \\ \bar{\alpha}_j = \alpha_j^0 e^{\sqrt{-1}\rho_j t}, & \bar{\beta}_j = \beta_j^0 e^{-\sqrt{-1}\rho_j t}, \qquad (j=1,2,\dots,n). \end{cases}$$

Then, turning back to the original variables the solution of our doubly curtailed system of differential equations can be solved in the form

$$(11) \quad \begin{cases} x_i = f_i(t;\mu;t_1,t_2,\dots,t_m), \\ y_i = g_i(t;\mu;t_1,t_2,\dots,t_m), \end{cases} \qquad (i=1,2,\dots,m),$$
$$\qquad \begin{cases} \xi_j = \varphi_j(t;\mu;\tau_1,\tau_2,\dots,\tau_n), \\ \eta_j = \psi_j(t;\mu;\tau_1,\tau_2,\dots,\tau_n), \end{cases} \qquad (j=1,2,\dots,n),$$

where f_i and g_i denote quasi-periodic functions with the corpus of periods $t_1, t_2, \dots, t_m$, and φ_j and ψ_j are quasi-periodic functions with the corpus of periods $\tau_1, \tau_2, \dots, \tau_n$.

Thus we have the following theorem:

In the original system of differential equations

$$(12) \quad \begin{cases} \dfrac{dx_i}{dt} = \dfrac{\partial H}{\partial y_i}, & \dfrac{dy_i}{dt} = -\dfrac{\partial H}{\partial x_i}, \qquad (i=1,2,\dots,m), \\ \dfrac{d\xi_j}{dt} = \dfrac{\partial H}{\partial \eta_j}, & \dfrac{d\eta_j}{dt} = -\dfrac{\partial H}{\partial \xi_j}, \qquad (j=1,2,\dots,n), \end{cases}$$

where H is a function of $2m+2n+1$ variables x_i, y_i, ξ_j, η_j, $(i=1,2,\dots,m;\ j=1,2,\dots,n)$, and t, and is analytic with regard to x_i, y_i, ξ_j, η_j, and t in a domain $|x_i|, |y_i| < D$ and $|\xi_j - A_j|, |\eta_j - B_j| < \Delta$, $(i=1,2,\dots,m;\ j=1,2,\dots,n)$. Assume that we have a solution $x_i = y_i = 0$, $\xi_j = A_j$, $\eta_j = B_j$, $(i=1,2,\dots,m;\ j=1,2,\dots,n)$, for all values of t, where A_j and B_j are constants, that the expansion of H in powers of x_i and y_i begins with the quadratic terms and the coefficients of these quadratic terms are constants, and that the m pairs of the characteristic numbers $\frac{1}{2}\lambda_i$, $(i=1,2,\dots,m)$, for the matrix formed of these coefficients are real, distinct and non-zero, without any linear homogeneous relation with rational coefficients among these λ_i 's and 1. Further assume that (6) is satisfied and that the matrix formed of the coefficients of the quadratic terms in $\alpha_j = \xi_j - A_j$, $\beta_j = \eta_j - B_j$, $(j=1,2,\dots,n)$, of the function $\Phi^{(s)\prime}$ obtained after the transformation described in the preceding note has n real, distinct, non-zero pairs of characteristic numbers $\frac{1}{2}\nu_l$'s without any linear homogeneous relation with rational coefficients among these characteristic numbers and $1/\mu^2$.

Then there exist solutions in the form of the quasi-periodic functions for both pairs of the variables x_i, y_i and ξ_j, η_j, when we cut short the terms beyond an arbitrary degree s with regard to x_i and y_i and beyond an arbitrary degree u with regard to ξ_j and η_j in the expansion of H.

In order that the errors committed in the solution thus formed should be less than an assigned positive constant δ, we ought to restrict the time interval so that

$$(13)\qquad |t-t_0|<\text{Min.}\left(\begin{array}{cc} G^{-1}(\delta), & H^{-1}(\delta) \\ \dfrac{1}{2m(s-1)N\varepsilon_0^{2s-1}}, & \dfrac{1}{2^5 n(s-1)N'\eta_0^{2u-1}\varepsilon_0^4} \end{array}\right),$$

where the operators G^{-1} and H^{-1} are defined as the inverses of the operators G and H, respectively, operated on z, such that

$$\begin{aligned} G(z) &= A\varepsilon_0^{2s}+B\eta_0^{2u}+C\eta_0^{2u}\varepsilon_0^4 z+D\varepsilon_0^{2s+1}z \\ &\quad +E\eta_0^{2u+2}\varepsilon_0^8 z^2+F\varepsilon_0^{2s+5}z^2, \\ H(z) &= A'\varepsilon_0^{2s}+B'\eta_0^{2u}+C'\eta_0^{2u+1}z+D'\varepsilon_0^{2s}z \\ &\quad +E'\eta_0^{2u+3}\varepsilon_0^5 z^2+F'\varepsilon_0^{2s+4}z^2, \end{aligned}$$

in which $A, B, \ldots, F, A', B', \ldots, F'$ and N, N' are positive constants and ε_0^2 and η_0^2 are respectively the initial values of $\sum_{i=1}^{m}(x_i^2+y_i^2)$ and $\sum_{j=1}^{n}[(\xi_j-A_j)^2+(\tau_j+B_j)^2]$ at $t=t_0$. Thus the original system of differential equations (12) is osculatingly represented by the quasi-periodic functions under the assumptions stated in the above, provided that the series (10) for r_i and ρ_j, $(i=1,2,\ldots,m;\ j=1,2,\ldots,n)$, converge as $s\to\infty$, $u\to\infty$.

As ρ_j 's, $(j=1,2,\ldots,n)$, are of the order of magnitude $1/\mu^2$ as compared with r_i 's, $(i=1,2,\ldots,m)$, the periods for ξ_j and η_j are long compared with those of x_i and y_i. Thus the solution of the system of differential equations of our dynamical system is osculatingly represented by quasi-periodic functions superposed on quasi-periodic functions of longer periods. Our theorem gives the maximum time interval (13) for which the true solution deviates from such quasi-periodic functions by less than a given amount δ.

A comparison of this result with the series employed in celestial mechanics by Delaunay, Newcomb, Lindstedt, Bohlin and Poincaré is of profound interest. The series for the principal function G in the contact transformation for x_i' and y_i', and for the principal function Γ for the contact transformation for α_j' and β_j' are generally not uniformly convergent, just as such series appearing in celestial mechanics, due to the presence of the so-called small divisors. Hence the series for the solution in the original variables x_i, y_i, ξ_j and η_j are generally not uniformly convergent. Our theorem gives the maximum time interval in which the curtailed series deviates from the true solution by less than an assigned amount δ.