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126. On the Osculating Representation for a Dynamical System with Slow Variation.

First Note.

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In a preceding note[1] the present author has given a theorem concerning the dynamical systems with slow variation and obtained the maximum time interval in which a semi-convergent representation of the solution deviates by less than a given amount from the true solution of the differential equations for the dynamical system in question. In the present note I have the privilege to report one of the results I have been able to reach in the case when the differential equations can be approximated, not necessarily convergently, by a quasi-periodic function of Bohl's class, as the general integrals usually adopted for the solution in the planetary theory, although not uniformly convergent as has been proved by Poincaré, are taken to be such a class of functions[2]. The dynamical system under consideration is meant for a simplification of the planetary and satellite systems existing in nature. I intend in future to extend the research towards the theory of the general form of the integrals and the stability of the planetary motion in general.

Consider a system of differential equations

$$(1)\quad \begin{cases} \dfrac{dx_i}{dt}=\dfrac{\partial H}{\partial y_i}, & \dfrac{dy_i}{dt}=-\dfrac{\partial H}{\partial x_i}, \qquad (i=1,2,\dots,m), \\[2ex] \dfrac{d\xi_j}{dt}=\dfrac{\partial H}{\partial \eta_j}, & \dfrac{d\eta_j}{dt}=-\dfrac{\partial H}{\partial \xi_j}, \qquad (j=1,2,\dots,n), \end{cases}$$

where H is a function of $2m+2n+1$ variables x_i, y_i, ξ_j, η_j, $(i=1,2,\dots,m;\ j=1,2,\dots,n)$, and t, and, together with its partial derivatives of the first and the second orders with respect to x_i, y_i, ξ_j and η_j, is Lipschitzian with regard to ξ_j and η_j, and is analytic with regard to x_i, y_i and t for all values of ξ_j, η_j and t and for all values of x_i and y_i in a domain

$$(2)\qquad |x_i|,\ |y_i|<D, \qquad (i=1,2,\dots,m),$$

with a finite positive constant D, and is periodic in t with period 2π. Assume that we have a solution $x_i=y_i=0$, $\xi_j=A_j$, $\eta_j=B_j$, $(i=1,2,\dots,m;$

1) Y. Hagihara, Proc. **7** (1931), 44.

2) See, for example, Delaunay, Théorie du Mouvement de la Lune. Mém. Acad. Sc. Inst. Imp. France. **27** (1860), **29** (1867); Newcomb, Journ. de Math. pure et appl., [ii] **16** (1871), 321; Smithsonian Contr. to Knowledge, 1874, 281; Lindstedt, Ann. École Norm. Sup., [iii] **1** (1884), 85; Bohlin, Bihang til Svenska Vet. Acad. Handlingar. **14** (1888); Poincaré, Les Méthodes Nouvelles de la Mécanique Céleste. **2** (1893).

$j=1,2,\dots,n$), for all values of t, where A_j and B_j are arbitrary constants. Assume further that the expansion of H in powers of x_i and y_i begins with quadratic terms and the coefficients of these quadratic terms are constants, that the m pairs of the characteristic numbers $\frac{1}{2}\lambda_i$, ($i=1,2,\dots,m$), for the matrix formed of these coefficients are real, distinct and non-zero[3], and that there is no linear homogeneous relation with rational coefficients among these λ_i 's and 1.

By a linear canonical transformation the system (1) can be transformed into a system for pairs of conjugate imaginary variables x'_i and y'_i accompanied by an associated system for real pairs ξ_j and η_j:

$$\text{(3)}\quad \begin{cases} \dfrac{dx'_i}{dt}=\dfrac{\partial F}{\partial y'_i}, \quad \dfrac{dy'_i}{dt}=-\dfrac{\partial F}{\partial x'_i}, \quad (i=1,2,\dots,m), \\ \dfrac{d\xi_j}{dt}=\dfrac{\sqrt{-1}}{2}\dfrac{\partial F}{\partial \eta_j}, \quad \dfrac{d\eta_j}{dt}=-\dfrac{\sqrt{-1}}{2}\dfrac{\partial F}{\partial \xi_j}, \quad (j=1,2,\dots,n), \\ F=-2\sqrt{-1}\,H-=\sqrt{-1}\cdot\sum_{k=1}^{m}\lambda_k x'_k y'_k+F_3+F_4+\cdots, \end{cases}$$

where $F_3, F_4, \dots$ are respectively the terms of the third, the fourth, ... degree in x'_i and y'_i, of which the coefficients are functions of ξ_j, η_j and t. F is also convergent.

Next apply the contact transformation[4]

$$\text{(4)}\qquad \bar{x}_i=\frac{\partial G}{\partial \bar{y}_i}, \qquad y'_i=\frac{\partial G}{\partial x'_i}, \qquad (i=1,2,\dots,m),$$

with

$$G=\sum_{k=1}^{m}x'_k\bar{y}_k+G_3+G_4+\cdots+G_u+G^{(u)}.$$

$G_3, G_4, \dots, G_u$ are homogeneous functions of $\bar{y}_i$ and x'_i of degree indicated by the suffixes and $G^{(u)}$ is the remainder term as yet undetermined but does not contain terms of degree $3,4,\dots,u$ in $\bar{y}_i$ and x'_i. By $s-2$ times' repetition of this transformation for $u=3,4,\dots,s$ we can bring F in the form

$$K=K^{(s)}+R^{(s)}, \qquad R^{(s)}=F_{s+1}+F_{s+2}+\cdots,$$

$$K^{(s)}=-\sqrt{-1}\cdot\sum_{k=1}^{m}\lambda_k\bar{x}_k\bar{y}_k+\sum f_{\alpha_1\alpha_2\dots\alpha_m}(\bar{x}_1\bar{y}_1)^{\alpha_1}(\bar{x}_2\bar{y}_2)^{\alpha_2}\cdots(\bar{x}_m\bar{y}_m)^{\alpha_m},$$

where the last sum extends to all integral values, positive or zero, of $\alpha_1, \alpha_2, \dots, \alpha_m$, satisfying

$$2<2(\alpha_1+\alpha_2+\cdots+\alpha_m)\leqq s, \qquad \text{if } s \text{ is even,}$$
$$2<2(\alpha_1+\alpha_2+\cdots+\alpha_m)\leqq s-1, \qquad \text{if } s \text{ is odd,}$$

and $f_{\alpha_1\alpha_1\dots\alpha_m}$'s are functions of ξ_j, η_j and t. Then (3) is transformed into

3) For such a transformation refer to: Whittaker, Analytical Dynamics. 1937. Chap. XVI.

4) G. D. Birkhoff, Dynamical Systems. 1927.

$$(5)\quad \begin{cases} \dfrac{d\bar{x}_i}{dt}=\dfrac{\partial K^{(s)}}{\partial \bar{y}_i}+\dfrac{\partial R^{(s)}}{\partial \bar{y}_i}, \quad \dfrac{d\bar{y}_i}{dt}=-\dfrac{\partial K^{(s)}}{\partial \bar{x}_i}-\dfrac{\partial R^{(s)}}{\partial \bar{x}_i}, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad (i=1,2,\dots,m), \\ \dfrac{d\xi_j}{dt}=\dfrac{\sqrt{-1}}{2}\left(\dfrac{\partial K^{(s)}}{\partial \eta_j}+\dfrac{\partial R^{(s)}}{\partial \eta_j}\right), \\ \dfrac{d\eta_j}{dt}=-\dfrac{\sqrt{-1}}{2}\left(\dfrac{\partial K^{(s)}}{\partial \xi_j}+\dfrac{\partial R^{(s)}}{\partial \xi_j}\right), \qquad (j=1,2,\dots,n). \end{cases}$$

Denote the result of substituting a system of arbitrary constants c_i for $\bar{x}_i\bar{y}_i$, $(i=1,2,\dots,m)$, in $K^{(s)}$, subtracted by its first degree terms with regard to c_i, by a parenthesis. And consider a curtailed system

$$(6)\quad \begin{cases} \dfrac{d\bar{x}_i}{dt}=\left\{-\sqrt{-1}\,\lambda_i+\left(\dfrac{\partial K^{(s)}}{\partial c_i}\right)\right\}\cdot\bar{x}_i, \\ \dfrac{d\bar{y}_i}{dt}=-\left\{-\sqrt{-1}\,\lambda_i+\left(\dfrac{\partial K^{(s)}}{\partial c_i}\right)\right\}\cdot\bar{y}_i, \quad (i=1,2,\dots,m), \\ \dfrac{d\xi_j}{dt}=\dfrac{\sqrt{-1}}{2}\left(\dfrac{\partial K^{(s)}}{\partial \eta_j}\right), \quad \dfrac{d\eta_j}{dt}=-\dfrac{\sqrt{-1}}{2}\left(\dfrac{\partial K^{(s)}}{\partial \xi_j}\right), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad (j=1,2,\dots,n), \end{cases}$$

in which $K^{(s)}$ is a finite power series arranged in ascending powers of the constants $c_1, c_2, \dots, c_m$, beginning with the terms of the second degree, the coefficients of the various powers of c_i 's being Lipschitzian functions of ξ_j and η_j and periodic with period 2π in t. The system (6) is said to be *normalised.*

Take the associated curtailed system

$$(7)\quad \dfrac{d\xi_j}{dt}=\dfrac{\sqrt{-1}}{2}\left(\dfrac{\partial K^{(s)}}{\partial \eta_j}\right), \quad \dfrac{d\eta_j}{dt}=-\dfrac{\sqrt{-1}}{2}\left(\dfrac{\partial K^{(s)}}{\partial \xi_j}\right), \qquad (j=1,2,\dots,n),$$

and carry out the transformation

$$(8)\qquad \tau-\tau_0=\mu^2(t-t_0), \qquad c_i=\mu\sigma_i, \qquad (i=1,2,\dots,m),$$

where μ is a small constant parameter of the order of magnitude of $|x_i|^2$ or $|y_i|^2$, or of the order of magnitude of D^2. Then we get

$$(9)\quad \begin{cases} \dfrac{d\xi_j}{d\tau}=\dfrac{\partial \Phi^{(s)}}{\partial \eta_j}, \quad \dfrac{d\eta_j}{d\tau}=-\dfrac{\partial \Phi^{(s)}}{\partial \xi_j}, \qquad (j=1,2,\dots,n), \\ \Phi^{(s)}=\Phi_0^{(s)}+\mu\Phi_1^{(s)}+\mu^2\Phi_2^{(s)}+\dots+\mu^v\Phi_v^{(s)}, \end{cases}$$

with

$$\begin{cases} v=\dfrac{s}{2}-2, & \text{if } s \text{ is even,} \\ v=\dfrac{s-1}{2}-2, & \text{if } s \text{ is odd,} \end{cases}$$

where $\Phi_r^{(s)}$, $(r=0,1,2,\dots,v)$, is a homogeneous polynomial of degree $r+2$ in $\sigma_1, \sigma_2, \dots, \sigma_m$, and Lipschitzian with regard to ξ_j and η_j $(j=$

$1, 2, \ldots, n)$, and analytic and periodic in τ with period $2\pi/\mu^2$.

We have to integrate this system with the initial condition that

$$(10) \qquad \xi_j = A_j, \quad \eta_j = B_j, \quad \text{for} \quad \tau = \tau_0, \quad (j=1, 2, \ldots, n),$$

or, by a further transformation

$$(11) \qquad \alpha_j = \xi_j - A_j, \qquad \beta_j = \eta_j - B_j, \qquad (j=1, 2, \ldots, n),$$

to integrate the system

$$(9)' \qquad \frac{d\alpha_j}{d\tau} = \frac{\partial \Phi^{(s)\prime}}{\partial \beta_j}, \qquad \frac{d\beta_j}{d\tau} = -\frac{\partial \Phi^{(s)\prime}}{\partial \alpha_j}, \qquad (j=1, 2, \ldots, n),$$

with the initial condition that

$$(10)' \qquad \alpha_j = \beta_j = 0, \qquad \text{for} \qquad \tau = \tau_0, \qquad (j=1, 2, \ldots, n),$$

where $\Phi^{(s)\prime}$ is the function $\Phi^{(s)}$ after the transformation (11).

Definition. Consider a system of differential equations

$$(12) \qquad \frac{dz_i}{dt} = \sum_{j=0}^{\infty} Z_j^{(i)}(z_i, t) \cdot \mu^j, \qquad (i=1, 2, \ldots, m),$$

in which the right hand members are either convergent or formal series arranged in ascending powers of a parameter μ. We cut short at the terms with the power k of μ on the right hand side and integrate this curtailed system of differential equations. Let the solution of the latter curtailed system be $z_i^{(k)}(t; \mu)$, $(i=1, 2, \ldots, m)$. Suppose that there exists a sequence of sets of functions $\zeta_i^{(k)}(t; \mu)$, $(i=1, 2, \ldots, m;$ $k=1, 2, \ldots)$, converging uniformly to a limiting set of functions $\zeta_i(t; \mu)$, $(i=1, 2, \ldots, m)$, respectively, as $k \to \infty$, and reducing to the same set of functions $z_j^{(k)}(t; \mu)$, $(i=1, 2, \ldots, m)$, for $t=t_0$, and further that for a given positive constant μ we have inequalities

$$|\zeta_i^{(k)}(t; \mu) - z_i^{(k)}(t; \mu)| < N\mu^{ak+b} \cdot |t-t_0|^p,$$

for any finite positive integral value of k in a non-vanishing interval of t with finite positive constants a, b and N and with a positive integer p, all depending on the nature of $\zeta_i^{(k)}(t; \mu)$ and $z_i^{(k)}(t; \mu)$, $(i=1, 2, \ldots, m)$. Then we say that the solution of the system of differential equations (12) is *osculatingly represented* by the set of functions $\zeta_i(t; \mu)$ to the *class* p and to the *genus* $ak+b$. This idea is a generalisation of Birkhoff's in his theory of divergent series at a singular point of an ordinary differential equations[5].

By repeating the method of proof in the preceding note[1] and by referring to Bohl-Esclangon's theorem[6] for the necessary and sufficient condition for a function to be represented by a quasi-periodic function,

5) G. D. Birkhoff, Sitzungsber. preuss. Akad. Wiss. Berlin. 1929, 171. *Cf.*, Bochner, Math. Ann., **96** (1926), 119.

6) P. Bohl, Dorpat Dissertation, 1893; Crelle Journ., **131** (1906), 268; Esclangon, Thèse Paris, 1904; Comptes Rendus Acad. Sc. Paris. **135** (1902), 891; **137** (1903), 305; Ann. Obs. Bordeaux **16** (1917), 53.

to Bohl's theorem[7] on the integrals of quasi-periodic functions, and to Kronecker's theorem on the diophantine approximation, we get the following theorem.

Theorem. If the solution of the associated curtailed system of differential equations is osculatingly represented by a set of quasi-periodic functions of Bohl's class with the corpus of periods $\tau_1, \tau_2, \dots, \tau_n$ and 2π with respect to t to the class 1 and to the genus l, where $s \geqq l+5$, then the solution of the original system of differential equations (1) can be osculatingly represented to the class 2 and to the genus $l+6$ by the set of quasi-periodic functions of Bohl's class with the corpus of periods $\tau_1, \tau_2, \dots, \tau_n$ and 2π, superposed on the corpus of periods, $t_1, t_2, \dots, t_m$ in t with

$$\frac{2\pi}{t_i} = \lambda_i + h_i\,, \qquad (i=1, 2, \dots, m)\,,$$

where $-\sqrt{-1}\,h_i$ is the secular constant[7] in $(\partial K^{(s)}/\partial c_i)$ expressed in the form of the first of these two quasi-periodic functions.

Here it is assumed that there is no linear homogeneous relation with rational coefficients among these two sets of periods and 2π and that for any term in $-\sqrt{-1}\,\lambda_i + (\partial K^{(s)}/\partial c_i)$ expressed in the form of a quasi-periodic function we have inequalities

$$\frac{k_1}{t_1} + \frac{k_2}{t_2} + \dots + \frac{k_m}{t_m} > L\mu^{\frac{p}{2}},$$

for all integral values, negative or positive or zero, of $k_1, k_2, \dots, k_m$ satisfying

$$|k_1| + |k_2| + \dots + |k_m| \leqq p\,,$$

were p is the sum of the absolute values of the powers of $e^{\sqrt{-1}\,t/\tau_1}$, $e^{\sqrt{-1}\,t/\tau_2}$, $\dots$, $e^{\sqrt{-1}\,t/\tau_m}$ in the term under consideration, and L is a finite positive constant.

The theorem can be extended to the case when the coefficients of the quadratic terms of H are functions of ξ_j and η_j, in which case the genus of the osculating representation should be written $l+4$ instead of $l+6$. The proof is then carried out by referring to the theory of a system of linear differential equations with quasi-periodic coefficients by Bohl[6]. Naturally the theorem can easily be generalised from the case of osculating representation by a quasi-periodic function to the case of an almost periodic function of Bohr's class by referring to the works of Bohr, Neugebauer and Favard[8].

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7) P. Bohl, Crelle Journ., **131** (1906), 268. For almost periodic functions refer to: H. Bohr, Det Danske Videnskabernes Selskab. Math.-fys. Medd., **10** (1930); Commentarii Math. Helvet., **4** (1932), 51; Bohr u. Jessen, Ann. Scuola Norm. Sup. Pisa. [ii] **1** (1932), 385; Jessen, Math. Ann., **111** (1935), 355.

8) H. Bohr and Neugebauer, Göttinger Nachrichten. 1926, 23; Favard, Acta Math., **51** (1927), 31; Comptes Rendus Acad. Sc. Paris. **182** (1926), 757, 1122; Walther, Göttinger Nachrichten. 1927, 196; Landau, Math. Ann., **102** (1929), 177; Bochner, Math. Ann., **102** (1929), 489; **103** (1930), 588; **104** (1931), 579.