142. *Subprojective Transformations, Subprojective Spaces and Subprojective Collineations.*

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§ 1. *The subpaths.*

Let A_n be an affinely connected space of n dimensions whose components of connection are $\Pi^\lambda_{\mu\nu}(x)$.

If we consider a curve $x^\lambda = x^\lambda(r)$ in this space, the derivative of $x^\lambda(r)$ with respect to the parameter r

$$\frac{\delta x^\lambda}{\delta r} = \frac{dx^\lambda}{dr}$$

defines the direction of the tangent at a point x^λ of the curve, but the covariant derivative

$$\frac{\delta^2 x^\lambda}{\delta r^2} = \frac{d^2 x^\lambda}{dr^2} + \Pi^\lambda_{\mu\nu}\frac{dx^\mu}{dr}\frac{dx^\nu}{dr}$$

of the tangent vector $\frac{dx^\lambda}{dr}$ does not define a direction uniquely. For, if we change the parameter r into $\bar{r}$, the vector $\frac{\delta^2 x^\lambda}{\delta \bar{r}^2}$ becomes a linear combination of $\frac{\delta^2 x^\lambda}{\delta r^2}$ and $\frac{\delta x^\lambda}{\delta r}$. Thus two vectors $\frac{\delta^2 x^\lambda}{\delta r^2}$ and $\frac{\delta x^\lambda}{\delta r}$ define, independently of the choice of the parameter r, a two dimensional linear space. We shall call it osculating plane defined along the curve. If the curve is a so-called path the osculating plane is indeterminate.

Now, we suppose that there is given a contravariant vector field $\xi^\lambda(x)$ in our affinely connected space A_n and shall consider a system of curves whose osculating planes contain always the contravariant vector field ξ^λ. The differential equations of such curves are

$$\frac{d^2 x^\lambda}{dr^2} + \Pi^\lambda_{\mu\nu}\frac{dx^\mu}{dr}\frac{dx^\nu}{dr} = \alpha\frac{dx^\lambda}{dr} + \beta\xi^\lambda. \tag{1.1}$$ [1]

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1) The equations of this type have first appeared in D. van Dantzig's projective geometry. See, for example, D. van Dantzig: Theorie des projektiven Zusammenhangs n-dimensionaler Räume. Math. Ann. **106** (1932), 400-454. J. A. Schouten and J. Haantjes: Zur allgemeinen projektiven Differentialgeometrie, Compositio Math. **3** (1936), 1-51. J. Haantjes: On the projective geometry of paths, Proc. of the Edinburgh Math. Soc. **5** (1937), 103-115. The paths in these theories are represented by subpaths in an affinely connected space A_{n+1} of $n+1$ dimensions which represents the projectively connected space P_n. The present author showed that the paths in O. Veblen's projective space may also be represented by subpaths in an affinely connected space A_{n+1} of $n+1$ dimensions which represents the projective space of n dimensions. See, K. Yano: Sur les équations des paths dans l'espace projcctif généralisé de M. O. Veblen. To appear in the Proc. Physico-Math. Soc. Japan, **26** (1944).

We shall call these curves subpaths of our affinely connected space with respect to the contravariant vector field $\xi^\lambda(x)$.

§ 2. *The subprojective change of affine connections.*

The differential equations of subpaths being given by (1.1), we shall seek for the most general transformations of the components $\Pi^\lambda_{\mu\nu}$ of affine connections which change the subpaths with respect to the contravariant vector field ξ^λ into the subpaths with respect to the same contravariant vector field ξ^λ.

The parameter r in the differential equations of the subpaths (1.1) being the most general one, we can write the differential equations of the subpaths, with respect to the new components of connection $\overline{\Pi}^\lambda_{\mu\nu}$ and with respect to the same contravariant vector field ξ^λ, in the form

$$\frac{d^2x^\lambda}{dr^2}+\overline{\Pi}^\lambda_{\mu\nu}\frac{dx^\mu}{dr}\frac{dx^\nu}{dr}=\overline{\alpha}\frac{dx^\lambda}{dr}+\overline{\beta}\xi^\lambda. \tag{2.1}$$

From the equations (1.1) and (2.1), we obtain

$$T^{\lambda}_{\cdot\mu\nu}\frac{dx^\mu}{dr}\frac{dx^\nu}{dr}=(\overline{\alpha}-\alpha)\frac{dx^\lambda}{dr}+(\overline{\beta}-\beta)\xi^\lambda, \tag{2.2}$$

where we have put

$$T^{\lambda}_{\cdot\mu\nu}=\overline{\Pi}^\lambda_{\mu\nu}-\Pi^\lambda_{\mu\nu}, \tag{2.3}$$

and consequently we know that $T^{\lambda}_{\cdot\mu\nu}$ is a symmetric tensor.

As the equations (2.2) must hold for any values of $\frac{dx^\lambda}{dr}$, we obtain

$$T^{\lambda}_{\cdot\mu\nu}=\delta^\lambda_\mu\varphi_\nu+\delta^\lambda_\nu\varphi_\mu+\varphi_{\mu\nu}\xi^\lambda, \tag{2.4}$$ [1]

where φ_ν and $\varphi_{\mu\nu}$ may be regarded as covariant vector and tensor respectively.

Conversely, the components of connection given by

$$\overline{\Pi}^\lambda_{\mu\nu}=\Pi^\lambda_{\mu\nu}+\delta^\lambda_\mu\varphi_\nu+\delta^\lambda_\nu\varphi_\mu+\varphi_{\mu\nu}\xi^\lambda \tag{2.5}$$

defines the same system of subpaths as that defined by the components of connection $\Pi^\lambda_{\mu\nu}$.

In this sense, we shall call this change of $\Pi^\lambda_{\mu\nu}$ the subprojective change of affine connections with respect to the contravariant vector field ξ^λ.

§ 3. *The concurrent vector field and subprojective transformations.*

The present author[2] has recently proved that, if a contravariant vector torse-forming along a curve with respect to an affine connection $\Pi^\lambda_{\mu\nu}$ is always torse-forming also with respect to another affine connection $\overline{\Pi}^\lambda_{\mu\nu}$, then, there must be a relation of the form

$$\overline{\Pi}^\lambda_{\mu\nu}=\Pi^\lambda_{\mu\nu}+\delta^\lambda_\mu\varphi_\nu+\delta^\lambda_\nu\psi_\mu \tag{3.1}$$

1) The equations of this type have first appeared also in D. van Dantzig's theory of projective spaces.

2) K. Yano: Über eine geometrische Deutung der projektive Transformationen nichtsymmetrischer affiner Übertragungen. Proc. **20** (1944), 284-287. See also, K. Yano: On the torse-forming directions in Riemannian spaces. Proc. **20** (1944), 340-345.

between the components of affine connections $\overline{\Pi}^\lambda_{\mu\nu}$ and $\Pi^\lambda_{\mu\nu}$. This fact gives us a geometrical interpretation of the projective change of asymmetric affine connections.

The subprojective change of the affine connections explained in §2 does not have this property. But, if the vector field ξ^λ is a torse-forming one, with respect to the affine connection $\Pi^\lambda_{\mu\nu}$, that is, if the vector field ξ^λ satisfies the equations of the form

$$\xi^\lambda_{;\nu} = \alpha\delta^\lambda_\nu + \beta_\nu\xi^\lambda\,, \tag{3.2}$$

the covariant derivative being taken with respect to the affine connection $\Pi^\lambda_{\mu\nu}$, the vector field ξ^λ is also torse-forming with respect to the affine connection

$$\overline{\Pi}^\lambda_{\mu\nu} = \Pi^\lambda_{\mu\nu} + \delta^\lambda_\mu\varphi_\nu + \delta^\lambda_\nu\varphi_\mu + \varphi_{\mu\nu}\xi^\lambda\,, \tag{3.3}$$

which is obtained, from $\Pi^\lambda_{\mu\nu}$, by a subprojective change with respect to ξ^λ.

For, denoting by $\xi^\lambda{}_{|\nu}$ the covariant derivative of ξ^λ with respect to $\overline{\Pi}^\lambda_{\mu\nu}$, we have

$$\begin{aligned}\xi^\lambda{}_{|\nu} &= \frac{\partial\xi^\lambda}{\partial x^\nu} + \overline{\Pi}^\lambda_{\mu\nu}\xi^\mu \\ &= \frac{\partial\xi^\lambda}{\partial x^\nu} + (\Pi^\lambda_{\mu\nu} + \delta^\lambda_\mu\varphi_\nu + \delta^\lambda_\nu\varphi_\mu + \varphi_{\mu\nu}\xi^\lambda)\xi^\mu \\ &= \xi^\lambda{}_{;\nu} + \varphi_\mu\xi^\mu\delta^\lambda_\nu + (\varphi_\nu + \varphi_{\mu\nu}\xi^\mu)\xi^\lambda.\end{aligned}$$

Thus, ξ^λ is torse-forming also with respect to $\overline{\Pi}^\lambda_{\mu\nu}$.

§ 4. *The subprojective spaces.*

Let us consider an arbitrary affine space E_n and take a system of linear coordinates x^λ. Then, the coordinates x^λ may be considered as defining a vector field in E_n. The components of affine connection $\Pi^\lambda_{\mu\nu}$ of this space being indentically zero, the covariant derivative of x^λ is δ^λ_ν, that is, x^λ is a concurrent vector field and consequently torse-forming vector field.

The subpaths of the affinely flat space E_n are given, in this special coordinates system, by

$$\frac{d^2x^\lambda}{dr^2} = \alpha\frac{dx^\lambda}{dr} + \beta x^\lambda\,. \tag{4.1}$$

If we effect a subprojective change to $\Pi^\lambda_{\mu\nu}$, we obtain new components of an affine connection

$$\overline{\Pi}^\lambda_{\mu\nu} = \delta^\lambda_\mu\varphi_\nu + \delta^\lambda_\nu\varphi_\mu + \varphi_{\mu\nu}x^\lambda\,, \tag{4.2}$$

and the subpaths (4.1) are naturally subpaths also with respect to the new affine connection.

The equations of the paths defined with respect to the new affine connection being

$$\frac{d^2x^\lambda}{dr^2} + \overline{\Pi}^\lambda_{\mu\nu}\frac{dx^\mu}{dr}\frac{dx^\nu}{dr} = \alpha\frac{dx^\lambda}{dr}\,,$$

or

$$\frac{d^2x^\lambda}{dr^2}+\left(2\varphi_\nu\frac{dx^\nu}{dr}-\alpha\right)\frac{dx^\lambda}{dr}+\varphi_{\mu\nu}\frac{dx^\mu}{dr}\frac{dx^\nu}{dr}x^\lambda=0\,,$$

we can conclude that the affinely connected space with the components of connection $\overline{\Pi}^\lambda_{\mu\nu}$ obtained, by a subprojective change, from an ordinary affine space E_n is a subprojective space in the sense of B. Kagan[1].

Conversely, if we have a subprojective space of B. Kagan, it may be always transformed to an ordinary affine space by a suitable subprojective change of affine connections.

§ 5. *The subprojective collineations.*

We shall consider, in this Paragraph, the infinitesimal transformation

$$\bar{x}^\lambda=x^\lambda+\varepsilon\xi^\lambda\,, \tag{5.1}$$

which transforms any subpath with respect to ξ^λ into a subpath with respect to the same vector field ξ^λ. Such an infinitesimal transformation may be called subprojective infinitesimal collineation.

Let

$$\frac{d^2x^\lambda}{dr^2}+\Pi^\lambda_{\mu\nu}\frac{dx^\mu}{dr}\frac{dx^\nu}{dr}=\alpha\frac{dx^\lambda}{dr}+\beta\xi^\lambda \tag{5.2}$$

be the differential equations of a subpath. This subpath is transformed into a curve by the infinitesimal transformation (5.1). The necessary and sufficient condition that the new curve be also a subpath with respect to the same vector field ξ^λ is that

$$\frac{d^2\bar{x}^\lambda}{dr^2}+\Pi^\lambda_{\mu\nu}(\bar{x})\frac{d\bar{x}^\mu}{dr}\frac{d\bar{x}^\nu}{dr}=\bar{\alpha}\frac{d\bar{x}^\lambda}{dr}+\bar{\beta}\xi^\lambda(\bar{x})\,. \tag{5.3}$$

Substituting (5.1) in (5.3), and taking account of the quantities containing only to the first order of ε, we find

$$\begin{aligned}&\frac{d^2x^\lambda}{dr^2}+\varepsilon\frac{\partial^2\xi^\lambda}{\partial x^\mu\partial x^\nu}\frac{dx^\mu}{dr}\frac{dx^\nu}{dr}+\varepsilon\frac{\partial\xi^\lambda}{\partial x^\alpha}\frac{d^2x^\alpha}{dr^2}\\ &\quad+\left(\Pi^\lambda_{\mu\nu}(x)+\varepsilon\frac{\partial\Pi^\lambda_{\mu\nu}}{\partial x^\omega}\xi^\omega\right)\left(\frac{dx^\mu}{dr}+\varepsilon\frac{\partial\xi^\mu}{\partial x^\sigma}\frac{dx^\sigma}{dr}\right)\left(\frac{dx^\nu}{dr}+\varepsilon\frac{\partial\xi^\nu}{\partial x^\tau}\frac{dx^\tau}{dr}\right)\\ &\quad=(\alpha+\varepsilon\alpha')\left(\frac{dx^\lambda}{dr}+\varepsilon\frac{\partial\xi^\lambda}{\partial x^\nu}\frac{dx^\nu}{dr}\right)+(\beta+\varepsilon\beta')\left(\xi^\lambda+\varepsilon\frac{\partial\xi^\lambda}{\partial x^\nu}\xi^\nu\right)\end{aligned}$$

Substituting (5.2) in this equation and equating the terms containing ε, we have

$$\left(\frac{\partial^2\xi^\lambda}{\partial x^\mu\partial x^\nu}-\frac{\partial\xi^\lambda}{\partial x^\alpha}\Pi^\alpha_{\mu\nu}+\frac{\partial\Pi^\lambda_{\mu\nu}}{\partial x^\omega}\xi^\omega+\Pi^\lambda_{\alpha\nu}\frac{\partial\xi^\alpha}{\partial x^\mu}+\Pi^\lambda_{\mu\alpha}\frac{\partial\xi^\alpha}{\partial x^\nu}\right)\frac{dx^\mu}{dr}\frac{dx^\nu}{dr}=\alpha'\frac{dx^\lambda}{dr}+\beta'\xi^\lambda.$$

1) B. Kagan: Über eine Erweiterung des Begriffes vom projektiven Raume und dem zugehörigen Absolute. Abhandlungen aus dem Seminar für Vektor- und Tensoranalysis samit Anwendungen auf Geometrie, Mechanik und Physik, **1** (1933), 12–96.

As these equations must hold for any values of $\frac{dx^\lambda}{dr}$, we obtain the equations of the form

$$\frac{\partial^2\xi^\lambda}{\partial x^\mu \partial x^\nu}-\frac{\partial \xi^\lambda}{\partial x^a}\Pi^a_{\mu\nu}+\frac{\partial \Pi^\lambda_{\mu\nu}}{\partial x^\omega}\xi^\omega+\Pi^\lambda_{a\nu}\frac{\partial \xi^a}{\partial x^\mu}+\Pi^\lambda_{\mu a}\frac{\partial \xi^a}{\partial x^\nu}=\delta^\lambda_\mu\varphi_\nu+\delta^\lambda_\nu\varphi_\mu+\varphi_{\mu\nu}\xi^\lambda ,$$

where φ_ν and $\varphi_{\mu\nu}$ are arbitrary covariant vector and tensor respectively.

Putting these equations in tensor form, we obtain

$$\xi^\lambda{}_{;\mu;\nu}+\Pi^\lambda_{.\mu\nu\omega}\xi^\omega=\delta^\lambda_\mu\varphi_\nu+\delta^\lambda_\nu\varphi_\mu+\varphi_{\mu\nu}\xi^\lambda .\text{[1]} \tag{5.4}$$

This is the necessary and sufficient condition that the infinitesimal transformation (5.1) transform any subpath with respect to the vector field ξ^λ into a subpath with respect to the same vector field ξ^λ, say, that the infinitesimal transformation be a subprojective collineation.

If we put $\beta=0$ in (5.2) and $\bar{\beta}=0$ in (5.3), we have, instead of (5.4),

$$\xi^\lambda{}_{;\mu;\nu}+\Pi^\lambda_{.\mu\nu\omega}\xi^\omega=\delta^\lambda_\mu\varphi_\nu+\delta^\lambda_\nu\varphi_\mu . \tag{5.5}$$

This is the well known condition that the infinitesimal transformation (5.1) be a projective collineation.

If we put $\alpha=\beta=0$ in (5.2) and $\bar{\alpha}=\bar{\beta}=0$ in (5.3), we have

$$\xi^\lambda{}_{;\mu;\nu}+\Pi^\lambda_{.\mu\nu\omega}\xi^\omega=0 . \tag{5.5}$$

In this case, the infinitesimal transformation (5.1) is an affine collineation.

§ 6. *The representation of the projective spaces.*

In a previous paper[2], we have proved the theorem: *In order that an affinely connected space of $n+1$ dimensions can represent a projective space of paths of n dimensions, it is necessary and sufficient that there exist, in the affinely connected space, a contravariant vector field ξ^λ such that the conditions*

$$\xi^\lambda{}_{;\mu;\nu}+\Pi^\lambda_{.\mu\nu\omega}\xi^\omega=\delta^\lambda_\mu\varphi_\nu+\delta^\lambda_\nu\varphi_\mu+\varphi_{\mu\nu}\xi^\lambda, \tag{6.1}$$

$$\xi^\lambda{}_{;\nu}=\alpha\delta^\lambda_\nu+\beta_\nu\xi^\lambda \tag{6.2}$$

are satisfied. But, the first condition represents that the affinely connected space admits a subprojective infinitesimal collineation in the direction ξ^λ, and the second says that the vector field ξ^λ is a torse-forming one.

Thus we can state the above theorem in the following form: *In order that an affinely connected space of $n+1$ dimensions can represent a projective space of paths of n dimensions, it is necessary and sufficient that there exist, in the affinely connected space, a torse-forming contravariant vector field ξ^λ in the direction of which the affinely connected space admits an infinitesimal subprojective transformation.*

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1) $\Pi^\lambda_{.\mu\nu\omega}$ denotes the curvature tensor formed with the components $\Pi^\lambda_{\mu\nu}$.

2) K. Yano: Sur les espaces à connexion affine qui peuvent représenter les espaces projectifs des paths. Proc., **20** (1944), 631-639.