Johnson pseudo-contractibility of various classes of Banach algebras

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Abstract

The notion of Johnson pseudo-contractibility for Banach algebras is introduced. We investigate this notion for Banach algebras defined on locally compact groups. For a compact metric space X and $\alpha > 0$, we show that the Lipschitz algebra $Lip_{\alpha}(X)$ is Johnson pseudo-contractible if and only if X is finite. We give some examples to distinguish our new notion with the classical ones.

1 Introduction and Preliminaries

The concept of amenable Banach algebras were studied and introduced by Johnson, see [17]. In fact a Banach algebra A is amenable if A has a virtual diagonal, that is, there exists an element M in $(A \otimes_p A)^{**}$ such that $a \cdot M = M \cdot a$ and $\pi_A^{**}(M)a = a$ where $\pi_A : A \otimes_p A \rightarrow A$ is the product morphism given by $\pi_A(a \otimes b) = ab$ for every $a, b \in A$.

Some new generalizations of amenability like pseudo-amenability and pseudo-contractibility have been introduced. A Banach algebra A is called pseudo-amenable (pseudo-contractible) if there exists a not necessarily bounded net (m_α) in $A \otimes_p A$ such that $a \cdot m_\alpha - m_\alpha \cdot a \rightarrow 0\,(a \cdot m_\alpha = m_\alpha \cdot a)$ and $\pi_A(m_\alpha)a - a \rightarrow 0$, for every $a \in A$, respectively, see [11]. Pseudo-amenability and pseudo-contractibility of Segal algebras, semigroup algebras and matrix algebras were studied in [2], [4], [7] and [8]. Using [6, Theorem 3.1] we can see

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that $M_I(\mathbb{C})$ (the Banach algebra of $I \times I$-matrices over $\mathbb{C}$, with finite ℓ^1-norm and matrix multiplication) has no bounded approximate identity whenever I is infinite. So $M_I(\mathbb{C})$ is not amenable. But by [8, Theorem 3.7] we can see that $M_I(\mathbb{C})$ is pseudo-amenable for each non-empty set I. But Essmaili *et al.* showed that $M_I(\mathbb{C})$ is pseudo-contractible if and only if I is finite. In fact $M_I(\mathbb{C})$ under the various notions of amenability like amenability and pseudo-contractibility is forced to have a finite index set while the notion of pseudo-amenability holds for an arbitrary index set I. The differences refer to the conditions $a \cdot m_\alpha - m_\alpha \cdot a \to 0$ and $a \cdot m_\alpha = m_\alpha \cdot a$ of the definition of these concepts.

Motivated by these considerations the question raised. "Is there any notion of amenability close to pseudo-amenability that forces the index set I to be finite? "

To answer this question we combined the condition of definitions of (Johnson) amenability and pseudo-contractibility to obtain a new notion, which is weaker than amenability and pseudo-contractibility but it is stronger than pseudo-amenability.

Definition 1.1. A Banach algebra A is called Johnson pseudo-contractible, if there exists a not necessarily bounded net (m_α) in $(A \otimes_p A)^{**}$ such that $a \cdot m_\alpha = m_\alpha \cdot a$ and $\pi_A^{**}(m_\alpha)a - a \to 0$, for every $a \in A$.

In this paper we study Johnson pseudo-contractibility for various classes of Banach algebras. We show that for a locally compact G, the measure algebra $M(G)$ (the group algebra $L^1(G)$) is Johnson pseudo-contractible if and only if G is discrete and amenable (amenable), respectively. Also for a compact metric space X and $\alpha > 0$, the Lipschitz algebra $Lip_\alpha(X)$ is Johnson pseudo-contractible if and only if X is finite. We give some examples to distinguish our new notion with the classic ones.

2 Johnson pseudo-contractibility

Let A be a Banach algebra. Throughout this work, the character space of A is denoted by $\Delta(A)$, that is, all non-zero multiplicative linear functionals on A. The projective tensor product $A \otimes_p A$ is a Banach A-bimodule via the following actions

$$a \cdot (b \otimes c) = ab \otimes c, \quad (b \otimes c) \cdot a = b \otimes ca \quad (a, b, c \in A).$$

For each $\phi \in \Delta(A)$ there exists a unique extension $\tilde{\phi}$ to A^{**} which is defined by $\tilde{\phi}(F) = F(\phi)$. It is easy to see that $\tilde{\phi} \in \Delta(A^{**})$.

Lemma 2.1. *Let A be an amenable Banach algebra. Then A is Johnson pseudo-contractible.*

Proof. Since A is amenable, it has a virtual diagonal, that is, an element $M \in (A \otimes_p A)^{**}$ such that $a \cdot M = M \cdot a$ and $\pi_A^{**}(M)a = a$, for every $a \in A$. Hence A is Johnson pseudo-contractible. ∎

Lemma 2.2. *Let A be a pseudo-contractible Banach algebra. Then A is Johnson pseudo-contractible.*

Proof. Since A is pseudo-contractible, it has a net (m_α) in $A \otimes_p A \hookrightarrow (A \otimes_p A)^{**}$ such that $a \cdot m_\alpha = m_\alpha \cdot a$ and $\pi_A(m_\alpha)a \to a$, for every $a \in A$. Hence A is Johnson pseudo-contractible. ∎

A Banach algebra A is called biflat if there exists a bounded A-bimodule morphism $\rho : A \to (A \otimes_p A)^{**}$ such that $\pi_A^{**} \circ \rho(a) = a$, for every $a \in A$. Note that A is amenable if and only if A is biflat and has a bounded approximate identity, see [17].

Proposition 2.3. *Suppose that A is a biflat Banach algebra with a central approximate identity. Then A is Johnson pseudo-contractible.*

Proof. Let (e_α) be a central approximate identity in A and let $\rho : A \to (A \otimes_p A)^{**}$ be a bounded A-bimodule morphism such that $\pi_A^{**} \circ \rho(a) = a$ for every $a \in A$. Then $M_\alpha = \rho(e_\alpha)$ is a net in $(A \otimes_p A)^{**}$ such that

$$a \cdot M_\alpha = a \cdot \rho(e_\alpha) = \rho(ae_\alpha) = \rho(e_\alpha a) = \rho(e_\alpha)a = M_\alpha \cdot a$$

and $\pi_A^{**}(M_\alpha)a = \pi_A^{**} \circ \rho(e_\alpha)a = e_\alpha a \to a$ for every $a \in A$. Thus A is Johnson pseudo-contractible. ∎

A Banach algebra A is called left ϕ-amenable if there exists an element $m \in A^{**}$ such that $am = \phi(a)m$ and $\tilde{\phi}(m) = 1$, for every $a \in A$, see [13].

Proposition 2.4. *Suppose that A is a Banach algebra with $\phi \in \Delta(A)$. If A is Johnson pseudo-contractible, then A is left ϕ-amenable.*

Proof. Since A is a Johnson pseudo-contractible Banach algebra, there exists a net (m_α) in $(A \otimes_p A)^{**}$ such that $a \cdot m_\alpha = m_\alpha \cdot a$ and $\pi_A^{**}(m_\alpha)a \to a$, for every $a \in A$. Define $T : A \otimes_p A \to A$ by $T(a \otimes b) = \phi(b)a$ for every $a, b \in A$. Since T^{**} is a w^*-continuous map, we have

$$T^{**}(a \cdot F) = a \cdot T^{**}(F), \quad \phi(a)T^{**}(F) = T^{**}(F \cdot a), \qquad (a \in A, F \in (A \otimes_p A)^{**}).$$

Thus

$$a \cdot T^{**}(m_\alpha) = T^{**}(a \cdot m_\alpha) = T^{**}(m_\alpha \cdot a) = \phi(a)T^{**}(m_\alpha)$$

for every $a \in A$. Using w^*-continuity of T^{**} one can see that $\tilde{\phi} \circ T^{**} = \tilde{\phi} \circ \pi_A^{**}$. So we have

$$\tilde{\phi} \circ T^{**}(m_\alpha) = \tilde{\phi} \circ \pi_A^{**}(m_\alpha) \to 1. \tag{2.1}$$

Then for sufficiently large values of α, $\tilde{\phi} \circ \pi_A^{**}(m_\alpha)$ stays away from zero. Replacing $T^{**}(m_\alpha)$ by $\frac{T^{**}(m_\alpha)}{\tilde{\phi} \circ T^{**}(m_\alpha)}$ we can assume that

$$aT^{**}(m_\alpha) = \phi(a)T^{**}(m_\alpha), \quad \tilde{\phi} \circ T^{**}(m_\alpha) = 1,$$

for every $a \in A$. It follows that A is left ϕ-amenable. ∎

Let A be a Banach algebra and I be a totally ordered set. The set of $I \times I$ upper triangular matrices with entries from A with the usual matrix operations and finite ℓ^1-norm defined by $||(a_{i,j})_{i,j\in I}|| = \sum_{i,j\in I} ||a_{i,j}|| < \infty$ is a Banach algebra and it is denoted by $UP(I, A)$.

Theorem 2.5. *Let A be a Banach algebra with $\phi \in \Delta(A)$ and let I be a finite set. Then $UP(I, A)$ is Johnson pseudo-contractible if and only if A is Johnson pseudo-contractible and $|I| = 1$.*

Proof. If $|I| = 1$, then $UP(I, A) = A$. Conversely, let $UP(I, A)$ be Johnson pseudo-contractible. We go towards a contradiction and suppose that $|I| > 1$. Take $I = \{1, 2, 3, ..., n\}$. Define $\psi_\phi((a_{i,j})) = \phi(a_{n,n})$. It is easy to see that ψ_ϕ is a nonzero character on $UP(I, A)$ and so by Proposition 2.4, $UP(I, A)$ is left ψ_ϕ-amenable. Put

$$J = \{(a_{i,j})_{i,j} \in UP(I, A) : a_{i,j} = 0 \quad \text{whenever} \quad j \neq n\}.$$

One can show that J is a closed ideal of $UP(I, A)$ and $\psi_\phi|_J \neq 0$, then by [13, Lemma 3.1] J is left ψ_ϕ-amenable. Using [13, Theorem 1.4] there exists a bounded net (j_α) in J such that

$$jj_\alpha - \psi_\phi(j)j_\alpha \to 0, \quad \psi_\phi(j_\alpha) = 1 \quad (j \in J). \tag{2.2}$$

Note that for every α the element j_α is of the form $\begin{pmatrix} 0 & \cdots & j_1^\alpha \\ \vdots & \cdots & \vdots \\ 0 & \cdots & j_n^\alpha \end{pmatrix}$, where $j_1^\alpha, \ldots, j_n^\alpha \in A$. So by (2.2) we have

$$\begin{pmatrix} 0 & \cdots & j_1 j_n^\alpha \\ \vdots & \cdots & \vdots \\ 0 & \cdots & j_n j_n^\alpha \end{pmatrix} - \begin{pmatrix} 0 & \cdots & \phi(j_n) j_n^\alpha \\ \vdots & \cdots & \vdots \\ 0 & \cdots & \phi(j_n) j_n^\alpha \end{pmatrix} = \begin{pmatrix} 0 & \cdots & j_1 j_n^\alpha - \phi(j_n) j_n^\alpha \\ \vdots & \cdots & \vdots \\ 0 & \cdots & j_n j_n^\alpha - \phi(j_n) j_n^\alpha \end{pmatrix} \to 0 \tag{2.3}$$

for every $j = \begin{pmatrix} 0 & \cdots & j_1 \\ \vdots & \cdots & \vdots \\ 0 & \cdots & j_n \end{pmatrix} \in J$, where $j_1, \ldots, j_n$ in A. Since ϕ is a character on A, there exists a_0 in A such that $\phi(a_0) = 1$. In the particular case $j_1 = j_2 = ... = j_{n-1} = a_0$ and $j_n = 0$ apply (2.3) we have $a_0 j_n^\alpha - \phi(j_n) j_n^\alpha = a_0 j_n^\alpha \to 0$, as $\phi(j_n) = 0$. Since ϕ is continuous and $\phi(a_0) = 1$, we have $\phi(j_n^\alpha) \to 0$. But $\psi_\phi(j_\alpha) = \phi(j_n^\alpha) = 1$ which is a contradiction. ■

Proposition 2.6. *Let A^{**} be a Johnson pseudo-contractible Banach algebra. Then A is pseudo-amenable.*

Proof. Since A^{**} is Johnson pseudo-contractible, there exists a net $(m_\alpha)_{\alpha\in I}$ in $(A^{**} \otimes_p A^{**})^{**}$ such that $a \cdot m_\alpha = m_\alpha \cdot a$ and $\pi^{**}_{A^{**}}(m_\alpha)a \to a$ for every $a \in A^{**}$. By [10, Lemma 1.7] there exists a bounded linear map $\psi : A^{**} \otimes_p A^{**} \to (A \otimes_p A)^{**}$ such that for $a, b \in A$ and $m \in A^{**} \otimes_p A^{**}$, the following holds

(i) $\psi(a \otimes b) = a \otimes b$,

(ii) $\psi(m) \cdot a = \psi(m \cdot a), \qquad a \cdot \psi(m) = \psi(a \cdot m)$,

(iii) $\pi_A^{**}(\psi(m)) = \pi_{A^{**}}(m)$.

Set $n_\alpha = \psi^{**}(m_\alpha)$, it is easy to see that (n_α) is a net in $(A \otimes_p A)^{****}$ which satisfies

$$a \cdot n_\alpha = a \cdot \psi^{**}(m_\alpha) = \psi^{**}(a \cdot m_\alpha) = \psi^{**}(m_\alpha \cdot a) = \psi^{**}(m_\alpha) \cdot a = n_\alpha \cdot a$$

and

$$\pi_A^{****}(n_\alpha)a - a = \pi_A^{****} \circ \psi^{**}(m_\alpha)a - a = \pi_{A^{**}}^{**}(m_\alpha)a - a \to 0, \quad (a \in A).$$

Suppose that $(y_\beta^\alpha)_{\beta \in \Theta}$ is a net in $(A \otimes_p A)^{**}$ such that $y_\beta^\alpha \xrightarrow{w^*} n_\alpha$. Then

$$w^* - \lim w^* - \lim a \cdot y_\beta^\alpha - y_\beta^\alpha \cdot a = w^* - \lim a \cdot n_\alpha - n_\alpha \cdot a = 0$$

and

$$w^* - \lim w^* - \lim \pi_A^{**}(y_\beta^\alpha)a - a = w^* - \lim w^* - \lim \pi_A^{****}(y_\beta^\alpha)a - a = \\ w^* - \lim \pi_{A^{**}}^{**}(m_\alpha)a - a = 0,$$

for every $a \in A$. Let $E = I \times \Theta^I$ be a directed set with product ordering defined by

$$(\alpha, \beta) \leq_E (\alpha', \beta') \Leftrightarrow \alpha \leq_I \alpha', \beta \leq_{\Theta^I} \beta' \qquad (\alpha, \alpha' \in I, \quad \beta, \beta' \in \Theta^I),$$

where Θ^I is the set of all functions from I into Θ and $\beta \leq_{\Theta^I} \beta'$ means that $\beta(d) \leq_\Theta \beta'(d)$ for every $d \in I$. Suppose that $\gamma = (\alpha, \beta_\alpha) \in E$ and $m_\gamma = y_{\beta_\alpha}^\alpha$. Applying iterated limit theorem [14, page 69] and above calculations, we can easily see that $a \cdot m_\gamma - m_\gamma \cdot a \xrightarrow{w^*} 0$ and $\pi_A^{****}(m_\gamma)a - a \xrightarrow{w^*} 0$, for every $a \in A$. So $a \cdot m_\gamma - m_\gamma \cdot a \xrightarrow{w} 0$ and $\pi_A^{**}(m_\gamma)a - a \xrightarrow{w} 0$ for every $a \in A$. By Mazur's lemma we can assume that

$$a \cdot m_\gamma - m_\gamma \cdot a \xrightarrow{||\cdot||} 0, \quad \pi_A^{**}(m_\gamma)a - a \xrightarrow{||\cdot||} 0,$$

for every $a \in A$. Use the similar method as above we can show that there exists a net (ξ_γ) in $A \otimes_p A$ such that

$$a \cdot \xi_\gamma - \xi_\gamma \cdot a \xrightarrow{||\cdot||} 0, \quad \pi_A(\xi_\gamma)a - a \xrightarrow{||\cdot||} 0, \quad (a \in A).$$

Then A is pseudo-amenable. ∎

The proof of the following corollary is similar to the previous Theorem so we omit it.

Corollary 2.7. *Let A be Johnson pseudo-contractible. Then A is pseudo-amenable.*

A Banach algebra A is said to be approximately amenable if for every A-bimodule X and every bounded derivation $D : A \to X^*$ there exists a net (x_α) in X^* such that $D(a) = \lim a \cdot x_\alpha - x_\alpha \cdot a$ for every $a \in A$.

Corollary 2.8. *Let A be a Johnson pseudo-contractible Banach algebra with a bounded approximate identity. Then A is approximately amenable.*

Proof. By hypothesis and previous Corollary A is pseudo-amenable. Now [11, Proposition 3.2] implies that A is approximately amenable. ∎

Proposition 2.9. *Let A and B be Banach algebras. Suppose that $T : A \to B$ is a continuous epimorphism. If A is Johnson pseudo-contractible, then B is Johnson pseudo-contractible.*

Proof. Since A is Johnson pseudo-contractible, there exists a bounded net (m_α) in $(A \otimes_p A)^{**}$ such that $a \cdot m_\alpha = m_\alpha \cdot a$ and $\pi_A^{**}(m_\alpha)a \to a$ for every $a \in A$. Define $T \otimes T : A \otimes_p A \to B \otimes_p B$ by $T \otimes T(x \otimes y) = T(x) \otimes T(y)$, for every $x, y \in A$. It is easy to see that $T \otimes T$ is a bounded linear map. So we have

$$T(a) \cdot (T \otimes T)^{**}(m_\alpha) - (T \otimes T)^{**}(m_\alpha) \cdot T(a) = \\ (T \otimes T)^{**}(a \cdot m_\alpha - m_\alpha \cdot a) = 0, \quad (a \in A).$$

Also

$$\pi_B^{**} \circ (T \otimes T)^{**}(m_\alpha)T(a) - T(a) = (\pi_B \circ (T \otimes T))^{**}(m_\alpha \cdot a) - T(a) = \\ T^{**}(\pi_A^{**}(m_\alpha)a - a) \to 0,$$

for every $a \in A$. Then B is Johnson pseudo-contractible. ∎

Corollary 2.10. *Let A be a Johnson pseudo-contractible Banach algebra and let I be a closed ideal of A. Then $\frac{A}{I}$ is Johnson pseudo-contractible.*

Proof. The quotient map is a bounded epimorphism from A onto $\frac{A}{I}$, now apply the previous Proposition. ∎

Theorem 2.11. *Fix $p \geq 1$. Let $A = \ell^p - \oplus_{i \in I} A_i$. If each A_i is Johnson pseudo-contractible, then A is Johnson pseudo-contractible.*

Proof. Let $\epsilon > 0$ and F be any finite subset of A. For each $a \in F$ there exists a finite set $J_\epsilon \subset I$ such that $||P_{J_\epsilon}(a) - a|| < \frac{\epsilon}{2}$, where P_{J_ϵ} is the associated projection from A onto $\ell^p - \oplus_{i \in J_\epsilon} A_i$. Since A_i is Johnson pseudo-contractible, there exists a net $(m_\alpha^i)_\alpha$ in $(A_i \otimes_p A_i)^{**}$ such that

$$x \cdot m_\alpha^i = m_\alpha^i \cdot x, \quad \pi_{A_i}^{**}(m_\alpha^i)x \to x \qquad (x \in A_i).$$

It is easy to see that we can naturally embed $A_i \otimes_p A_i$ in $A \otimes_p A$. So there exists a bounded A-bimodule morphism $L_i : A_i \otimes_p A_i \to A \otimes_p A$. Set $U_\alpha = \sum_{i \in J_\epsilon} L_i^{**}(m_\alpha^i)$. We regard that for every $a \in F$,

$$\begin{aligned} a \cdot U_\alpha = \sum_{i \in J_\epsilon} a \cdot L_i^{**}(m_\alpha^i) &= \sum_{i \in J_\epsilon} L_i^{**}(a \cdot m_\alpha^i) \\ &= \sum_{i \in J_\epsilon} L_i^{**}(P_{J_\epsilon}(a) \cdot m_\alpha^i) \\ &= \sum_{i \in J_\epsilon} L_i^{**}(m_\alpha^i \cdot P_{J_\epsilon}(a)) \\ &= \sum_{i \in J_\epsilon} L_i^{**}(m_\alpha^i \cdot a) = \sum_{i \in J_\epsilon} L_i^{**}(m_\alpha^i) \cdot a = U_\alpha \cdot a. \end{aligned} \tag{2.4}$$

Also for sufficiently large values of α, we have

$$\begin{aligned} ||\pi_A^{**}(U_\alpha)a - a|| = \\ ||\pi_A^{**}(U_\alpha \cdot a) - a|| &= ||\pi_A^{**}(U_\alpha \cdot P_{J_\epsilon}(a)) - a|| \\ &= ||\pi_A^{**}(U_\alpha \cdot P_{J_\epsilon}(a)) - P_{J_\epsilon}(a) + P_{J_\epsilon}(a) - a|| < \frac{\epsilon}{2} + \frac{\epsilon}{2} \end{aligned} \tag{2.5}$$

It follows that A is Johnson pseudo-contractible. ∎

3 Some applications

In this section we characterized the Johnson pseudo-contractibility for algebras related to a locally compact group and next for Lipschitz algebras. Throughout this section G is a locally compact group. A linear subspace $S(G)$ of $L^1(G)$ is said to be a Segal algebra on G if it satisfies the following conditions

(i) $S(G)$ is dense in $L^1(G)$,

(ii) $S(G)$ with a norm $||\cdot||_{S(G)}$ is a Banach space and $||f||_{L^1(G)} \leq ||f||_{S(G)}$ for every $f \in S(G)$,

(iii) for $f \in S(G)$ and $y \in G$, we have $L_y f \in S(G)$ the map $y \mapsto L_y(f)$ from G into $S(G)$ is continuous, where $L_y(f)(x) = f(y^{-1}x)$,

(iv) $||L_y(f)||_{S(G)} = ||f||_{S(G)}$ for every $f \in S(G)$ and $y \in G$,

for more information see [15].

Proposition 3.1. *If $S(G)$ is Johnson pseudo-contractible, then G is amenable.*

Proof. Since $S(G)$ is Johnson pseudo-contractible, by Corollary 2.7, $S(G)$ is pseudo-amenable. Applying [18, Theorem 3.1] G is amenable. ∎

Remark 3.2. In the case $S(G) = L^1(G)$ by previous proposition and the Johnson theorem, we can easily see that $L^1(G)$ is Johnson pseudo-contractible if and only G is amenable.

Proposition 3.3. *The measure algebra $M(G)$ is Johnson pseudo-contractible if and only if G is discrete and amenable.*

Proof. Suppose that $M(G)$ is Johnson pseudo-contractible. Then by Corollary 2.7 $M(G)$ is pseudo-amenable. Using [11, Proposition 4.2] G is discrete and amenable.

For converse, let G be discrete and amenable. By the main result of [3] $M(G)$ is amenable. Then by Proposition 2.1 $M(G)$ is Johnson pseudo-contractible. ∎

Proposition 3.4. *$L^1(G)^{**}$ is Johnson pseudo-contractible if and only if G is finite.*

Proof. Suppose that $L^1(G)^{**}$ is Johnson pseudo-contractible. Then by Corollary 2.7 $L^1(G)^{**}$ is pseudo-amenable. Using [11, Proposition 4.2] G is finite.

Converse is clear. ∎

At the following we characterize Johnson pseudo-contractibility of Lipschitz algebras.

Let A be a Banach algebra and $\phi \in \Delta(A)$. An element $m \in A^{**}$ that satisfies $am = \phi(a)m$ and $\tilde{\phi}(m) = 1$, is called ϕ-mean. Suppose that $m \in A^{**}$ is a ϕ-mean for A. Since $||\phi|| = 1$, we have $||m|| \geq 1$. So for $C \geq 1$, A is called C-ϕ-amenable if A has a ϕ-mean m which $||m|| \leq C$. Note that every amenable Banach algebra A is left ϕ-amenable for every $\phi \in \Delta(A)$ but the converse is not true, see [13].

Lemma 3.5. *Let A be a Johnson pseudo-contractible Banach algebra with an identity and $\Delta(A) \neq \emptyset$. Then A is C-ϕ-amenable for every $\phi \in \Delta(A)$.*

Proof. Since A is Johnson pseudo-contractible, there exists a net (m_α) in $(A \otimes_p A)^{**}$ such that $a \cdot m_\alpha = m_\alpha \cdot a$ and $\pi_A^{**}(m_\alpha)a \to a$ for every $a \in A$. So for every $\epsilon > 0$ there exists α_ϵ such that $||\pi_A^{**}(m_{\alpha_\epsilon})e - e|| < \epsilon$ and $a \cdot m_{\alpha_\epsilon} = m_{\alpha_\epsilon} \cdot a$, where e is an identity for A. Let $T : A \otimes_p A \to A$ be a map defined by $T(a \otimes b) = \phi(b)a$ for every $a, b \in A$. Since $||T|| \leq 1$, we have

$$|\tilde{\phi} \circ T^{**}(m_{\alpha_\epsilon}) - 1| = |\tilde{\phi}(\pi_A^{**}(m_{\alpha_\epsilon})) - 1| = |\tilde{\phi}(\pi_A^{**}(m_{\alpha_\epsilon})e - e)| < \epsilon,$$

Take $\epsilon = \frac{1}{2}$, we have $\frac{1}{2} < |\tilde{\phi} \circ T^{**}(m_{\alpha_\epsilon})| < \frac{3}{2}$. So $||\frac{T^{**}(m_{\alpha_\epsilon})}{\tilde{\phi}(T^{**}(m_{\alpha_\epsilon}))}|| < 2||T^{**}(m_{\alpha_\epsilon})|| \leq 2||m_{\alpha_\epsilon}||$ for every $\phi \in \Delta(A)$. Using the similar method as in the proof of Proposition 2.4, one can see that A is $||m_{\alpha_\epsilon}||$-ϕ-amenable, for every $\phi \in \Delta(A)$. ∎

Let X be a compact metric space and $\alpha > 0$. Set

$$Lip_\alpha(X) = \{f : X \to \mathbb{C} : p_\alpha(f) < \infty\},$$

where

$$p_\alpha(f) = \sup\{\frac{|f(x) - f(y)|}{d(x,y)^\alpha} : x, y \in X, x \neq y\}$$

and also

$$\ell ip_\alpha(X) = \{f \in Lip_\alpha(X) : \frac{|f(x) - f(y)|}{d(x,y)^\alpha} \to 0 \quad \text{as} \quad d(x,y) \to 0\}.$$

Define

$$||f||_\alpha = ||f||_\infty + p_\alpha(f),$$

with pointwise multiplication and norm $||\cdot||_\alpha$, $Lip_\alpha(X)$ and $\ell ip_\alpha(X)$ become Banach algebras. It is well-known that each nonzero multiplicative linear functional on $Lip_\alpha(X)$ or $\ell ip_\alpha(X)$ has a form ϕ_x, where $\phi_x(f) = f(x)$ for every $x \in X$. For further information about Lipschitz algebras see [1] and [20].

Theorem 3.6. *Let X be a compact metric space and let A be $Lip_\alpha(X)$ or $\ell ip_\alpha(X)$. Then the following statements are equivalent*

(i) A is Johnson pseudo-contractible;

(ii) X is finite;

(iii) A is amenable.

Proof. (i)⇒(ii) Suppose that A is Johnson pseudo-contractible. Since A has an identity, by previous Lemma there exists $C \geq 1$ such that A is C-ϕ-amenable for each $\phi \in \Delta(A)$. By [5, Proposition 2.1] for every distinct element $x, y \in X$, we have $||\phi_x - \phi_y|| > C^{-1}$. Also since

$$||\phi_x - \phi_y|| = \sup_{||f||_\alpha \leq 1} ||\phi_x(f) - \phi_y(f)|| = \sup_{||f||_\alpha \leq 1} |f(x) - f(y)| < d(x,y)^\alpha,$$

it follows that $d(x,y)^\alpha > C^{-1}$. This yields X is discrete and compact. So X is finite.

(ii)⇒(iii) It is clear by [12, Theorem 3].

(iii)⇒(i) It is valid by Lemma 2.1. ∎

4 Examples

Example 4.1. (i) Let G be the integer Heisenberg group. It is well-known that G is discrete and amenable. Also G is not a finite extension of an abelian group. Using the main result of [9], we can see that the Fourier algebra $A(G)$ is not amenable. On the other hand by Leptin theorem [17, Theorem 7.1.3] amenability of G implies that $A(G)$ has a central approximate identity. Using [18, Theorem 4.2] we see that $A(G)$ is pseudo-contractible. Then by Lemma 2.2 $A(G)$ is Johnson pseudo-contractible. Thus we find a Johnson pseudo-contractible Banach algebra which is not amenable.

(ii) It is well-known that $\ell^1(\mathbb{Z})$ is an amenable Banach algebra. So by Lemma 2.1 $\ell^1(\mathbb{Z})$ is Johnson pseudo-contractible. We claim that $\ell^1(\mathbb{Z})$ is not pseudo-contractible. Suppose conversely that $\ell^1(\mathbb{Z})$ is pseudo-contractible. Since $\ell^1(\mathbb{Z})$ is unital by [11, Theorem 2.4] $\ell^1(\mathbb{Z})$ must be contractible. It follows that $\mathbb{Z}$ is finite which is impossible. Hence we get a Johnson pseudo-contractible Banach algebra which is not pseudo-contractible.

(iii) In this part we give a pseudo-amenable Banach algebra, which is not Johnson pseudo-contractible.

Suppose that $A = M_{\mathbb{N}}(\mathbb{C})$ is the set of all $\mathbb{N} \times \mathbb{N}$-matrices over $\mathbb{C}$ with finite ℓ^1-norm and matrix multiplication. By [16, Proposition 2.7] $A = M_{\mathbb{N}}(\mathbb{C})$ is biflat. Since $\mathbb{C}$ is unital, by [8, Proposition 3.6] A has an approximate identity. So [8, Proposition 3.6] implies that A is pseudo-amenable. We claim that A is not Johnson pseudo-contractible. We go toward a contradiction and suppose that A is Johnson pseudo-contractible. It follows that there exists a net (m_α) in $(A \otimes_p A)^{**}$ such that $a \cdot m_\alpha = m_\alpha \cdot a$ and $\pi_A^{**}(m_\alpha)a \to a$ for every $a \in A$. Let a be any non-zero element of A. By Hahn-Banach theorem there exists a bounded linear functional Υ in A^* such that $\Upsilon(a) \neq 0$. It follows that $\pi_A^{**}(m_\alpha)a(\Upsilon) \to a(\Upsilon)$. Then $\pi_A^{**}(m_\alpha)(a \cdot \Upsilon) \to \Upsilon(a) \neq 0$. So we can assume that $\pi_A^{**}(m_\alpha)((a \cdot \Upsilon)) \neq 0$ for every α. By Alaoglu's Theorem we have a bounded net (x_α^β) with bound $||m_\alpha||$ in $A \otimes_p A$ such that $x_\alpha^\beta \xrightarrow{w^*} m_\alpha$ and $\pi_A^{**}(x_\alpha^\beta) \xrightarrow{w^*} \pi_A^{**}(m_\alpha)$, since π_A^{**} is a w^*-continuous map.

It is clear that $a \cdot x_\alpha^\beta \xrightarrow{w^*} a \cdot m_\alpha$ and $x_\alpha^\beta \cdot a \xrightarrow{w^*} m_\alpha \cdot a$ for each $a \in A$. It follows that $a \cdot x_\alpha^\beta - x_\alpha^\beta \cdot a \xrightarrow{w^*} 0$ (and also since (x_α^β) is a net in $A \otimes_p A$ we have $a \cdot x_\alpha^\beta - x_\alpha^\beta \cdot a \xrightarrow{w} 0$). Thus $\pi_A(a \cdot x_\alpha^\beta - x_\alpha^\beta \cdot a) = a\pi_A^{**}(x_\alpha^\beta) - \pi_A^{**}(x_\alpha^\beta)a \xrightarrow{w^*} 0$. Therefore $a\pi_A^{**}(x_\alpha^\beta) - \pi_A^{**}(x_\alpha^\beta)a \xrightarrow{w} 0$. Put $y_\beta = \pi_A^{**}(x_\alpha^\beta)$. So (y_β) is a bounded net which satisfies $ay_\beta - y_\beta a \xrightarrow{w} 0$ and $y_\beta \xrightarrow{w^*} \pi_A^{**}(m_\alpha)$ for every $a \in A$. Suppose that $y_\beta = (y_\beta^{i,j})$. We denote $\varepsilon_{i,j}$ for a matrix which (i,j)-entry is 1 and 0 elsewhere. Since the product of weak topology on $\mathbb{C}$ coincides with the weak topology on A, [19, Theorem 4.3, p 137] and $\varepsilon_{1,j}y_\beta - y_\beta\varepsilon_{1,j} \xrightarrow{w} 0$, we have $y_\beta^{j,j} - y_\beta^{1,1} \xrightarrow{w} 0$ and $y_\beta^{i,j} \xrightarrow{w} 0$, whenever $i \neq j$. Since $||y_\beta|| \leq ||m_\alpha||$, it follows that $(y_\beta^{1,1})$ is a bounded net in $\mathbb{C}$. Then $(y_\beta^{1,1})$ has a convergence subnet $(y_{\beta_k}^{1,1})$, with w-limit point l. Since $(y_{\beta_k}^{1,1})$ is a net in $\mathbb{C}$, we can assume that $(y_{\beta_k}^{1,1})$ converges to l with respect to $|\cdot|$. Using $y_\beta^{j,j} - y_\beta^{1,1} \xrightarrow{w} 0$, we have $y_\beta^{j,j} - y_\beta^{1,1} \xrightarrow{|.|} 0$. It follows that $y_{\beta_k}^{j,j} - y_{\beta_k}^{1,1} \xrightarrow{|.|} 0$, so $y_{\beta_k}^{j,j} \xrightarrow{|.|} l$ for every $j \in \mathbb{N}$. We claim that $l \neq 0$, otherwise by
[19, Theorem 4.3] we have $y_\beta \xrightarrow{w} 0$. Then $(a \cdot \mathrm{Y})(y_\beta) \to 0$. On the other hand $(a \cdot \mathrm{Y})(y_\beta) = y_\beta(a \cdot \mathrm{Y}) \to \pi_A^{**}(m_\alpha)(a \cdot \mathrm{Y}) \neq 0$, which reveals a contradiction. So $l \neq 0$. Since $y_{\beta_k}^{j,j} - y_\beta^{1,1} \xrightarrow{w} 0$ and $y_{\beta_k}^{i,j} \xrightarrow{w} 0$ by [19, Theorem 4.3,p 137] we have $y_{\beta_k} \xrightarrow{w} y_0$, where y_0 is a matrix which each entry of main diagonal is l and 0 elsewhere. Thus $y_0 \in \overline{Conv(y_\beta)}^{w} = \overline{Conv(y_\beta)}^{||\cdot||}$. So $y_0 \in A$. But $\infty = \sum_{j\in\mathbb{N}} |l| = \sum_{j\in\mathbb{N}} |y_0^{j,j}| = ||y_0|| < \infty$, which is a contradiction. Therefore A is not Johnson pseudo-contractible.

(iv) A Banach algebra A is called approximately biprojective if there exists a net (ρ_α) of bounded linear A-bimodule morphisms from A into $A \otimes_p A$ such that $\pi_A \circ \rho_\alpha(a) - a \to 0$, for every $a \in A$, see [21]. Suppose that $A = \ell^2(\mathbb{N})$. With the pointwise multiplication A becomes a Banach algebra. By the main result of [4], A is not approximately amenable. But by [21, Example p-3239], A is approximately biprojective Banach algebra with a central approximate identity. Then by [11, Proposition 3.8], A is pseudo-contractible. So by Lemma 2.2 A is a Johnson pseudo-contractible Banach algebra. So there exists a Johnson pseudo-contractible Banach algebra which is not approximately amenable.

Example 4.2. Consider a commutative Banach algebra $A = C^1[0,1]$. It is well-known that $\Delta(A) = \{\phi_t : t \in [0,1]\}$, where $\phi_t(f) = f(t)$ for each $f \in A$. Define $D : A \to \mathbb{C}$ by $D(f) = f'(t)$. Clearly D satisfies $D(fg) = \phi_t(f)g + \phi_t(g)f$. So D is a non-zero point derivation at $\{\phi_t\}$. Thus by [13, Remark 2.4] A is not left ϕ_t-amenable. Thus by Proposition 2.4 A is not Johnson pseudo-contractible.

Example 4.3. Let A be a Banach space with $\dim A > 1$ and $\phi \in A^* \setminus \{0\}$. Define $a * b = \phi(b)a$. It is easy to see that $(A, *)$ becomes a Banach algebra.

We claim that A is not Johnson pseudo-contractible. We go toward a contradiction and suppose that A is Johnson pseudo-contractible. Then by Corollary 2.7, A is pseudo-amenable. So A has an approximate identity, say (e_α). Then

$$\phi(a)e_\alpha - a = e_\alpha * a - a \to 0 \quad (a \in A).$$

Take $a_0 \in A$ such that $\phi(a_0) = 1$. So $e_\alpha - a_0 = \phi(a_0)e_\alpha - a_0 = e_\alpha * a_0 - a_0 \to 0$. So a_0 is an identity for A, that is, $a = a * a_0 = a_0 * a = \phi(a)a_0 \quad (a \in A)$. It follows that $\dim A = 1$ which is a contradiction.

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