A note on nontrivial intersection for selfmaps of complex Grassmann manifolds*

Thaís F. M. Monis Northon C. L. Penteado Sérgio T. Ura
Peter Wong

Abstract

Let $G(k,n)$ be the complex Grassmann manifold of k-planes in $\mathbb{C}^{k+n}$. In this note, we show that for $1 < k < n$ and for any selfmap $f : G(k,n) \to G(k,n)$, there exists a k-plane $V^k \in G(k,n)$ such that $f(V^k) \cap V^k \neq \{0\}$.

1 Introduction

The problem of determining the fixed point property (f.p.p.) for Grassmann manifolds has been studied by many authors (for example [7], [5], [6]).

Let

$$\mathbb{F}M(n_1,\ldots,n_k) = \frac{U_{\mathbb{F}}(n)}{U_{\mathbb{F}}(n_1) \times \cdots \times U_{\mathbb{F}}(n_k)},$$

$n_1 + \cdots + n_k = n$. Here, $\mathbb{F}$ stands for one of the fields $\mathbb{R}$, $\mathbb{C}$ or the skew field $\mathbb{H}$, and

$$U_{\mathbb{F}}(n) = \begin{cases} O(n) \text{ the orthogonal group of order } n \text{ if } \mathbb{F} = \mathbb{R}, \\ U(n) \text{ the unitary group of order } n \text{ if } \mathbb{F} = \mathbb{C}, \\ Sp(n) \text{ the symplectic group of order } n \text{ if } \mathbb{F} = \mathbb{H}. \end{cases}$$

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In [4], Glover and Homer have given the following necessary condition for $\mathbb{F}M(n_1, \ldots, n_k)$ to have the f.p.p..

Theorem 1 ([4], Theorem 1)**.** *If $\mathbb{F}M(n_1, \ldots, n_k)$ has the f.p.p., then $n_1, \ldots, n_k$ are distinct integers and, if $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, at most one is odd.*

The above theorem gives rise to the following conjectures:

Conjecture 1. *If $n_1, \ldots, n_k$ are all distinct then $\mathbb{H}M(n_1, \ldots, n_k)$ has the f.p.p..*

Conjecture 2. *If $n_1, \ldots, n_k$ are all distinct and at most one is odd then $\mathbb{F}M(n_1, \ldots, n_k)$ has the f.p.p., for $\mathbb{F} = \mathbb{R}$ and $\mathbb{F} = \mathbb{C}$.*

The above conjectures were already proved to be true in the following cases:

- Projective spaces ($\mathbb{F}M(1, n-1)$);
- If n_2 and n_3 are distinct positive even integers and $n_3 \geq 2n_2^2 - 1$ then $\mathbb{C}M(1, n_2, n_3)$ has the f.p.p. ([4]).
- If $1, n_2$ and n_3 are distinct positive integers and $n_3 \geq 2n_2^2 - 1$, then $\mathbb{H}M(1, n_2, n_3)$ has the f.p.p. ([4]).
- If $n_2 < n_3$ are even integers greater than 1 and either $n_2 \leq 6$ or $n_3 \geq n_2^2 - 2n_2 - 2$, then $\mathbb{R}M(1, n_2, n_3)$ has the f.p.p. ([4]).
- If n_1, n_2, n_3 are positive integers such that at most one is odd, $n_1 \leq 3$, $n_3 \geq n_2^2 - 1$, and $[n_1/2] < [n_2/2] < [n_3/2]$, then $\mathbb{R}M(n_1, n_2, n_3)$ has the f.p.p. ([4]).
- If $\mathbb{F} = \mathbb{C}$ or $\mathbb{H}$, $\mathbb{F}M(2, q)$ has the f.p.p. for all $q > 2$ ([7]).
- $\mathbb{R}M(2, q)$ has the f.p.p. for all $q = 4k$ or $q = 4k + 1$, $k = 1, 2, 3, \ldots$ ([7]).
- For $p \leq 3$ and $q > p$ or $p > 3$ and $q \geq 2p^2 - p - 1$, $\mathbb{C}M(p, q)$ has the f.p.p. iff pq is even ([5]).
- For $p \leq 3$ and $q > p$ or $p > 3$ and $q \geq 2p^2 - p - 1$, $\mathbb{H}M(p, q)$ always has the f.p.p. ([5]).

The main tool used to prove the above results is the calculation of the Lefschetz number of a self-map of such a space. Let's focus on the case of complex Grassmann manifolds $\mathbb{C}M(k, n) = G(k, n)$, the space of k-planes in $\mathbb{C}^{k+n}$. Let γ^k be the canonical k-plane bundle over $G(k, n)$. If

$$ch(\gamma^k) = 1 + c_1 + \cdots + c_k, \quad c_i \in H^{2i}(G(k, n); \mathbb{Q}),$$

is the total Chern class of γ^k, then the cohomology ring $H^*(G(k, n); \mathbb{Q})$ is given by:

$$H^*(G(k, n); \mathbb{Q}) = \mathbb{Q}[c_1, \ldots, c_k] / I_{k,n},$$

where $I_{k,n}$ is the ideal generated by the elements $(c^{-1})_{n+1}, \ldots, (c^{-1})_{n+k}$. Here, $(c^{-1})_q$ is the part of the formal inverse of c in dimension $2q$ (see [6], Theorem 2.1). Then, c_1 is the only generator in dimension 2. Therefore, given a self-map $f : G(k,n) \to G(k,n)$, $f^*(c_1) = mc_1$ for some coefficient m.

Theorem 2 ([5], Theorem 1)**.** *Let $k \leq 3$ and $n > k$ or $k > 3$ and $n \geq 2k^2 - k - 1$. Then every graded ring endomorphism of $H^*(G(k,n);\mathbb{Q})$ is an Adams endomorphism*[1]*. Consequently, if $f : G(k,n) \to G(k,n)$ is a self-map with $f^*(c_1) = mc_1$ then $f^*(c_i) = m^i c_i$, $i = 1, \ldots, k$.*

The classification of the graded ring endomorphisms of $H^*(G(k,n);\mathbb{Q})$ is fundamental in the study of f.p.p. for $G(k,n)$ because of the following.

Proposition 1. *An Adams endomorphism of $H^*(G(k,n);\mathbb{Q})$ has Lefschetz number zero if and only if its degree is -1 and kn is odd.*

Proof. See [4], Proposition 4. ■

In [6], M. Hoffman was able to prove the following.

Theorem 3 ([6], Theorem 1.1)**.** *Let $k < n$ and h be a graded ring endomorphism of $H^*(G(k,n);\mathbb{Q})$ with $h(c_1) = mc_1$, $m \neq 0$. Then $h(c_i) = m^i c_i$, $1 \leq i \leq k$.*

If $k < n$ and h is a graded ring endomorphism of $H^*(G(k,n);\mathbb{Q})$ with $h(c_1) = 0$, it is still unclear about what h looks like in general. The conjecture is that, in this case, h must be the null homomorphism. If one can prove such conjecture then the problem of determining the f.p.p. for $G(k,n)$ will be completely solved.

In this note, we prove a much more modest result for complex Grassmann manifolds than a fixed point theorem. Our main theorem is the following.

Theorem 4 (Main Result)**.** *Let $k > 1$ and $k < n$. Then for every continuous map $f : G(k,n) \to G(k,n)$ there exists a k-plane $V^k \in G(k,n)$ such that $V^k \cap f(V^k) \neq \{0\}$.*

The motivation for this work is the paper [8] where the author gave an alternative proof for the f.p.p. of $\mathbb{C}P^{2n}$ using characteristic classes. In fact, a closer look at the proof of the main result in [8] indicates that the same argument would also yield an alternative proof of the f.p.p. for $\mathbb{R}P^{2n}$ by replacing Chern classes with Stiefel-Whitney classes. We should also point out that a non-trivial intersection result similar to Theorem 4 has been obtained in [1] for maps between two *different* Grassmann manifolds.

[1]An Adams endomorphism of $H^*(G(k,n);\mathbb{Q})$ is an endomorphism φ of the form $\varphi(x) = \lambda^i x$ for $x \in H^{2i}(G(k,n);\mathbb{Q})$. The coefficient λ is called the degree of φ.

2 Proof of the Main Theorem

Throughout this paper, $G(k,n)$ denotes the complex Grassmann manifold of k-planes in $\mathbb{C}^{k+n}$.

Note that, since $G(k,n)$ and $G(n,k)$ are homeomorphic, γ^k and γ^n can be seen as subbundles of the trivial bundle $G(k,n) \times \mathbb{C}^{k+n}$, which is denoted by ϵ^{k+n}, and, under such identification,

$$\gamma^k \oplus \gamma^n = \epsilon^{k+n}.$$

Lemma 1. *Let $ch(\gamma^n) = 1 + \bar{c}_1 + \cdots + \bar{c}_n$ be the total Chern class of the bundle γ^n. Then, a general formula for the class $\bar{c}_i$ in terms of the Chern classes of γ^k is given by*

$$\bar{c}_i = \sum_{\|\alpha\|=i} (-1)^{|\alpha|} \frac{|\alpha|!}{\alpha!} ch(\gamma^k)^{\alpha},$$

where α represents the k-uple $\alpha = (a_1, \ldots, a_k)$, $\|\alpha\| = a_1 + 2a_2 + \cdots + ka_k$, $|\alpha| = a_1 + a_2 + \cdots + a_k$, $\alpha! = a_1!a_2! \cdots a_k!$ and $ch(\gamma^k)^{\alpha} = c_1^{a_1} \smile c_2^{a_2} \smile \cdots \smile c_k^{a_k}$.

Proof. The proof is given recursively in the index i.

As $\gamma^k \oplus \gamma^n = \epsilon^{k+n}$, we have

$$ch(\gamma^k) \smile ch(\gamma^n) = ch(\epsilon^{k+n}) = 1$$

in $H^*(G(k,n); \mathbb{Z})$. So

$$(1 + c_1 + \cdots + c_k) \smile (1 + \bar{c}_1 + \cdots + \bar{c}_n) = 1$$

and then

$$\begin{aligned} 1 &= 1 \\ 0 &= c_1 + \bar{c}_1 \\ 0 &= c_2 + c_1 \smile \bar{c}_1 + \bar{c}_2 \\ &\cdots \end{aligned}$$

Then

$$\bar{c}_j = -\sum_{i=1}^{j} c_i \smile \bar{c}_{j-i}$$

for all $j = 1, \ldots, n$, with the convention $c_i = 0$ when $i > k$. Thus,

(i) $\bar{c}_1 = -c_1$;

(ii) $\bar{c}_2 = -(c_1 \smile -c_1) - c_2 = c_1^2 - c_2$;

(iii) Suppose

$$\bar{c}_j = \sum_{||\alpha||=j} (-1)^{|\alpha|} \frac{|\alpha|!}{\alpha!} ch(\gamma^k)^{\alpha},$$

for $j = 1, \ldots, m-1 < n$.

Then

$$\begin{aligned}
\bar{c}_m &= -\sum_{i=1}^{m} c_i \smile \bar{c}_{m-i} \\
&= -\sum_{i=1}^{m} \left(c_i \smile \sum_{||\alpha||=m-i} (-1)^{|\alpha|} \frac{|\alpha|!}{\alpha!} ch(\gamma^k)^{\alpha} \right) \\
&= \sum_{i=1}^{m} \left(c_i \smile \sum_{||\alpha||=m-i} (-1)^{|\alpha|+1} \frac{|\alpha|!}{\alpha!} ch(\gamma^k)^{\alpha} \right) \\
&= \sum_{i=1}^{m} \left(\sum_{||\alpha||=m-i} (-1)^{|\alpha|+1} \frac{|\alpha|!}{\alpha!} ch(\gamma^k)^{\alpha} \smile c_i \right) \\
&= \sum_{i=1}^{m} \sum_{||\alpha||=m-i} (-1)^{|\alpha+e_i|} \frac{|\alpha|!}{\alpha!} ch(\gamma^k)^{\alpha+e_i} \qquad (e_i = (0,\ldots,0,1,0,\ldots 0)) \\
&= \sum_{||\beta||=m} (-1)^{|\beta|} X(\beta) ch(\gamma^k)^{\beta} \qquad (\beta = \alpha + e_i)
\end{aligned}$$

where

$$\begin{aligned}
X(\beta) &= \sum_{b_i \neq 0} \frac{|\beta - e_i|!}{(\beta - e_i)!} \\
&= \sum_{b_i \neq 0} \frac{(|\beta|-1)! b_i}{\beta!} \\
&= \sum_{i=1}^{m} \frac{(|\beta|-1)! b_i}{\beta!} \\
&= \frac{(|\beta|-1)! \sum_{i=1}^{m} b_i}{\beta!} \\
&= \frac{(|\beta|-1)! |\beta|}{\beta!} \\
&= \frac{|\beta|!}{\beta!}
\end{aligned}$$

■

2.1 Proof of Theorem 4

Suppose, to the contrary, there exists a continuous map $f : G(k,n) \to G(k,n)$ such that $V^k \cap f(V^k) = \{0\}$ for every k-plane $V^k \in G(k,n)$. Then the direct sum $\gamma^k \oplus f^*\gamma^k$ can be seen as a subbundle of the trivial bundle ϵ^{k+n}. Let η^{n-k} be the normal bundle of $\gamma^k \oplus f^*\gamma^k$ in ϵ^{k+n}. Then

$$ch(\gamma^k) \smile ch(f^*\gamma^k) \smile ch(\eta^{n-k}) = 1. \tag{2.1}$$

It follows that

$$ch(f^*\gamma^k) \smile ch(\eta^{n-k}) = 1 + \bar{c}_1 + \cdots + \bar{c}_n. \tag{2.2}$$

Let

$$ch(f^*\gamma^k) = 1 + \tilde{c}_1 + \cdots + \tilde{c}_k,\ \tilde{c}_i \in H^{2i}(G(k,n);\mathbb{Q}), \tag{2.3}$$

and

$$ch(\eta^{n-k}) = 1 + t_1 + \cdots + t_{n-k},\ t_j \in H^{2j}(G(k,n);\mathbb{Q}). \tag{2.4}$$

We will show that it is impossible for

$$\bar{c}_n = \tilde{c}_k \smile t_{n-k}. \tag{2.5}$$

The proof of the impossibility of the above equality will be split into several cases.

Case 1: $1 < k \leq 3$**.** Since $c_1 \in H^2(G(k,n);\mathbb{Q})$ is the only generator in dimension 2, $f^*(c_1)$ is a multiple of c_1, let's say $f^*(c_1) = mc_1$. Following [7] and [5], for $k \leq 3$ and $k < n$, every endomorphism of the ring $H^*(G(k,n);\mathbb{Q})$ that preserves dimension is an Adams endomorphism. Therefore, if $f^*(c_1) = mc_1$ then $f^*(c_2) = m^2c_2, \ldots, f^*(c_k) = m^k c_k$. Thus

$$ch(f^*\gamma^k) = f^*(ch(\gamma^k)) = 1 + mc_1 + m^2c_2 + \cdots + m^k c_k.$$

It follows that

$$\bar{c}_n = m^k c_k \smile t_{n-k},$$

in contradiction with Lemma 1.

Case 2: $k > 3$**.** This case will be split in four cases.

Case 2(i): $n = l(k-1) + r$ with remainder $r \neq 1$, that is, $1 < r < k-1$ or $r = 0$. In this case, r is of the form $r = 2i$ or $r = 2i + 3$, for some integer $i \geq 0$. In case of $r = 2i$, the class $c_{k-1}^l \smile c_2^i$ does not appear in $\tilde{c}_k \smile t_{n-k}$ but, by Lemma 1, it appears in $\bar{c}_n$, contradicting $\bar{c}_n = \tilde{c}_k \smile t_{n-k}$. In case of $r = 2i+3$, the class $c_{k-1}^l c_2^i c_3$ does not appear in $\tilde{c}_k \smile t_{n-k}$ but, by Lemma 1, it appears in $\bar{c}_n$, contradicting $\bar{c}_n = \tilde{c}_k \smile t_{n-k}$.

Case 2(ii): $k > 4$ **and** $n = (l+1)(k-1) + 1$**.** In this case, we have

$$\begin{aligned} n &= (l+1)(k-1) + 1 \\ &= l(k-1) + k \end{aligned}$$

and, since $n > k$, $l \geq 1$. We can write $n = (l+1)(k-1) + 1$ in the form

$$n = (l-1)(k-1) + 2(k-2) + 3$$

and, since we are supposing $k > 4$, $k - 2 > 2$. With these information, one can check that the class $c_{k-1}^{m-1} \smile c_{k-2}^2 \smile c_3$ cannot appear in $\tilde{c}_k \smile t_{n-k}$. On the other hand, by Lemma 1, the class $c_{k-1}^{m-1} \smile c_{k-2}^2 \smile c_3$ appears in $\bar{c}_n$. Therefore, $\bar{c}_n = \tilde{c}_k \smile t_{n-k}$ is impossible.

Case 2(iii): $k = 4$, $n = (l+1)(k-1)+1$ and l even, say $l = 2j$. In this case, $n - k = 3l$ and, since $n > 1$, $l \geq 1$. Let

$$\begin{aligned}\tilde{c}_4 &= c_1^4 + \alpha c_2^2 + \theta c_4 + \text{other terms}\\ t_{3l} &= c_1^{3l} + \alpha' c_2^{3j} + \beta c_3^l + \text{other terms.}\end{aligned}$$

Thus, in the product $\tilde{c}_4 \smile t_{3l}$, $\alpha\alpha'$ is the coefficient of c_2^{3j+2}, $\alpha\beta$ is the coefficient of $c_2^2 \smile c_3^l$ and $\theta\beta$ is the coefficient of $c_4 \smile c_3^l$. From Lemma 1 together with the fact that $\tilde{c}_4 \smile t_{3l} = \bar{c}_n$, it follows that

$$\begin{aligned}\alpha\alpha' &= \frac{(3j+2)!}{(3j+2)!1!}\\ \alpha\beta &= \frac{(l+2)!}{l!2!}\\ \theta\beta &= \frac{(l+1)!}{l!1!}.\end{aligned}$$

Thus

$$\begin{aligned}\alpha\alpha' &= 1\\ \alpha\beta &= \frac{(l+2)(l+1)}{2}\\ \theta\beta &= l+1.\end{aligned}$$

Then, we conclude that $\alpha = \pm 1$, $\beta = \pm\dfrac{(l+2)(l+1)}{2}$ and $|\beta| = \dfrac{(l+2)(l+1)}{2}$ divides $\theta\beta = l+1$. It follows that $l = 0$, but $l \geq 1$, a contradiction!

Case 2(iv): $k = 4$, $n = (l+1)(k-1)+1$ and l odd, say $l = 2j+1$. Again, $n - k = 3l$ and, since $n > 1$, $l \geq 1$. Let

$$\begin{aligned}\tilde{c}_4 &= c_1^4 + \alpha c_2^2 + \theta c_4 + \gamma c_1 c_3 + \text{other terms}\\ t_{3l} &= c_1^{3l} + \alpha' c_1 c_2^{3j+1} + \beta c_3^l + \text{other terms.}\end{aligned}$$

It follows that, in the product $\tilde{c}_4 \smile t_{3l}$, $\alpha\alpha'$ is the coefficient of $c_1 \smile c_2^{3j+3}$, $\alpha\beta$ is the coefficient of $c_2^2 \smile c_3^l$, $\theta\beta$ is the coefficient of $c_4 \smile c_3^l$ and $\gamma\beta$ is the coefficient of $c_1 \smile c_3^{l+1}$. Since $\bar{c}_n = \tilde{c}_4 \smile t_{3l}$, together with Lemma 1,

$$\begin{aligned}\alpha\alpha' &= \frac{(3j+4)!}{1!(3j+3)!}\\ \alpha\beta &= \frac{(l+2)!}{l!2!}\\ \theta\beta &= \frac{(l+1)!}{l!1!}\\ \gamma\beta &= \frac{(l+2)!}{1!(l+1)!}.\end{aligned}$$

Thus

$$\begin{aligned}\alpha\alpha' &= 3j+4\\ \alpha\beta &= \frac{(l+2)(l+1)}{2}\\ \theta\beta &= l+1\\ \gamma\beta &= l+2.\end{aligned}$$

From the two last equalities above, it follows that β divides $l+1$ and $l+2$. Therefore, $\beta = 1$. It follows that $\alpha = \frac{(l+2)(l+1)}{2}$ and, since α divides $3j+4$,

$$\frac{(l+2)(l+1)}{2} \leq 3j+4 = \frac{3l+5}{2}.$$

Therefore, $l^2 \leq 3$. Since l is an integer not smaller than 1, it follows that $l = 1$. Then, $3j+4 = \frac{3l+5}{2} = 4$ is divisible by $\frac{(l+2)(l+1)}{2} = 3$, a contradiction! ■

References

[1] Chakraborty, Prateep and Sankaran, Parameswaran *Maps between certain complex Grassmann manifolds*. Topology Appl. **170** (2014), 119–123.

[2] Duan, Haibao, *Self-maps of the Grassmannian of complex structures*. Compositio Math. **132** (2002), no. 2, 159–175.

[3] Glover, Henry and Homer, William, *Self-maps of flag manifolds*. Trans. Amer. Math. Soc. **267** (1981), no. 2, 423–434.

[4] Glover, Henry and Homer, William, *Fixed points on flag manifolds*, Pacific J. Math. **101** (1982), no. 2, 303–306.

[5] Glover, Henry and Homer, William, *Endomorphisms of the cohomology ring of finite Grassmann manifolds*. Lecture Notes in Math., vol. 657, Springer-Verlag, Berlin and New York, 1978, 179–193.

[6] Hoffman, Michael, *Endomorphisms of the cohomology of complex Grassmannians*. Trans. Amer. Math. Soc. **281** (1984), 745–740.

[7] O'Neill, Larkin S., *On the f.p.p. for Grassmann manifolds*. Ph.D. Thesis, Ohio State University, 1974.

[8] Taghavi, Ali, *An alternative proof for the f.p.p. of* $\mathbb{C}P^{2n}$. Expo. Math. **33** (2015), 105–107.

São Paulo State University (UNESP),
Institute of Geosciences and Exact Sciences (IGCE).
Av. 24A, 1515 Bela Vista, Rio Claro-SP, Brazil.
emails: tfmonis@rc.unesp.br, northoncanevari@gmail.com, sergioura@gmail.com

Department of Mathematics, Bates College, Lewiston, ME 04240, USA.
e-mail: pwong@bates.edu