Maps between Sol 3-manifolds and coincidence Nielsen numbers*

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Abstract

Let M_A be the torus bundle over S^1 obtained using as gluing map an Anosov matrix A. In this paper we discuss maps from M_{A^r} to M_A and compute the coincidence Nielsen numbers for such maps, moreover we use that such manifolds are double covers of torus semi-bundles and compute the coincidence Nielsen number for selfmaps of Sol 3-manifolds which are torus semi-bundles.

1 Introduction

Maps between torus bundles over the circle and Nielsen theory for such spaces were studied by many authors (e.g. [Sa, SWW, GW, Vi, JL]). In some of these works the authors are concerned with the description of possible maps between such spaces (specially the non-trivial maps) and in others they try to compute Nielsen numbers for some maps. In this work, following some ideas from [GW], we discuss maps between Sol 3-manifolds which are torus bundles, in particular we study maps from a torus bundle obtained using a gluing map that is a power of the Anosov matrix used in the target space, such situation explore some covering maps. In the end, using [Je2], we compute the coincidence Nielsen numbers for those maps.

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Let T denote the torus obtained as the quotient space $\dfrac{\mathbb{R}\times\mathbb{R}}{\mathbb{Z}\times\mathbb{Z}}$. For each homeomorphism A on T, induced by a linear operator in $\mathbb{R}\times\mathbb{R}$ that preserves $\mathbb{Z}\times\mathbb{Z}$, we identify A with an integer matrix with determinant either 1 or -1.

We constructed $M_A = \dfrac{T\times\mathbb{R}}{((x,y),t)\sim(A^n(x,y),t-n)}$ which is a torus bundle over S^1. If A is an Anosov matrix (*i.e.* either $\det(A) = 1$ and $|tr(A)| > 2$ or $\det(A) = -1$ and $tr(A) \neq 0$ [Sa]), we have that M_A is a 3-manifold with Sol-geometry [[SWW], (1.3)].

The present paper is organized in two sections besides this Introduction. In Section 2 we present some general facts about maps between Sol-torus bundles, in special we describe the possible maps from M_{A^r} to M_A (such situation includes many covering maps) and we compute the Nielsen coincidence numbers for said maps. In Section 3 we use the fact that torus bundles are double covers of Sol-torus semi-bundles (also named sapphire manifolds) to compute the coincidence Nielsen number for these manifolds.

2 Coincidence Nielsen numbers

As observed above, $T \to M_A \xrightarrow{p} S^1$ is a fiber bundle where p is the projection given by $p[((x,y),t)] = [t] \in \dfrac{\mathbb{R}}{\mathbb{Z}} \simeq \dfrac{[0,1]}{0\sim 1} \simeq S^1$.

Let $f,g : M_{A^r} = \dfrac{T\times\mathbb{R}}{((x,y),t)\sim((A^r)^n(x,y),t-n)} \to$

$$M_A = \frac{T\times\mathbb{R}}{((x,y),t)\sim(A^n(x,y),t-n)},$$

where $r\in\mathbb{N},\ n\in\mathbb{Z}$.

By [[Je2], (5.5)], we have that the pair (f,g) is homotopic to a fiber pair, so the following diagram is commutative:

$$\begin{tikzcd} T \arrow[r] \arrow[d, "f'"', "g'"] & M_{A^r} \arrow[r] \arrow[d, "f"', "g"] & S^1 \arrow[d, "\bar{f}"', "\bar{g}"] \\ T \arrow[r] & M_A \arrow[r] & S^1 \end{tikzcd} \tag{2.1}$$

The theorem below describe the possible maps in such context (that includes many covering maps). This characterization will be useful later.

Theorem 2.1. *Let $f : M_{A^r} \to M_A$ be a map between Sol-torus bundles M_{A^r} and M_A with Anosov matrices A^r and A, respectively, $A\in GL(2,\mathbb{Z})$, $r\in\mathbb{N}$. Let $f' : T\to T$ be the induced map on the fiber such that $f'_\# = B = \begin{pmatrix} m & n \\ p & q \end{pmatrix}$, and let $\bar{f} : S^1 \to S^1$. Suppose that $A^r = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$. Then:*

$$B = \begin{cases} \begin{pmatrix} q + \left[\dfrac{a'-d'}{c'}\right] p & \left[\dfrac{b'}{c'}\right] p \\ p & q \end{pmatrix} & , \ if \deg \bar{f} = r \\ \begin{pmatrix} -q & \left[\dfrac{a'-d'}{c'}\right] q - \left[\dfrac{b'}{c'}\right] p \\ p & q \end{pmatrix} & , \ if \deg \bar{f} = -r \\ \begin{pmatrix} -q & \left[\dfrac{b'}{d'}\right] q \\ -\left[\dfrac{c'}{a'}\right] q & q \end{pmatrix} & , \ if \deg \bar{f} = -r; r \ odd \ and \ \det A = -1 \\ 0 & , \ if \deg \bar{f} \neq \pm r \end{cases},$$

where $\left[\dfrac{a'-d'}{c'}\right] p, \left[\dfrac{b'}{c'}\right] p, \left[\dfrac{a'-d'}{c'}\right] q, \left[\dfrac{b'}{d'}\right] q, \left[\dfrac{c'}{a'}\right] q \in \mathbb{Z}.$

Proof. Suppose $\deg \bar{f} = r$. The commutative diagram (2.1) implies that $BA^r = A^r B$. By solving this matrix equation we get to the first case.

If $\deg \bar{f} = -r$, then the commutative diagram (2.1) implies that $BA^r = A^{-r}B$, and the result follows solving this matrix equation.

Suppose now that $\deg \bar{f} = k \neq \pm r$.

Since A is an Anosov matrix, A is diagonalizable. So, there exists $P \in GL(2, \mathbb{R})$ such that $P^{-1}AP = \bar{A} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, where λ_1 and λ_2 are the eigenvalues of A.

Let $\bar{B} = P^{-1}BP = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$. By the commutative diagram (2.1) we have that $BA^r = A^k B$.

It follows that $\bar{B}\bar{A}^r = \bar{A}^k \bar{B}$, that is, $\begin{pmatrix} x\lambda_1^r & y\lambda_2^r \\ z\lambda_1^r & w\lambda_2^r \end{pmatrix} = \begin{pmatrix} \lambda_1^k x & \lambda_1^k y \\ \lambda_2^k z & \lambda_2^k w \end{pmatrix}$.

Since $\lambda_1\lambda_2 = \det \bar{A} = \pm 1$, $k \neq r$ and $\lambda_1^{|k-r|} \neq 1 \neq \lambda_2^{|k-r|}$, we conclude that $x = 0 = w$. Also $y = 0 = z$ since $\lambda_1 = \pm \dfrac{1}{\lambda_2}$, $\lambda_2^{r+k} \neq \pm 1$, and $k \neq -r$.

Thus, $\bar{B} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and consequently, $B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. ■

REMARK 1: This form of the matrix B coincides with the one given by [SWW] when f is a selfmap, that is, $r = 1$.

The next theorem follows from Corollary (5.5) of [Je2].

Theorem 2.2. *Let $f, g : M_{A^r} \to M_A$ be maps between Sol-torus bundles M_{A^r} and M_A with Anosov matrices A^r and A, respectively, $A \in GL(2, \mathbb{Z})$, $r \in \mathbb{N}$. Let $f'g' : T \to T$ be the induced maps on the fiber such that $f'_\# = B$ and $g'_\# = C$. Let $\bar{f}, \bar{g} : S^1 \to S^1$ such that $\deg \bar{f} = k$ and $\deg \bar{g} = l$. Then*

$$N(f,g) = \begin{cases} 0 & , \text{ if } k = l \\ \displaystyle\sum_{i=0}^{|k-l|-1} \left|\det(A^{sign(k)i}B - C)\right| & , \text{ if } k \neq l \end{cases}$$

where $sign(k) := \begin{cases} -1 & , \text{ if } k < 0 \\ 1 & , \text{ if } k > 0. \end{cases}$

REMARK 2: Under the hypotheses of Theorem 2.2, if $A \in SL(2, \mathbb{Z})$ and $f, g : M_{A^r} \to M_A$, when B and C are of the form of Theorem 2.1, then a straightforward calculation shows that $\det(A^i B - C) = \det(B) + \det(C)$.

Corollary 2.1. *Under the hypotheses of Theorem 2.2, if $A \in SL(2, \mathbb{Z})$, then*

$$N(f,g) = |k - l| |\det(B) + \det(C)|.$$

Corollary 2.2. *Under the hypotheses of Theorem 2.2, if $f, g : M_A \to M_A$ are self homeomorphisms, then either $N(f,g) = 0$ or $N(f,g) = 4$.*

Proof. First, we observe that since f and g are homeomorphisms, $k = deg(\bar{f}) = \pm 1$ and $l = \deg(\bar{g}) = \pm 1$.

If $\det(A) = -1$, then by [[Sa],Lemma 1.7 (3)] we have that there exists no matrix B with $\det(B) = \pm 1$ such that $BA = A^{-1}B$. Thus, $\deg \bar{f} = 1$, that is, $\bar{f}$ must induce the identity homomorphism on $\pi_1(S^1)$. Analogously, $\bar{g}$ also induces the identity homomorphism on $\pi_1(S^1)$. Therefore, $N(f,g) = 0$.

If $\det(A) = 1$, then in the cases where $k = 1 = l$ and $k = -1 = l$, we get $N(f,g) = 0$. For $k = -1$ and $l = 1$ (or the symmetric case) we have, by Corollary 2.1, that $N(f,g) = 2|\det(B) + \det(C)|$.

So, we see that

- $N(f,g) = 0$ when either $\det(B) = 1$ and $\det(C) = -1$ or $\det(B) = -1$ and $\det(C) = 1$;
- $N(f,g) = 4$ when $\det(B) = \det(C)$. ∎

REMARK 3: This corollary generalizes to coincidence the Theorem 2.2 of [GW].

3 Coincidence Nielsen numbers for selfmaps on sapphires

The family of 3-dimensional manifolds with Sol-geometry has two subfamilies, one of them consists of the torus bundles with an Anosov gluing map, the other one contains the torus semi-bundles (also named sapphire manifolds) (see [Mo]).

The construction of torus semi-bundles can be found in [SWW] or [Mo]. We will follow the approach of [GW] about such spaces, and we will use the same notation found there.

Let N_ϕ be a sapphire space that is not a torus bundle over S^1. By [SWW] we have that N_ϕ admits a Sol-geometry if and only if $\det \phi_* = \pm 1$ and $xyzw \neq 0$ where $\phi_* = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$. By [[GW],(3.2)] we have that N_ϕ is double-covered by a torus bundle M over S^1, that has Anosov gluing map, and the Lemma 3.3 of the same paper shows that the fundamental group $\pi_1(M)$ is fully-invariant in $\pi_1(N_\phi)$. This torus bundle, M, is always orientable [[SWW],(2.8)].

In this section we compute the coincidence Nielsen number of selfmaps $f, g : N_\phi \to N_\phi$, beginning with self homeomorphisms.

Theorem 3.1. *Let N_ϕ be a sapphire space that is not a torus bundle over S^1. If N_ϕ supports Sol-geometry, then for every pair of self homeomorphisms $(f, g) : N_\phi \to N_\phi$, we have either $N(f, g) = 0$ or $N(f, g) = 4$.*

Proof. Since $f, g : N_\phi \to N_\phi$ are self homeomorphisms, they can be lifted to $(f_1, g_1), (\alpha f_1, \alpha g_1), (\alpha f_1, g_1), (f_1, \alpha g_1) : M \to M$, where $\alpha : M \to M$ is a deck transformation. Besides, we have that $\deg f = \pm 1$ and $\deg g = \pm 1$. So we have to analyze three possibilities:

(i) $\deg f = 1 = \deg g$;

(ii) $\deg f = -1 = \deg g$;

(iii) $\deg f = -1$ and $\deg g = 1$.

Note that if $\bar{f}_1$ induces the identity on $\pi_1(S^1)$, then $\bar{f}_1$ is homotopic to the identity map on S^1 and since α induces $-id$ on the base S^1 [GW], we have that $\overline{\alpha f_1}$ induces $-id$ on $\pi_1(S^1)$.

So, we have four possibilities for the maps $\bar{f}_1, \bar{g}_1, \overline{\alpha f_1}, \overline{\alpha g_1}$:

- $\bar{f}_1$ and $\bar{g}_1$ induce id_{S^1} while $\overline{\alpha f_1}$ and $\overline{\alpha g_1}$ induce $-id_{S^1}$;
- $\bar{f}_1$ induces id_{S^1} and $\bar{g}_1$ induces $-id_{S^1}$ while $\overline{\alpha f_1}$ induces $-id_{S^1}$ and $\overline{\alpha g_1}$ induces id_{S^1};
- $\bar{f}_1$ induces $-id_{S^1}$ and $\bar{g}_1$ induces id_{S^1} while $\overline{\alpha f_1}$ induces id_{S^1} and $\overline{\alpha g_1}$ induces $-id_{S^1}$;
- $\bar{f}_1$ and $\bar{g}_1$ induce $-id_{S^1}$ while $\overline{\alpha f_1}$ and $\overline{\alpha g_1}$ induce id_{S^1}.

The first and last cases are symmetric, as well as the second and the third cases, so we need to analyze just the third and fourth cases.

(i) If $\deg f = 1 = \deg g$, then $\deg f_1 = 1 = \deg \alpha f_1$ and $\deg g_1 = 1 = \deg \alpha g_1$.

Also, if $\bar{f}_1, \bar{g}_1 \simeq -id_{S^1}$ and $\overline{\alpha f_1}, \overline{\alpha g_1} \simeq id_{S^1}$, then $\deg \bar{f}_1 = \deg \bar{g}_1 = -1$ and $\deg \overline{\alpha f_1} = \deg \overline{\alpha g_1} = 1$. So, by Theorem 2.2, we have that $N(f_1, g_1) = 0$ and $N(f_1, \alpha g_1) = \sum_{i=0}^{1} \left| \det(A^{-i}B - \alpha_\# C) \right| \overset{\text{Corollary 2.1}}{=} 2|\det(B) + \det(\alpha_\# C)|$.

We only need to analyze these two pairs of lifts because $(\alpha f_1, \alpha g_1) = \alpha(f_1, g_1)$ and $(\alpha f_1, g_1) = \alpha(f_1, \alpha g_1)$, which means that the two first pairs of lifts $(\alpha f_1, \alpha g_1)$ and (f_1, g_1) are conjugated and so are the two last $(\alpha f_1, g_1)$ and $(f_1, \alpha g_1)$.

Now, $f'_{1_\#} = B$ with $\det(B) = \pm 1$ and $g'_{1_\#} = C$ with $\det(C) = \pm 1$. Since α induces $\alpha_\# = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ on the fiber T [GW], $\det(\alpha_\# C) = -\det(C)$. Besides, it can not happen that $\det(\alpha_\# C) = -1$ and $\det(B) = 1$, because for $\det(\alpha_\# C) = -1$ we must have $\deg g'_1 = 1$ and since $\deg \bar{g}_1 = -1$, we obtain $\deg g_1 = -1$, a contradiction. The same occurs to $\det(B) = 1$. Therefore, we can only have $\det(B) = -1$ and $\det(\alpha_\# C) = 1$, which implies $|\det(B) + \det(\alpha_\# C)| = 0$, that is, $N(f_1, \alpha g_1) = 0$.

Now, since the coincidence Nielsen numbers of the lifts are null, we conclude that $N(f, g) = 0$.

As for the other case, that is, $\bar{f}_1, \overline{\alpha g_1} \simeq -id_{S^1}$ and $\bar{g}_1, \overline{\alpha f_1} \simeq id_{S^1}$, we can argue in the same way as above to conclude that we can only have $\det(B) = -1$ and $\det(C) = 1$, which means that $|\det(B) + \det(C)| = 0$, that is, $N(f_1, g_1) = 0$. And since $N(f_1, \alpha g_1) = 0$, we obtain that $N(f, g) = 0$.

(ii) If $\deg f = -1 = \deg g$, then $\deg f_1 = -1 = \deg \alpha f_1$ and $\deg g_1 = -1 = \deg \alpha g_1$. Following the same procedure as above, we find $N(f_1, g_1) = 0$ and $N(f_1, \alpha g_1) = 0$ for the first case, and for the other case we have $N(f_1, g_1) = 0$ and $N(f_1, \alpha g_1) = 0$. So, we conclude that $N(f, g) = 0$.

(iii) If $\deg f = -1$ and $\deg g = 1$, then $\deg f_1 = -1 = \deg \alpha f_1$ and $\deg g_1 = 1 = \deg \alpha g_1$. Thus, $N(f_1, g_1) = 0$ and $N(f_1, \alpha g_1) = 4$ for the first case, and $N(f_1, g_1) = 4$ and $N(f_1, \alpha g_1) = 0$ for the other case. So, we conclude that $N(f, g) = 4$. ■

Now, we will need of some definitions from [DJ].

In our context, the lift f_1 will be called *odd* if $f_1(\alpha \tilde{x}) = \alpha f_1(\tilde{x})$ and will be called *even* if $f_1(\alpha \tilde{x}) = f_1(\tilde{x})$, for all $\tilde{x} \in M$ and a deck transformation $\alpha : M \to M$.

We already know that $(\alpha f_1, \alpha g_1)$ and (f_1, g_1) are in the same lifting class; and the same for $(\alpha f_1, g_1)$ and $(f_1, \alpha g_1)$.

Now:

$\alpha(\alpha f_1, g_1) = (f_1, \alpha g_1)$.

$$(\alpha f_1, g_1)\alpha = (\alpha f_1 \alpha, g_1 \alpha) = \begin{cases} (\alpha f_1, g_1) & \text{if } f_1 \text{ and } g_1 \text{ are both even;} \\ (f_1, \alpha g_1) & \text{if } f_1 \text{ and } g_1 \text{ are both odd;} \\ (\alpha f_1, \alpha g_1) & \text{if } f_1 \text{ is even and } g_1 \text{ is odd;} \\ (f_1, g_1) & \text{if } f_1 \text{ is odd and } g_1 \text{ is even.} \end{cases}$$

$$\alpha(\alpha f_1, g_1)\alpha = (f_1\alpha, \alpha g_1\alpha) = \begin{cases} (f_1, \alpha g_1) & \text{if} f_1 \text{ and } g_1 \text{ are both even;} \\ (\alpha f_1, g_1) & \text{if } f_1 \text{ and } g_1 \text{ are both odd;} \\ (f_1, g_1) & \text{if } f_1 \text{ is even and } g_1 \text{ is odd;} \\ (\alpha f_1, \alpha g_1) & \text{if } f_1 \text{ is odd and } g_1 \text{ is even.} \end{cases}$$

Thus, if f_1 and g_1 are simultaneously even or odd, then we have two lifting classes, $\{(f_1, g_1); (\alpha f_1, \alpha g_1)\}$ and $\{(\alpha f_1, g_1); (f_1, \alpha g_1)\}$. If one of f_1, g_1 is even and the other is odd, then all four pairs of lifts form one lifting class.

Let us denote $C(f_\#, g_\#)_x = \{\beta \in \pi_1(N_\phi, x) : f_\#\beta = g_\#\beta\}$, for $x \in Coin(f, g)$.

Theorem 3.2. *Let N_ϕ be a sapphire space that is not a torus bundle over S^1. Suppose N_ϕ supports Sol-geometry, then for every pair of selfmaps $(f, g) : N_\phi \to N_\phi$, let $(f_1, g_1), (f_1, \alpha g_1) : M \to M$ be the lifts to the torus bundle M which is a two fold cover of N_ϕ. Then*

$$N(f,g) = \begin{cases} \dfrac{N(f_1, g_1) + N(f_1, \alpha g_1)}{2} & \text{if } C(f_\#, g_\#)_{p\tilde{x}} \subseteq p_\#\pi_1(M, \tilde{x}), \\ & \qquad \forall \tilde{x} \in Coin(f_1, g_1); \\ N(f_1, g_1) + N(f_1, \alpha g_1) & \text{if } C(f_\#, g_\#)_{p\tilde{x}} \not\subseteq p_\#\pi_1(M, \tilde{x}). \end{cases}$$

Proof. Suppose that f_1 and g_1 are simultaneously even or odd.

If $C(f_\#, g_\#)_{p\tilde{x}} \subseteq p_\#\pi_1(M, \tilde{x})$ for any $\tilde{x} \in Coin(f_1, g_1)$, then following [[DJ],(2.5)], we obtain that if a Nielsen class $A \subset Coin(f, g)$ satisfies $A \subset p(Coin(f_1, g_1))$, then $p^{-1}A$ is the sum of two Nielsen classes of (f_1, g_1) both of the same index as A. Therefore, these classes are essential if and only if A is essential. The same is true for any class in $p(Coin(f_1, \alpha g_1))$ and since the pair of lifts (f_1, g_1) and $(f_1, \alpha g_1)$ aren't conjugated, the sets $p(Coin(f_1, g_1))$ and $p(Coin(f_1, \alpha g_1))$ are disjoint [[DJ],(2.3)].

Thus, $N(f, g) = \dfrac{N(f_1, g_1) + N(f_1, \alpha g_1)}{2}$.

If $C(f_\#, g_\#)_{p\tilde{x}} \not\subseteq p_\#\pi_1(M, \tilde{x})$, then there exists $w \in C(f_\#, g_\#)_{x_0}$ that lifts to the open path $\tilde{w}$ such that $f_1\tilde{w} \simeq g_1\tilde{w}$. Such w establishes the Nielsen relation between $\tilde{x_0}$ and $\alpha\tilde{x_0}$; since p is a local homeomorphism and α is orientation preserving [GW], the index on $\tilde{x_0}$ and $\alpha\tilde{x_0}$ are equal, so this class has the double of the index of the class A, therefore it is essential if and only if A is essential. Since we have two lifting classes, $N(f, g) = N(f_1, g_1) + N(f_1, \alpha g_1)$.

Now, let us assume that f_1 is even and g_1 is odd.

If $C(f_\#, g_\#)_{p\tilde{x}} \subseteq p_\#\pi_1(M, \tilde{x})$ for any $\tilde{x} \in Coin(f_1, g_1)$, then $p : Coin(f_1, g_1) \to Coin(f, g)$ is a bijection preserving Nielsen relation. So, for each class $A \in Coin(f, g)$, we have that $p^{-1}A$ is the sum of two Nielsen classes, $\tilde{A}_1 \subset Coin(f_1, g_1)$ and $\tilde{A}_2 \subset Coin(f_1, \alpha g_1)$, both of the same index as A, thus $N(f, g) = \dfrac{N(f_1, g_1) + N(f_1, \alpha g_1)}{2}$.

The case $C(f_\#, g_\#)_{p\tilde{x}} \not\subseteq p_\#\pi_1(M, \tilde{x})$ does not happen when f_1 and g_1 don't have the same parity.

Suppose otherwise; then there exists $w \in C(f_\#, g_\#)_{x_0}$ that lifts to the open path $\tilde{w}$. Since f_1 is even and g_1 is odd, fw lifts to a loop and gw lifts to a open path [DJ], which is a contradiction because $fw \simeq gw$ relative to the endpoints. ∎

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