On the curvature ellipse of minimal surfaces in $N^3(c) \times R$

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Abstract

We discuss the curvature ellipse of minimal surfaces in the product space $N^3(c) \times R$, where $N^3(c)$ is the 3-dimensional simply connected space form of constant curvature c.

1 Introduction

Let $N^n(c)$ denote the n-dimensional simply connected space form of constant curvature c. When $c > 0$, $N^n(c)$ is the n-dimensional sphere $S^n(c)$ of constant curvature c. When $c < 0$, $N^n(c)$ is the n-dimensional hyperbolic space $H^n(c)$ of constant curvature c. When $c = 0$, $N^n(c)$ is the n-dimensional Euclidean space R^n.

For a surface M in a Riemannian manifold, the curvature ellipse at $p \in M$ is defined as

$$E(p) = \{h(X, X) | X \in T_pM, |X| = 1\},$$

where h is the second fundamental form of M. The notion of the curvature ellipse plays an important role in the geometry of surfaces in $N^n(c)$ (cf. [9], [14]). In particular, when the curvature ellipse is a circle at any point, the surface is called isotropic or superconformal (cf. [2], [5]). On the other hand, surfaces and submanifolds in the product space $N^n(c) \times R$ have been studied actively (cf. [1], [3], [4], [6], [7], [8], [12], [13]).

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In this paper we discuss the curvature ellipse of minimal surfaces in $N^3(c) \times R$. The results are stated as follows:

Theorem 1.1. *Let M be a minimal surface in $N^3(c) \times R$ where $c \neq 0$. If the curvature ellipse is a circle at any point, then M is totally geodesic.*

Remark. When $c = 0$, for a minimal surface in R^4, the curvature ellipse is a circle at any point if and only if the surface is a complex curve in R^4 with respect to some orthogonal complex structure. It can be seen, for example, by combining [11, Th.A] and [10, Th.5.3 or 5.4].

Theorem 1.2. *There exists no minimal surface in $N^3(c) \times R$ with $c \neq 0$ such that the semi-major axis and the semi-minor axis of the curvature ellipse are both positive constant.*

Remark. There are minimal surfaces in $S^3(c) \times R$ such that the semi-major axis of the curvature ellipse is positive constant and the semi-minor axis of the curvature ellipse is zero, which are minimal constant angle surfaces (cf. [3]).

2 Preliminaries

Let $N^3(c)$ be the 3-dimensional simply connected space form of constant curvature c. The curvature tensor $\bar{R}$ of $N^3(c) \times R$ satisfies

$$\begin{aligned}&\langle \bar{R}(X,Y)Z, W\rangle = \\ &\qquad c\{\langle d\pi(Y), d\pi(Z)\rangle\langle d\pi(X), d\pi(W)\rangle - \langle d\pi(X), d\pi(Z)\rangle\langle d\pi(Y), d\pi(W)\rangle\},\end{aligned}$$

where $\pi : N^3(c) \times R \to N^3(c)$ is the projection map. Let ξ denote the unit vector along R. Then we can see that

$$\begin{aligned}\bar{R}(X,Y)Z = c\{&\langle Y,Z\rangle X - \langle X,Z\rangle Y - \langle Y,\xi\rangle\langle Z,\xi\rangle X + \langle X,\xi\rangle\langle Z,\xi\rangle Y \\ &+ \langle X,Z\rangle\langle Y,\xi\rangle\xi - \langle Y,Z\rangle\langle X,\xi\rangle\xi\}. \end{aligned} \tag{2.1}$$

We recall the method of moving frames for surfaces in $N^3(c) \times R$. Unless otherwise stated, we use the following convention on the ranges of indices:

$$1 \leq A, B, \cdots \leq 4, \quad 1 \leq i, j, \cdots \leq 2, \quad 3 \leq \alpha, \beta, \cdots \leq 4.$$

Let $\{e_A\}$ be a local orthonormal frame field in $N^3(c) \times R$, and $\{\omega^A\}$ the dual coframe field. The connection forms satisfy

$$de_B = \sum_A \omega_B^A e_A. \tag{2.2}$$

Then $\omega_B^A + \omega_A^B = 0$. The structure equations are given by

$$d\omega^A = -\sum_B \omega_B^A \wedge \omega^B, \tag{2.3}$$

$$d\omega_B^A = -\sum_C \omega_C^A \wedge \omega_B^C + \frac{1}{2}\sum_{C,D} \bar{R}_{ABCD}\omega^C \wedge \omega^D, \tag{2.4}$$

where

$$\bar{R}_{ABCD} = \langle \bar{R}(e_C, e_D)e_B, e_A \rangle. \tag{2.5}$$

Let M be a surface in $N^3(c) \times R$. We choose the frame $\{e_A\}$ so that $\{e_i\}$ are tangent to M. Then $\omega^\alpha = 0$ along M. In the following our argument will be restricted to M. By (2.3),

$$0 = -\sum_i \omega_i^\alpha \wedge \omega^i.$$

So there is a symmetric tensor $\{h_{ij}^\alpha\}$ such that

$$\omega_i^\alpha = \sum_j h_{ij}^\alpha \omega^j, \tag{2.6}$$

where h_{ij}^α are the components of the second fundamental form of M.

We decompose the unit vector ξ along R as

$$\xi = T + \eta,$$

where T is tangent to M and η is normal to M. We say that M is a constant angle surface (or a helix surface), if the tangent planes of M make a constant angle with ξ, which is equivalent to that the length $|T|$ of T is constant (cf. [3], [7], [12]). If $T = 0$, then M is a surface in a slice $N^3(c) \times \{*\}$. If $\eta = 0$, then M is a part of a cylinder, that is, a product of a regular curve in $N^3(c)$ and the factor R.

The Gaussian curvature K and the normal curvature K_ν are given by

$$d\omega_2^1 = K\omega^1 \wedge \omega^2, \quad d\omega_4^3 = K_\nu \omega^1 \wedge \omega^2. \tag{2.7}$$

By (2.1), (2.4), (2.5), (2.6) and (2.7), we get

$$K = c(1 - |T|^2) + h_{11}^3 h_{22}^3 - (h_{12}^3)^2 + h_{11}^4 h_{22}^4 - (h_{12}^4)^2, \tag{2.8}$$

and

$$K_\nu = h_{11}^3 h_{12}^4 - h_{12}^3 h_{11}^4 + h_{12}^3 h_{22}^4 - h_{22}^3 h_{12}^4. \tag{2.9}$$

The mean curvature vector of M is defined by

$$H = \frac{1}{2}\sum_\alpha (h_{11}^\alpha + h_{22}^\alpha) e_\alpha.$$

We say that M is minimal if $H = 0$ on M.

We assume that M is minimal. Then by (2.8) and (2.9),

$$K = c(1 - |T|^2) - (h_{11}^3)^2 - (h_{12}^3)^2 - (h_{11}^4)^2 - (h_{12}^4)^2, \tag{2.10}$$

and

$$K_\nu = 2(h_{11}^3 h_{12}^4 - h_{12}^3 h_{11}^4). \tag{2.11}$$

3 On the structure equations

Let M be a minimal surface in $N^3(c) \times R$. We use the notations in Section 2. Suppose that either the curvature ellipse is not a circle at any point, or the curvature ellipse is a circle of positive radius at any point. Then we can choose the frame field $\{e_A\}$ so that

$$(h^3_{ij}) = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}, \quad (h^4_{ij}) = \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix},$$

for some functions a and b with $|a| \geq |b|$. The semi-major axis of the curvature ellipse is $|a|$, and the semi-minor axis of the curvature ellipse is $|b|$. So, we have $|a| > |b|$ when the curvature ellipse is not a circle, and $|a| = |b| > 0$ when the curvature ellipse is a circle of positive radius. Then by (2.6), (2.10) and (2.11),

$$\omega^3_1 = a\omega^1, \quad \omega^3_2 = -a\omega^2, \quad \omega^4_1 = b\omega^2, \quad \omega^4_2 = b\omega^1, \tag{3.1}$$

$$K = c(1 - |T|^2) - a^2 - b^2, \quad K_v = 2ab. \tag{3.2}$$

Using (2.3), (2.4) and (3.1), we have

$$\begin{aligned} d\omega^3_1 &= da \wedge \omega^1 - a\omega^1_2 \wedge \omega^2 \\ &= -\omega^3_2 \wedge \omega^2_1 - \omega^3_4 \wedge \omega^4_1 + \bar{R}_{3112}\omega^1 \wedge \omega^2 \\ &= a\omega^2 \wedge \omega^2_1 - b\omega^3_4 \wedge \omega^2 + \bar{R}_{3112}\omega^1 \wedge \omega^2. \end{aligned}$$

We can write

$$T = T^1 e_1 + T^2 e_2, \quad \eta = \eta^3 e_3 + \eta^4 e_4.$$

Then by (2.5) and (2.1), $\bar{R}_{3112} = cT^2\eta^3$. So, using the notation like

$$da = a_1\omega^1 + a_2\omega^2, \quad db = b_1\omega^1 + b_2\omega^2,$$

$$\omega^1_2 = (\omega^1_2)_1\omega^1 + (\omega^1_2)_2\omega^2, \quad \omega^3_4 = (\omega^3_4)_1\omega^1 + (\omega^3_4)_2\omega^2,$$

we get

$$-a_2 - 2a(\omega^1_2)_1 + b(\omega^3_4)_1 = cT^2\eta^3. \tag{3.3}$$

Similarly, from the exterior derivatives of ω^3_2, ω^4_1 and ω^4_2,

$$a_1 - 2a(\omega^1_2)_2 + b(\omega^3_4)_2 = cT^1\eta^3, \tag{3.4}$$

$$b_1 - 2b(\omega^1_2)_2 + a(\omega^3_4)_2 = cT^2\eta^4, \tag{3.5}$$

$$-b_2 - 2b(\omega^1_2)_1 + a(\omega^3_4)_1 = -cT^1\eta^4. \tag{3.6}$$

Let $\bar{\nabla}$ denote the Levi-Civita connection on $N^3(c) \times R$. Then, using (2.2), we have

$$\begin{aligned} 0 = \bar{\nabla}\xi &= \bar{\nabla}\left(\sum_i T^i e_i + \sum_\alpha \eta^\alpha e_\alpha\right) \\ &= \sum_i \left(dT^i + \sum_j T^j\omega^i_j - \sum_\alpha \eta^\alpha\omega^\alpha_i\right) e_i \end{aligned}$$

$$+\sum_{\alpha}\left(d\eta^{\alpha}+\sum_{i}T^{i}\omega_{i}^{\alpha}+\sum_{\beta}\eta^{\beta}\omega_{\beta}^{\alpha}\right)e_{\alpha}.$$

So, using the notation like

$$dT^{i}=\sum_{j}T_{j}^{i}\omega^{j},\quad d\eta^{\alpha}=\sum_{j}\eta_{j}^{\alpha}\omega^{j},$$

and (3.1), we have

$$T_1^1 = -T^2(\omega_2^1)_1 + a\eta^3,\quad T_2^1 = -T^2(\omega_2^1)_2 + b\eta^4, \tag{3.7}$$

$$T_1^2 = T^1(\omega_2^1)_1 + b\eta^4,\quad T_2^2 = T^1(\omega_2^1)_2 - a\eta^3, \tag{3.8}$$

$$\eta_1^3 = -\eta^4(\omega_4^3)_1 - aT^1,\quad \eta_2^3 = -\eta^4(\omega_4^3)_2 + aT^2, \tag{3.9}$$

$$\eta_1^4 = \eta^3(\omega_4^3)_1 - bT^2,\quad \eta_2^4 = \eta^3(\omega_4^3)_2 - bT^1. \tag{3.10}$$

4 Proof of Theorems

In this section, using the notations in Sections 2 and 3, we prove Theorems 1.1 and 1.2.

Proof of Theorem 1.1. Let M be a minimal surface in $N^3(c) \times R$ where $c \neq 0$, and assume that the curvature ellipse at any point is a circle.

Suppose that M is not totally geodesic. Then there exists a point $p \in M$ such that the curvature ellipse at p is a circle of positive radius. So the curvature ellipse is a circle of positive radius on a neighborhood U of p. We can use the argument in Section 3 on U, and $|a| = |b| > 0$ on U. By the continuity, either $b = a \neq 0$ on U, or $b = -a \neq 0$ on U. In the case where $b = a \neq 0$ on U, by (3.3)-(3.6) and that $c \neq 0$, we have

$$T^1\eta^3 - T^2\eta^4 = 0,\quad T^2\eta^3 + T^1\eta^4 = 0$$

on U. Hence, noting also that $|T|^2 + |\eta|^2 = 1$, we have either $T = 0$ on U or $\eta = 0$ on U. In either case, since $|T|^2 + |\eta|^2 = 1$ and $a \neq 0$ on U, we have a contradiction from (3.7)-(3.10). In the case where $b = -a \neq 0$ on U, similarly, we have a contradiction.

Therefore, M has to be totally geodesic.

Proof of Theorem 1.2. Let M be a minimal surface in $N^3(c) \times R$ where $c \neq 0$. Suppose that the semi-major axis and the semi-minor axis of the curvature ellipse are both positive constant. By Theorem 1.1, the curvature ellipse cannot be a circle of positive constant radius. So the curvature ellipse is not a circle at any point, and we can use the argument in Section 3, so that a and b are constant with $|a| > |b| > 0$. Then (3.3)-(3.6) become

$$-2a(\omega_2^1)_1 + b(\omega_4^3)_1 = cT^2\eta^3, \tag{4.1}$$

$$-2a(\omega_2^1)_2 + b(\omega_4^3)_2 = cT^1\eta^3, \tag{4.2}$$

$$-2b(\omega_2^1)_2 + a(\omega_4^3)_2 = cT^2\eta^4, \tag{4.3}$$

$$-2b(\omega_2^1)_1 + a(\omega_4^3)_1 = -cT^1\eta^4. \tag{4.4}$$

By (4.1)-(4.4) we obtain

$$-2a\omega_2^1 + b\omega_4^3 = c\eta^3(T^2\omega^1 + T^1\omega^2), \tag{4.5}$$

$$-2b\omega_2^1 + a\omega_4^3 = c\eta^4(-T^1\omega^1 + T^2\omega^2), \tag{4.6}$$

and

$$2(\omega_2^1)_1 = -\frac{c}{a^2 - b^2}(aT^2\eta^3 + bT^1\eta^4), \tag{4.7}$$

$$2(\omega_2^1)_2 = \frac{c}{a^2 - b^2}(bT^2\eta^4 - aT^1\eta^3), \tag{4.8}$$

$$(\omega_4^3)_1 = -\frac{c}{a^2 - b^2}(bT^2\eta^3 + aT^1\eta^4), \tag{4.9}$$

$$(\omega_4^3)_2 = \frac{c}{a^2 - b^2}(aT^2\eta^4 - bT^1\eta^3). \tag{4.10}$$

Taking the exterior derivative of (4.5), and using (2.3), (2.7), (3.7)-(3.10), (4.7)-(4.10), we can get

$$-2aK + bK_\nu = ca\left(\frac{c}{a^2 - b^2}|T|^2|\eta|^2 + 2(\eta^3)^2 - |T|^2\right). \tag{4.11}$$

By (4.11), (3.2) and that $a \neq 0$,

$$-2c(1 - |T|^2) + 2a^2 + 4b^2 = \frac{c^2}{a^2 - b^2}|T|^2|\eta|^2 + c\{2(\eta^3)^2 - |T|^2\}. \tag{4.12}$$

Similarly, by the exterior derivative of (4.6), we get

$$-2bK + aK_\nu = cb\left(2(\eta^4)^2 - |T|^2 - \frac{c}{a^2 - b^2}|T|^2|\eta|^2\right). \tag{4.13}$$

By (4.13), (3.2) and noting that $b \neq 0$, we have

$$-2c(1 - |T|^2) + 4a^2 + 2b^2 = c\{2(\eta^4)^2 - |T|^2\} - \frac{c^2}{a^2 - b^2}|T|^2|\eta|^2. \tag{4.14}$$

By (4.12)+(4.14),

$$8c|T|^2 = 6c - 6(a^2 + b^2).$$

So $|T|$ is constant, and M is a constant angle surface. Since $|a| > |b| > 0$, $0 < |T| < 1$.

When $c < 0$, by [12, Th. 3.2], M is totally geodesic, which is a contradiction to $|a| > |b| > 0$. When $c > 0$, by [3, Lemma 2], we have $b = 0$, which is also a contradiction to $|a| > |b| > 0$. Thus we have proved Theorem 1.2.

Remark. By [3, Sect.4], for any minimal constant angle surface in $S^3(c) \times R$ with $0 < |T| < 1$, $|a|$ is positive constant and b is zero. It is not certain if the converse is true. So we can consider the following:

Question. Except for minimal constant angle surfaces, does there exist a minimal surface in $N^3(c) \times R$ with $c \neq 0$ such that the semi-major axis of the curvature ellipse is positive constant and the semi-minor axis of the curvature ellipse is zero?

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