Composition operators related to the Dirichlet space

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Dedicated to the memory of Junzo Wada

Abstract

The Hilbert-Schmidtness of composition operators acting between the classical Hilbert Hardy space and the Dirichlet space is known. We here consider boundedness and compactness of composition operators acting between their spaces.

1 Introduction

Throughout this paper, let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$. We denote by $\mathcal{S}(\mathbb{D})$ the set of analytic self-maps of $\mathbb{D}$. Each $\varphi \in \mathcal{S}(\mathbb{D})$ induces the composition operator C_φ defined by $C_\varphi f = f \circ \varphi$ for analytic function f on $\mathbb{D}$. Properties of composition operators have been actively investigated during these decades. In [9], Shapiro and Taylor considered the Hilbert-Schmidtness of composition operators on the Hilbert Hardy space and moreover characterized results related to the Dirichlet space. The classical Hilbert Hardy space H^2 is the space of analytic functions f on $\mathbb{D}$ such that

$$\|f\|_{H^2}^2 = \sup_{0\le r<1} \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = \frac{1}{2\pi}\int_0^{2\pi} |f^*(e^{i\theta})|^2 d\theta < \infty,$$

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where $f^*(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta})$ a.e. on the boundary $\partial\mathbb{D}$ of $\mathbb{D}$. Let $\mathcal{D}$ denote the Dirichlet space of analytic functions f on $\mathbb{D}$ for which

$$\int_{\mathbb{D}} |f'(z)|^2 \, dA(z) < \infty,$$

where dA is the normalized area measure on $\mathbb{D}$. The norm is defined by

$$\|f\|_{\mathcal{D}}^2 = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 dA(z).$$

A linear operator T from a Hilbert space X to another Hilbert space Y is called a Hilbert-Schmidt operator if there exists an orthonormal basis $\{e_n\}$ in X such that

$$\sum_n \|Te_n\|_Y < \infty.$$

The following results are presented in [9].

Theorem A. (i) C_φ is a Hilbert-Schmidt operator from $\mathcal{D}$ to H^2 if and only if

$$\int_0^{2\pi} \log(1 - |\varphi^*(e^{i\theta})|)d\theta > -\infty.$$

(ii) C_φ is a Hilbert-Schmidt operator from H^2 to $\mathcal{D}$ if and only if

$$\int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^3} dA(z) < \infty.$$

It is known by de Leeuw and Rudin [3] that φ is not an extreme point of the unit ball of the space of bounded analytic functions on $\mathbb{D}$ if and only if φ satisfies the condition in (i) above.

During the past decades, composition operators on $\mathcal{D}$ have been investigated in [4, 5, 7, 10]. But there is no information on boundedness or compactness of composition operators acting between H^2 and $\mathcal{D}$ in literature. So we will consider them. In the next section we will see that $C_\varphi : \mathcal{D} \to H^2$ is always compact. In section 3, we characterize the boundedness and compactness of composition operators C_φ acting from H^2 to $\mathcal{D}$. Furthermore we will present examples concerning boundedness and compactness.

Throughout the paper, C will stand for positive constants whose values may change from one occurrence to another.

2 $C_\varphi : \mathcal{D} \to H^2$

As $\mathcal{D} \subset H^2$ and C_φ is bounded on H^2, it is trivial that C_φ is bounded from $\mathcal{D}$ to H^2.

In the proof of characterization of compactness we usually need the so-called "weak convergence theorem" by adapting the proof of [2, Proposition 3.11].

Lemma 2.1. *Let X, Y be H^2 or $\mathcal{D}$. For $\varphi \in \mathcal{S}(\mathbb{D})$, suppose that $C_\varphi : X \to Y$ is bounded. Then C_φ is a compact operator from X to Y if and only if $\|C_\varphi f_n\|_Y \to 0$ for any bounded sequence $\{f_n\}$ in X such that f_n converges to 0 uniformly on every compact subset of $\mathbb{D}$.*

Theorem 2.2. *For $\varphi \in \mathcal{S}(\mathbb{D})$, C_φ is always a compact operator from $\mathcal{D}$ to H^2.*

Proof. If $\varphi(0) \neq 0$, put $\lambda = \varphi(0)$ and $\alpha_\lambda(z) = (\lambda - z)/(1 - \overline{\lambda}z)$. Let $\psi = \alpha_\lambda \circ \varphi$. Then $\psi \in \mathcal{S}(\mathbb{D})$ and $\psi(0) = 0$. We will show that C_ψ is compact.

By the change-of-variable formula, for $f \in \mathcal{D}$ we have

$$\|C_\psi f\|_{H^2}^2 = |f(\psi(0))|^2 + 2\int_{\mathbb{D}} |f'(w)|^2 N_\psi(w)\,dA(w)$$

where N_ψ is the Nevanlinna counting function of ψ (see [2, Theorem 2.31] and [8, p. 179], for instance). As $\psi(0) = 0$, it holds that

$$N_\psi(w) \le \log \frac{1}{|w|} \quad \text{for} \quad w \in \mathbb{D}$$

([8, p. 188, Corollary]).

So, for any $\varepsilon > 0$, there is a constant $R, 0 < R < 1$, such that

$$0 < \log \frac{1}{|w|} < \varepsilon \quad \text{whenever} \quad R < |w| < 1.$$

Let $\{f_n\}$ in $\mathcal{D}$ such that $\|f_n\|_{\mathcal{D}} \le 1$ and f_n converges to 0 uniformly on every compact subset of $\mathbb{D}$. Then

$$\begin{aligned}
\|C_\psi f_n\|_{H^2}^2 &= |f_n(0)|^2 + 2\int_{\mathbb{D}} |f_n'(w)|^2 N_\psi(w)\,dA(w) \\
&= |f_n(0)|^2 + 2\Big(\int_{\{|w|\le R\}} |f_n'(w)|^2 N_\psi(w)\,dA(w) \\
&\qquad + \int_{\{|w|>R\}} |f_n'(w)|^2 N_\psi(w)\,dA(w)\Big) \\
&\le |f_n(0)|^2 + 2\Big(\sup_{\{|w|\le R\}} |f_n'(w)|^2 \int_{\mathbb{D}} N_\psi(w)\,dA(w) \\
&\qquad + \varepsilon \int_{\mathbb{D}} |f_n'(w)|^2\,dA(w)\Big).
\end{aligned}$$

Thus

$$\lim_{n\to\infty} \|C_\psi f_n\|_{H^2}^2 \le \varepsilon.$$

As ε is arbitrary, $\lim\limits_{n\to\infty} \|C_\psi f_n\|_{H^2}^2 = 0$. By Lemma 2.1, C_ψ is compact from $\mathcal{D}$ to H^2. ∎

Here we recall that C_φ is a Hilbert-Schmidt operator from $\mathcal{D}$ to H^2 if and only if

$$\int_0^{2\pi} \log(1 - |\varphi^*(e^{i\theta})|)\,d\theta > -\infty.$$

So each inner function induces a bounded and compact composition operator acting from $\mathcal{D}$ to H^2, but does not satisfy the Hilbert-Schmidt condition. Let $\varphi(z) = (z+1)/2$. This φ satisfies the Hilbert-Schmidt condition.

3 $C_\varphi : H^2 \to \mathcal{D}$

Let $n_\varphi(w)$ be the cardinality of $\varphi^{-1}(w)$. Then, for $f \in H^2$ we have

$$\begin{aligned}\|C_\varphi f\|_{\mathcal{D}}^2 &= |f(\varphi(0))|^2 + \int_{\mathbb{D}} |(f\circ\varphi)'(z)|^2 dA(z)\\ &= |f(\varphi(0))|^2 + \int_{\mathbb{D}} |f'(\varphi(z))|^2 |\varphi'(z)|^2 dA(z)\\ &= |f(\varphi(0))|^2 + \int_{\mathbb{D}} |f'(w)|^2 n_\varphi(w) dA(w).\end{aligned}$$

Let $d\mu = n_\varphi dA$. Then C_φ is bounded from H^2 to $\mathcal{D}$ if and only if it holds that

$$\int_{\mathbb{D}} |f'(w)|^2 d\mu(w) \le C\|f\|_{H^2}^2$$

for some constant $C > 0$. Such inequalities were characterized by Luecking [6].

For any $\zeta = e^{i\theta} \in \partial\mathbb{D}$ and $h > 0$, let

$$S(\theta, h) = \{z = re^{it} \in \mathbb{D} : 1 - h \le r < 1, |t - \theta| < h\}.$$

Then $S(\theta, h)$ is called a Carleson square at $\zeta \in \partial\mathbb{D}$. It is clear that the area of $S(\theta, h)$ is comparable to h^2 (uniformly in ζ) as $h \to 0$.

For any $\lambda \in \mathbb{D}$, let $\alpha_\lambda(z) = (\lambda - z)/(1 - \overline{\lambda}z)$. Then we have the following.

Theorem 3.1. *Let $d\mu = n_\varphi dA$. Then the following are equivalent.*

(i) *C_φ is bounded from H^2 to $\mathcal{D}$.*

(ii) *There exists a constant $C > 0$ such that*

$$\mu(S(\theta, h)) \le Ch^3$$

for $0 < h < 1$ and $0 \le \theta < 2\pi$.

(iii) *There exists a constant $C > 0$ such that*

$$\int_{\mathbb{D}} |\alpha_\lambda'(z)|^3 d\mu(z) \le C$$

for all $\lambda \in \mathbb{D}$.

(iv)

$$\sup_{\lambda\in\mathbb{D}} \int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^3} \left(1 - |\alpha_\lambda(\varphi(z))|^2\right)^3 dA(z) < \infty.$$

Proof. The equivalence between conditions (i) and (ii) is due to [6, Theorem 3.1] and the equivalence between conditions (ii) and (iii) is due to [1, Theorem 1.3] (Also see [10]).

Moreover, we have

$$
\begin{aligned}
&\int_{\mathbb{D}} |\alpha_\lambda'(z)|^3 d\mu(z) \\
&= \int_{\mathbb{D}} \Big(\frac{1-|\lambda|^2}{|1-\overline{\lambda}\varphi(z)|^2}\Big)^3 |\varphi'(z)|^2 dA(z) \\
&= \int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^3} \Big(\frac{(1-|\lambda|^2)(1-|\varphi(z)|^2)}{|1-\overline{\lambda}\varphi(z)|^2}\Big)^3 dA(z) \\
&= \int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^3} \Big(1-|\alpha_\lambda(\varphi(z))|^2\Big)^3 dA(z).
\end{aligned} \tag{3.1}
$$

So we obtain the equivalence between (iii) and (iv). ■

Example 3.2. (1) Let Ω be a simply connected region in $\mathbb{D}$ touching $\partial\mathbb{D}$ only at 1 and suppose that near 1 the boundary of Ω is a piece of the curve $(x-1)^4 - y^2 = 0 \ (z = x+iy)$.

Let φ be a univalent map of $\mathbb{D}$ onto Ω. Then

$$
\begin{aligned}
\int_{S(1,h)} n_\varphi(z) dA(z) &= |\varphi(\mathbb{D})) \cap \mathcal{S}(1,h)| \\
&\simeq \int_{1-h}^{1} (x-1)^2 dx = \frac{h^3}{3},
\end{aligned}
$$

where $|E|$ is the area of a subset E. So C_φ is bounded from H^2 to $\mathcal{D}$.

(2) Note that if C_φ is bounded from H^2 to $\mathcal{D}$, C_φ is bounded from $\mathcal{D}$ to $\mathcal{D}$. Let $\varphi(z) = (z+1)/2$.

$$
\begin{aligned}
\int_{S(1,h)} n_\varphi(z) dA(z) &\simeq 2\int_0^h \int_{1-h}^{\cos\theta} r \, dr d\theta \\
&= \frac{h}{2} + \frac{\sin 2h}{4} - (1-h)^2 h \\
&\fallingdotseq \frac{h}{2} + \frac{2h}{4} - (1-h)^2 h \quad \text{(whenever } h \text{ is so small)} \\
&= h^2(2-h).
\end{aligned}
$$

So C_φ is bounded on $\mathcal{D}$ but C_φ is not bounded from H^2 to $\mathcal{D}$.

Next we consider the compactness.

Theorem 3.3. *Let $d\mu = n_\varphi dA$. Then the following are equivalent.*

(i) C_φ *is compact from* H^2 *to* $\mathcal{D}$.

(ii) $\displaystyle\lim_{h\to 0} \sup_{\theta\in[0,2\pi)} \frac{\mu(S(\theta,h))}{h^3} = 0.$

(iii) $\lim_{|\lambda|\to 1}\int_{\mathbb{D}} |\alpha'_\lambda(z)|^3 d\mu(z) = 0.$

(iv) $\lim_{|\lambda|\to 1}\int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^3}\Big(1-|\alpha_\lambda(\varphi(z))|^2\Big)^3 dA(z) = 0.$

Proof. First we show the implication (i)⇒(iv). Suppose that C_φ is compact from H^2 to $\mathcal{D}$. For $\lambda \in \mathbb{D}$, let $k_\lambda(z) = \sqrt{1-|\lambda|^2}/(1-\overline{\lambda}z)$. Then $k_\lambda \in H^2$, $\|k_\lambda\|_{H^2} = 1$ and k_λ converges to 0 weakly in H^2 as $|\lambda| \to 1$. So $\|C_\varphi k_\lambda\|_{\mathcal{D}} \to 0$ as $|\lambda| \to 1$.

$$
\begin{aligned}
&\|C_\varphi k_\lambda\|_{\mathcal{D}}^2 \\
&\geq \int_{\mathbb{D}} \frac{(1-|\lambda|^2)|\lambda|^2|\varphi'(z)|^2}{|1-\overline{\lambda}\varphi(z)|^4} dA(z) \\
&= \int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^3} \frac{(1-|\lambda|^2)^3|\lambda|^2(1-|\varphi(z)|^2)^3}{|1-\overline{\lambda}\varphi(z)|^6} \frac{|1-\overline{\lambda}\varphi(z)|^2}{(1-|\lambda|^2)^2} dA(z) \\
&\geq \int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^3} \Big(\frac{(1-|\lambda|^2)(1-|\varphi(z)|^2)}{|1-\overline{\lambda}\varphi(z)|^2}\Big)^3 |\lambda|^2 \frac{(1-|\lambda|)^2}{(1-|\lambda|^2)^2} dA(z) \\
&\geq \frac{1}{4}\int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^3} \Big(\frac{(1-|\lambda|^2)(1-|\varphi(z)|^2)}{|1-\overline{\lambda}\varphi(z)|^2}\Big)^3 |\lambda|^2 dA(z).
\end{aligned}
$$

So we obtain condition (iv), that is,

$$\lim_{|\lambda|\to 1}\int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^3}\Big(1-|\alpha_\lambda(\varphi(z))|^2\Big)^3 dA(z) = 0.$$

The implication (iv)⇒(iii) could be checked by the equalities (3.1) and the equivalence between (iii) and (ii) is due to [10, Theorem 3.4].

Finally we see the implication (ii)⇒(i). Let $\{f_n\}$ be a bounded sequence in H^2 that converges to 0 uniformly on compact sets. To show the compactness of C_φ, it is sufficient to see that $\|C_\varphi f_n\|_{\mathcal{D}} \to 0$ as $n \to \infty$ by Lemma 2.1.

For $w \in \mathbb{D}$ and $0 < r < 1$, let $\Delta(w,r) = \{z \in \mathbb{D} : |z-w| < r\}$. As the absolute values of analytic functions are subharmonic,

$$
\begin{aligned}
|f_n'(w)|^2 &\leq \frac{C}{|\Delta(w,\frac{1-|w|}{2})|}\int_{\Delta(w,\frac{1-|w|}{2})} |f_n'(z)|^2 dA(z) \\
&\leq \frac{C}{(1-|w|)^2}\int_{\Delta(w,\frac{1-|w|}{2})} |f_n'(z)|^2 dA(z).
\end{aligned}
$$

So

$$
\begin{aligned}
&\int_{\mathbb{D}} |f_n'(\varphi(z))|^2|\varphi'(z)|^2 dA(z) \\
&= \int_{\mathbb{D}} |f_n'(w)|^2 d\mu(w) \\
&\leq \int_{\mathbb{D}} \frac{C}{(1-|w|)^2}\Big(\int_{\Delta(w,\frac{1-|w|}{2})} |f_n'(z)|^2 dA(z)\Big) d\mu(w) \\
&= C\int_{\mathbb{D}} |f_n'(z)|^2 \Big(\int_{\mathbb{D}} \frac{\chi_{\Delta(w,\frac{1-|w|}{2})}(z)}{(1-|w|)^2} d\mu(w)\Big) dA(z).
\end{aligned}
$$

Here, if $|w-z| < \dfrac{1-|w|}{2}$, then $\dfrac{1-|w|}{2} < 1-|z|$ and so

$$\begin{aligned}|w-e^{i\theta}| &< |w-z| + \Big|z - \frac{z}{|z|}\Big| \\ &< \frac{1-|w|}{2} + |z|\frac{1-|z|}{|z|} \\ &< 2(1-|z|),\end{aligned}$$

where $z = |z|e^{i\theta}$. Thus $w \in S(\theta, s(1-|z|))$ for some $s > 0$ and also if $|w-z| < \dfrac{1-|w|}{2}$, then $\dfrac{1}{1-|w|} < \dfrac{3}{2}\dfrac{1}{1-|z|}$. Therefore

$$\begin{aligned}\|C_\varphi f_n\|_{\mathcal{D}}^2 &\le C\int_{\mathbb{D}} \frac{|f_n'(z)|^2}{(1-|z|)^2}\Big(\int_{S(\theta,s(1-|z|))} d\mu(w)\Big)dA(z) \\ &= C\Big(\int_{|z|\le 1-\delta} + \int_{|z|>1-\delta}\Big)\frac{|f_n'(z)|^2}{(1-|z|)^2}\mu(S(\theta,s(1-|z|)))dA(z)\end{aligned}$$

for $0<\delta<1$. By condition (ii), For any $\varepsilon > 0$,

$$\int_{S(\theta,h)} d\mu(w) = \mu(S(\theta,h)) < \varepsilon h^3$$

for h close enough to 0. So, for $0<\delta<h/s$,

$$\|C_\varphi f_n\|_{\mathcal{D}}^2 \le C\Big(\frac{1}{\delta^2}\sup_{|z|\le 1-\delta}|f_n'(z)|^2 + \varepsilon\|f_n\|_{H^2}^2\Big).$$

Consequently

$$\lim_{n\to\infty}\|C_\varphi f_n\|_{\mathcal{D}}^2 \le C\varepsilon.$$

As ε is arbitrary, $\lim_{n\to\infty}\|C_\varphi f_n\|_{\mathcal{D}}^2 = 0$. ∎

Example 3.4. Let Ω be a simply connected region in $\mathbb{D}$ touching $\partial\mathbb{D}$ only at 1 and suppose that near 1 the boundary of Ω is a piece of the curve $(x-1)^6 - y^2 = 0$ $(z = x+iy)$.

Let φ be a univalent map of $\mathbb{D}$ onto Ω. Then

$$\begin{aligned}\int_{S(1,h)} n_\varphi(z)dA(z) &= |\varphi(\mathbb{D}))\cap \mathcal{S}(1,h)| \\ &\simeq -2\int_{1-h}^{1}(x-1)^3dx = \frac{h^4}{2}.\end{aligned}$$

So C_φ is compact from H^2 to $\mathcal{D}$.

Finally we make a comparison amongst the known results on the Hilbert-Schmidtness of composition operators related to our case (refer to [4, 5, 9, 10]).

Theorem 3.5. *For $\varphi \in \mathcal{S}(\mathbb{D})$, the following hold.*

(i) C_φ *is Hilbert-Schmidt on* H^2 *if and only if*

$$\int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^3}(1-|z|^2)\, dA(z) < \infty.$$

(ii) C_φ *is Hilbert-Schmidt on* $\mathcal{D}$ *if and only if*

$$\int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^2}\, dA(z) < \infty.$$

(iii) C_φ *is Hilbert-Schmidt from* H^2 *to* $\mathcal{D}$ *if and only if*

$$\int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^3}\, dA(z) < \infty.$$

Thus, if C_φ is Hilbert-Schmidt from H^2 to $\mathcal{D}$, then C_φ is Hilbert-Schmidt on $\mathcal{D}$ and so on H^2.

Let $\varphi(z) = (z+1)/2$. It is known that C_φ is neither Hilbert-Schmidt on H^2 nor on $\mathcal{D}$ ([4]). So C_φ is not Hilbert-Schmidt from H^2 to $\mathcal{D}$.

A function φ in Example 3.4 induces a Hilbert-Schmidt operator C_φ from H^2 to $\mathcal{D}$.

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