Some restrictions on the Betti numbers of a nilpotent Lie algebra

Peyman Niroomand Francesco G. Russo

Abstract

We have recently shown that a nilpotent Lie algebra L of dimension $n \geq 1$ satisfies the inequality $\dim H_2(L,\mathbb{Z}) \leq \frac{1}{2}(n+m-2)(n-m-1)+1$, where $\dim L^2 = m \geq 1$ and $H_2(L,\mathbb{Z})$ is the 2-nd integral homology Lie algebra of L. Our first main result correlates this bound with the i-th Betti number $\dim H^i(L,\mathbb{C}^\times)$ of L, where $H^i(L,\mathbb{C}^\times)$ denotes the i-th complex cohomology Lie algebra of L. Our second main result describes a more general restriction, which follows an idea of Ellis in [G. Ellis, The Schur multiplier of a pair of groups, Appl. Categ. Structures 6 (1998), 355–371].

1 Statement of the results

Given a nilpotent Lie algebra L of dimension $\dim L = n$, it is well–known that the second homology Lie algebra $H_2(L,\mathbb{Z})$ of L with coefficients in $\mathbb{Z}$ is again a finite dimensional Lie algebra. Usually, $H_2(L,\mathbb{Z})$ is called *Schur multiplier* of L (and denoted by $M(L)$), following the terminology of Schur (see [4, 5, 6, 7, 18, 19, 20, 21, 22, 25]). Now $\dim H_2(L,\mathbb{Z})$ is upper bounded by a function depending only on n, due to Batten and others [4, 5]:

$$\dim M(L) \leq \frac{1}{2}n(n-1), \tag{1.1}$$

where the bound is exact if and only if L is abelian. On the other hand, the notion of Schur multiplier may be reformulated from the perspective of the cohomology,

Received by the editors in September 2013.
Communicated by Y. Félix.
2010 *Mathematics Subject Classification* : 17B30; 17B60; 17B99.
Key words and phrases : Heisenberg algebras, cohomology, nilpotent Lie algebras, Betti numbers, Schur multiplier.

since there exists the isomorphism of Lie algebras

$$H^2(L,\mathbb{C}^\times) \simeq H_2(L,\mathbb{Z}) \tag{1.2}$$

which follows from the *Poincaré duality* [16, Theorem 6.10] (note that this happens for groups in [22, Theorem 10.31]). Now (1.2) is very useful not only for computational aspects, but also for studying the numerical restrictions on dim $M(L)$ in a more appropriate context of the literature. In fact some authors [8, 10, 15, 16] call dim $H^i(L,\mathbb{C}^\times)$ the *i-th Betti number of* L, and, by means of (1.2), if we have bounds on the second Betti number of a nilpotent Lie algebra, then we have bounds on its Schur multiplier, and viceversa. This means that most of the results in [1, 2, 3, 18, 19, 20, 21, 25] are connected with [8, 10, 24], and viceversa, but we will give more details in the rest of the paper. For the moment, we note that [8, 10, 15, 16] mention that the behaviour of the Betti numbers is still unknown and several problems remain unsolved. However the following three bounds are true for a nilpotent Lie algebra L of dim $L = n$ (see [8, 10, 15, 16, 24]):

$$2 \leq \dim H^i(L,\mathbb{C}^\times) \leq \frac{n!}{(n-i)!i!} - \frac{(n-2)!}{(n-i-1)!(i-1)!}, \quad \forall i = 1,\ldots,n-1 \text{ with } n \geq 3, \tag{1.3}$$

$$1 \leq \sum_{k=0}^{i} (-1)^{k+i} \dim H^k(L,\mathbb{C}^\times), \tag{1.4}$$

$$\dim H^1(L,\mathbb{C}^\times) \cdot \dim H^1(L,\mathbb{C}^\times) \leq 4 \cdot \dim H^2(L,\mathbb{C}^\times), \tag{1.5}$$

while the following is a conjecture of Halperin [15]:

$$2^{\dim Z(L)} \leq \sum_{i=0}^{n} \dim H^i(L,\mathbb{C}^\times), \tag{1.6}$$

where $Z(L)$ denotes the center of L. Originally, (1.3) is due to Cairns and others [8], (1.4) is an isoperimetric type inequality of homological nature (see [14, 15, 16, 22]) and (1.5) is of type Golod-Shafarevich (see [16, 22]). Finally, (1.6) is known as the *toral rank conjecture* (see [15, 24]).

In order to understand the motivation of our investigations, we note that the idea of classifying nilpotent Lie algebras of finite dimension by restrictions on their Schur multipliers goes back to [4, 5] and continued in [2, 6, 7, 11] under different perspectives. These authors proved inequalities on dim $M(L)$, involving invariants related with the presentation of L. We know that $L \simeq F/R$ by the short exact sequence $0 \to R \rightarrowtail F \twoheadrightarrow L \to 0$, where F is the *free Lie algebra* F on n generators, R an ideal of F and the Witt's Formula (see [2, 14, 16, 22]) shows

$$\dim F^d/F^{d+1} = \frac{1}{d}\sum_{r|d} \mu(r)\, n^{\frac{d}{r}} \equiv l_n(d), \tag{1.7}$$

where

$$\mu(r) = \begin{cases} 1, & \text{if } r = 1, \\ 0, & \text{if } r \text{ is divisible by a square,} \\ (-1)^s, & \text{if } r = p_1 \ldots p_s \text{ for distinct primes } p_1,\ldots,p_s \end{cases}$$

is the celebrated *Möbius function*. Now [2, Theorem 2.5] and similar results of [2, 6, 7, 11] provide inequalities of the same nature of (1.1), but based on (1.7) and the main problem is to give an explicit expression for $l_n(d)$. For instance, if c denotes the class of nilpotence of L, then [7, Theorem 4.1] shows

$$\dim M(L) \leq \sum_{j=1}^{c} l_n(j+1) = \sum_{j=1}^{c} \left(\frac{1}{j+1} \sum_{i|j+1} \mu(i)\, n^{\frac{j+1}{i}} \right) \tag{1.8}$$

and [7, Examples 4.3, 4.4] provide explicit values for $\mu(i)$ in order to evaluate numerically (1.8) and then to compare with (1.1). It is in fact hard to describe the behaviour of the Möbius function from a general point of view and so (1.7) is not very helpful, when we do not evaluate the coefficients $\mu(i)$. It is still more interesting to observe that the exactness of certain upper bounds on dim $M(L)$ implies theorems of splitting in the sense of [18, Theorem 3.1] and [19, Theorems 2.2, 3.1, 3.5, 3.6, 4.2], but these happen when the dimensions are small enough and we are far from controlling the general cases. Notice that Chao [9] and Seeley [23] proved that there exist uncountably many non–isomorphic nilpotent Lie algebras of finite dimension, beginning already from dimension 10, and this illustrates the complexity of the problem. Now we may understand the importance of being as much concrete as possible in the study of the upper bounds for dim $M(L)$.
Denoting by $L^i = \underbrace{[L, L, \ldots, L]}_{i-\text{times}}$ the i-th term of the lower central series of L, Yankosky [25] sharpened (1.1) by

$$\dim M(L) \leq \frac{1}{2}(n^2 - n - m^2 - m), \tag{1.9}$$

where the role of dim $L^2 = m$ is significant, but we showed [18] that

$$\dim M(L) \leq \frac{1}{2}(n + m - 2)(n - m - 1) + 1 \tag{1.10}$$

is better than (1.1) and (1.9) for all $n \geq 4$ (see [19, Corollary 3.4]). More recently, Degrijse presents a different argument for proving (1.9) and uses spectral sequences for bounding dim $H^2(L, \mathbb{C}^\times)$ (hence dim $M(L)$) in [10, Theorem 4.6]. In order to state one of his results, we recall that the *type* of a finite dimensional nilpotent Lie algebra L of class c is defined as the c-tuple $(m_1, m_2, \ldots, m_c)$ where $m_1 = \dim L/L^2$, $m_2 = \dim L^2/L^3$, ..., $m_i = \dim L^i/L^{i+1}$, ..., $m_c = \dim L^c/L^{c+1} = \dim L^c$. Modifying slightly [10, Definition 1.4], a *free c-step nilpotent extension* of a finite dimensional Lie algebra L is a short exact sequence $0 \to N \rightarrowtail F_{m_1,c} \twoheadrightarrow L \to 0$ such that $N \subseteq F^2_{m_1,c}$, where $F_{m_1,c}$ is the free nilpotent Lie algebra of class c on m_1 generators. We say that L has *depth d*, if d is the largest integer such that $N \subseteq F^d_{m_1,c}$. With the present notations, [10, Theorem 4.6] becomes

$$\dim M(F_{m_1,d-1}) - m_d \leq \dim M(L) \leq \frac{1}{2}(n^2 - n - m^2 - m) \tag{1.11}$$

and, in virtue of (1.10), this may be improved as

$$\dim M(F_{m_1,d-1}) - m_d \leq \dim M(L) \leq \frac{1}{2}(n + m - 2)(n - m - 1) + 1. \tag{1.12}$$

An important inequality of the same nature of (1.9) and (1.10) is given by

$$\dim M(L) \leq \dim M(L/L^2) + \dim L^2(\dim L/Z(L) - 1) \tag{1.13}$$

and can be found in [2, Corollary 3.3], where it is shown that it is better than (1.1). At this point, we may state the first main result.

Theorem 1.1. *Let L be a nilpotent Lie algebra of* $\dim L = n$, $\dim L^2 = m$ *and* $\dim Z(L) = z$. *If L is nonabelian, then*

(i) (1.10) *is better than* (1.13) *for all* $n \geq 3$ *and* $m \leq \left\lfloor \frac{n-2}{z+1} \right\rfloor$.

(ii) (1.10) *is better than* (1.3) *for all* $n \geq 3$ *and* $m \geq 1$ *(when in* (1.3) $i = 2$*).*

(iii) (1.12) *is better than* (1.11) *for all* $n \geq 4$.

(iv) $\frac{1}{4}(n-m)^2 \leq \dim M(L) \leq \frac{1}{2}(n+m-2)(n-m-1)+1$ *for all* $n, m \geq 1$.

Some notions of homological algebra should be recalled from [3, 13], in order to formulate the next result. The *Schur multiplier of the pair* (L, N), where L is a Lie algebra with ideal N, is the abelian Lie algebra $M(L, N)$ which appears in the following natural exact sequence of Mayer–Vietoris type

$$H_3(L,\mathbb{Z}) \longrightarrow H_3(L/N,\mathbb{Z}) \longrightarrow M(L,N) \longrightarrow M(L) \longrightarrow M(L/N) \longrightarrow \tag{1.14}$$
$$\longrightarrow \frac{L}{[L,N]} \longrightarrow \frac{L}{L^2} \longrightarrow \frac{L}{L^2+N} \longrightarrow 0.$$

We also recall that $\Phi(L)$ denotes the *Frattini* subalgebra of L, that is, the intersection of all maximal subalgebras of L (see [17, 21]). It is easy to see that $\Phi(L)$ is an ideal of L, when L is finite dimensional and nilpotent. To convenience of the reader, given an ideal I of a finite dimensional Lie algebra L and a subalgebra J of L, we recall that I is said to be a complement of J in L if $L = I + J$ and $I \cap J = 0$.

Theorem 1.2. *Let L be a finite dimensional nilpotent Lie algebra and N an ideal of L of* $\dim N = k$ *and* $\dim L/N = u$. *Then*

$$\dim M(L,N) \ + \ \dim\,[L,N] \leq \frac{1}{2}k(2u+k-1).$$

Conversely, if $\dim L/N + \Phi(L) = s$ *and* $\dim N/N \cap \Phi(L) = t$, *then*

$$\frac{1}{2}t(2s+t-1) \leq \dim M(L,N) \ + \ \dim\,[L,N].$$

Furthermore, if N has a complement in L and both L and L/N are nonabelian, then

$$\frac{1}{4}((n-m)^2 - (u-v)^2) \leq \dim M(L,N) \leq$$
$$\frac{1}{2}((n+m-2)(n-m-1) - (u+v-2)(u-v-1)),$$

where $m = \dim L^2$, $\dim L = n$ *and* $v = \dim L^2 + N/N$.

When $N = L$ in Theorem 1.2, we get $u = 0$ and find again (1.1) so that Theorem 1.2 is a generalization of (1.1). On the other hand, Theorem 1.2 improves most of the bounds in [11, Theorem B], where L is assumed to be factorized.

2 Proofs of the results

We recall that $A(n)$ denotes the abelian Lie algebra of dimension n and the main results of [2, 5, 6, 7, 11, 18, 19] illustrate that many inequalities on $\dim M(L)$ become equalities if and only if L splits in the sums of $A(n)$ and of a Heisenberg algebra $H(m)$ (here $m \geq 1$ is a given integer). Most of the proofs of these results are based on the following property, which we will use largely.

Lemma 2.1 (See [22], Theorem 11.31, Künneth Formula)**.** *Two finite dimensional Lie algebras H and K satisfy the condition*

$$M(H \oplus K) = M(H) \oplus M(K) \oplus (H/H^2 \otimes K/K^2).$$

In particular,

$$\dim M(H \oplus K) = \dim M(H) \ + \ \dim M(K) \ + \ \dim H/H^2 \otimes K/K^2.$$

The dimension of the Schur multiplier of abelian Lie algebras is a classic.

Lemma 2.2 (See [5], Lemma 3)**.** $L \simeq A(n)$ *if and only if* $\dim M(L) = \frac{1}{2}n(n-1)$.

Now we may specify (1.13).

Lemma 2.3. *If a nilpotent Lie algebra L of* $\dim L = n$ *has* $\dim L^2 = m$ *and* $\dim Z(L) = z$*, then* (1.13) *becomes*

$$\dim M(L) \leq \frac{1}{2}(n-m)(n-m-1) + m(n-z-1).$$

Proof. This is an application of Lemma 2.2, noting that $\dim L/L^2 = \dim L - \dim L^2 = \dim A(n-m) = n-m$ and $\dim L/Z(L) = \dim L - \dim Z(L) = m - z$. ∎

Proof of Theorem 1.1. (i). From Lemma 2.3, (1.13) becomes

$$\dim M(L) \leq \frac{1}{2}(n-m)(n-m-1) + m(n-z-1)$$

$$= \frac{1}{2}(n^2 - nm - n - nm + m^2 + m) + mn - zm - m = \frac{1}{2}(n^2 + m^2 + m - n) - zm - m$$

$$= \frac{1}{2}(n^2 + m^2) + \frac{1}{2}m - m - zm - \frac{1}{2}n = \frac{1}{2}(n^2 + m^2) - \frac{1}{2}m - zm - \frac{1}{2}n$$

$$= \frac{1}{2}(n^2 + m^2) - \left(z + \frac{1}{2}\right)m - \frac{1}{2}n.$$

On the other hand, (1.10) becomes

$$\dim M(L) \leq \frac{1}{2}(n+m-2)(n-m-1) + 1$$

$$= \frac{1}{2}(n^2 - nm - n + nm - m^2 - m - 2n + 2m + 2) + 1 = \frac{1}{2}(n^2 - m^2) + \frac{1}{2}m - \frac{3}{2}n + 2.$$

Of course, the first terms satisfy $\frac{1}{2}(n^2 - m^2) \leq \frac{1}{2}(n^2 + m^2)$ for all $m, n \geq 1$, but the remaining terms satisfy

$$\frac{1}{2}m - \frac{3}{2}n + 2 \leq -\left(z + \frac{1}{2}\right)m - \frac{1}{2}n \Leftrightarrow 0 \leq -(z+1)m + n - 2$$

$$\Leftrightarrow 0 \geq (z+1)m - n + 2 \Leftrightarrow m \leq \left\lfloor \frac{n-2}{z+1} \right\rfloor.$$

It follows that (1.10) is better than (1.13) for these values of m.
(ii). When $i = 2$ the upper bound of (1.3) becomes $\frac{1}{2}(n^2 - 3n + 4)$ and (1.2) allows us to compare it with (1.10). We discover that

$$0 \geq +m - m^2 \Rightarrow n^2 - 3n + 4 \geq n^2 - 3n + m - m^2 + 4 - nm + nm$$

$$\Rightarrow n^2 - 3n + 4 \geq (n+m-2)(n-m-1) + 2 \Rightarrow \frac{1}{2}(n^2 - 3n + 4) \geq \frac{1}{2}(n+m-2)(n-m-1) + 1.$$

(iii). This is straightforward from [19, Corollary 3.4].
(iv). The upper bound is (1.10). In order to derive the lower bound, we need to observe from [16, Theorem 6.10] and [16, Example (2), p. 168] that even a stronger form of (1.2) is true, that is, the Poincaré duality allows us to conclude that there exists an isomorphism of finite dimensional Lie algebras $H^1(L, \mathbb{C}^\times) \simeq (L/L^2)^*$, where $*$ denotes the dual Lie algebra with respect to L/L^2 (see again [16, Chapter 6] for the definition of $*$) and so

$$\dim H^1(L, \mathbb{C}^\times) = \dim\,(L/L^2)^* = \dim L/L^2 = n - m, \tag{2.1}$$

where the equality $\dim L/L^2 = \dim\,(L/L^2)^*$ is true by the fact that the dimension is invariant under $*$. Now the lower bound follows from (1.5). ■

In order to prove Theorem 1.2, we recall an important construction of algebraic topology called nonabelian tensor product (see for instance [12, 19]). Let L and K be two arbitrary Lie algebras (on the same field F). By an action of L on K, we mean and F-bilinear map $(l,k) \in L \times K \longmapsto {}^{l}k \in K$ satisfying ${}^{[l,l']}k = {}^{l}({}^{l'}k) - {}^{l'}({}^{l}k)$ and ${}^{l}[k,k'] = [{}^{l}k, k'] + [k, {}^{l}k']$ for all $l, l' \in L$ and $k, k' \in K$. Clearly, if L is a subalgebra of some Lie algebra P and K is an ideal in P, then the Lie multiplication in P induces and action of L on K. In fact, L acts on K via ${}^{l}k = [l,k]$. Now let L and K be two Lie algebras acting on each other, and on themselves by Lie multiplications. The actions are said to be *compatible* if ${}^{{}^{k}l}k' = {}^{k'}({}^{l}k)$ and ${}^{{}^{l}k}l' = {}^{l'}({}^{k}l)$ for all $l, l' \in L$ and $k, k' \in K$. It is obvious that if L and K are both ideals of a bigger Lie algebra M, then the Lie multiplication gives rise to compatible actions inside M. The *nonabelian tensor product* $L \otimes K$ of L and K is the Lie algebra generated by the symbols $l \otimes k$ with defining relations $c(l \otimes k) = cl \otimes k = l \otimes ck$, $(l + l') \otimes k = l \otimes k + l' \otimes k$, $l \otimes (k + k') = l \otimes k + l \otimes k'$, ${}^{l}l' \otimes k = l \otimes {}^{l'}k - l' \otimes {}^{l}k$, $l \otimes {}^{k}k' = {}^{k'}l \otimes k - {}^{k}l \otimes k'$, $[l \otimes k, l' \otimes k'] = -{}^{k}l \otimes {}^{l'}k'$, where $c \in F$, $l, l' \in L$ and

$k, k' \in K$. In case $L = K$ and all actions are given by Lie multiplication, $L \otimes L$ is called *nonabelian tensor square* of L. One notes that the nonabelian tensor product always exists (see [12, Proposition 1.2 (iii)]) and, in particular, we find the usual abelian tensor product $L \otimes_{\mathbb{Z}} K$, when L and K are abelian and the actions are compatible and trivial.

When K is an ideal of the Lie algebra L, there are compatible actions such that we may form $L \otimes K$ and $L \wedge K$ and (see for instance, [12, 19]) the following is an epimorphism of Lie algebras

$$\kappa_{L,K} : l \otimes k \in L \otimes K \longmapsto \kappa_{L,K}(l \otimes k) = [l, k] \in [L, K]$$

such that $\ker \kappa_{L,K}$ is a central ideal of $L \otimes K$. Now $L \square K = \langle l \otimes l \mid l \in L \cap K \rangle$ is an ideal of $L \otimes K$ contained in $Z(L \otimes K)$ and we have automatically a natural epimorphism $\varepsilon : l \otimes k \in L \otimes K \mapsto (l \otimes k) + L \square K \in L \otimes K / L \square K$ which allows us to form the Lie algebra quotient

$$L \wedge K = \frac{L \otimes K}{L \square K} = \langle l \otimes k + (L \square K) \mid l \in L, k \in K \rangle = \langle l \wedge k \mid l \in L, k \in K \rangle,$$

called *nonabelian exterior product* of L and K. Here we have the following epimorphism of Lie algebras

$$\kappa'_{L,K} : l \wedge k \in L \wedge K \longmapsto \kappa'_{L,K}(l \wedge k) = [l, k] \in [L, K]$$

such that $\ker \kappa'_{L,K}$ is a central ideal of $L \wedge K$. It is easy to see that $\ker_{L,K} = L \square K$ and that $\ker'_{L,K} = M(L, K)$ and that the following diagram is exact and formed by central extension of Lie algebras:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L \square K & \longrightarrow & L \otimes K & \xrightarrow{\kappa_{L,K}} & [L, K] & \longrightarrow & 0 \\ & & \downarrow & & \varepsilon \downarrow & & \| & & \\ 0 & \longrightarrow & M(L, K) & \longrightarrow & L \wedge K & \xrightarrow{\kappa'_{L,K}} & [L, K] & \longrightarrow & 0 \end{array} \quad . \tag{2.2}$$

Now we may prove Theorem 1.2.

Proof of Theorem 1.2. We begin to prove the lower bound. We claim that

$$\dim M\left(\frac{L}{\Phi(L)}, \frac{N}{N \cap \Phi(L)}\right) \leq \dim M(L, N) + \dim\, [L, N]. \tag{2.3}$$

Note from [17, Corollary 2, p.420] that $\Phi(L) = L^2$ is always true for nilpotent Lie algebras. Then $L/\Phi(L)$ and $N/N \cap \Phi(L) \simeq N + \Phi(L)/\Phi(L) \subseteq L/\Phi(L)$ are abelian. In our situation,

$$M\left(\frac{L}{\Phi(L)}, \frac{N}{N \cap \Phi(L)}\right) \simeq \frac{L}{\Phi(L)} \wedge \frac{N}{N \cap \Phi(L)} \simeq \frac{L}{L^2} \wedge \frac{N}{N \cap L^2}.$$

We form the nonabelian exterior product $L \wedge N$ and deduce from the the diagram (2.2) that

$$\dim L \wedge N = \dim M(L, N) + \dim\, [L, N].$$

On the other hand,

$$x \wedge y \in L \wedge N \longmapsto x + L^2 \wedge y + (N \cap L^2) \in \frac{L}{L^2} \wedge \frac{N}{N \cap L^2}$$

is also an epimorphism of Lie algebras and it implies

$$\dim L \wedge N \geq \dim \frac{L}{L^2} \wedge \frac{N}{N \cap L^2}$$

so that

$$\dim M\left(\frac{L}{\Phi(L)}, \frac{N}{N \cap \Phi(L)}\right) = \dim \frac{L}{L^2} \wedge \frac{N}{N \cap L^2} \leq \dim L \wedge N.$$

The claim (2.3) follows. Consequently, it will be enough to prove

$$\dim M\left(\frac{L}{\Phi(L)}, \frac{N}{N \cap \Phi(L)}\right) = \frac{1}{2}t(2s + t - 1) \tag{2.4}$$

in order to conclude

$$\frac{1}{2}t(2s + t - 1) \leq \dim M(L, N) \ + \ \dim\,[L, N].$$

Since $N/N \cap \Phi(L) \simeq A(t)$ is a direct factor of the abelian Lie algebra $L/\Phi(L) \simeq A(s + t) \simeq A(s) \oplus A(t)$, Lemma 2.2 implies

$$\dim M\left(\frac{L}{\Phi(L)}\right) = \frac{1}{2}(s + t)(s + t - 1) \text{ and } \dim M\left(\frac{N}{N \cap \Phi(L)}\right) = \frac{1}{2}t(t - 1).$$

On the other hand, we are dealing with abelian Lie algebras and $N/N \cap \Phi(L) \simeq N + \Phi(L)/\Phi(L)$ has a complement in $L/\Phi(L)$. As in [3, p.174, Line +15], we may conclude that

$$M\left(\frac{L}{\Phi(L)}\right) \simeq M\left(\frac{L}{\Phi(L)}, \frac{N}{N \cap \Phi(L)}\right) \oplus \ M\left(\frac{L/\Phi(L)}{N/N \cap \Phi(L)}\right),$$

hence

$$\dim M\left(\frac{L}{\Phi(L)}, \frac{N}{N \cap \Phi(L)}\right) = \dim M\left(\frac{L}{\Phi(L)}\right) - \dim M\left(\frac{L/\Phi(L)}{N + \Phi(L)/\Phi(L)}\right)$$

$$= \dim M\left(\frac{L}{\Phi(L)}\right) - \dim M\left(\frac{L/\Phi(L)}{N/N \cap \Phi(L)}\right) = \frac{1}{2}(s + t)(s + t - 1) - \frac{1}{2}s(s - 1)$$

$$= \frac{1}{2}t(2s + t - 1),$$

and so (2.4) is proved.
Now we prove the upper bound

$$\dim M(L, N) \leq \frac{1}{2}k(2u + k - 1).$$

Proceed by induction on k. Of course, the above inequality is true when $k = 0$. Suppose that it holds whenever N is of dimension strictly less than k, and suppose that $\dim N = k$. Let C be a Lie algebra of dimension one in the center of N. Again we use the nonabelian exterior product $L \wedge C$ in order to conclude that

$$\dim L \wedge C = \dim M(L, C) + \dim [L, C].$$

Having in mind (2.2), we have

$$\dim M(L, C) \leq \dim L \wedge C = \dim L \otimes C - \dim L\square C$$
$$= \dim L \cdot \dim C - \dim L\square C = (u + k) \cdot 1 - 1 = u + k - 1.$$

Now we observe that the Schur multiplier of a pair (L, N) induces the following exact sequence (since $C \subseteq Z(N)$):

$$\longrightarrow M(L, C) \longrightarrow M(L, N) \longrightarrow M(L/C, N/C) \longrightarrow 0.$$

Therefore

$$\dim M(L, N) \leq \dim M(L, C) + \dim M \left(\frac{L}{C}, \frac{N}{C}\right)$$
$$\leq (u + k - 1) + \frac{1}{2}(k - 1)(2u + k - 2) = \frac{1}{2}k(2u + k - 1),$$

as wished.

Finally, if N possesses a complement in L and both L and L/N are nonabelian, [3, p.174, Line +15] implies

$$M(L) \simeq M(L, N) \oplus M(L/N), \tag{2.5}$$

hence $\dim M(L, N) = \dim M(L) - \dim M(L/N)$ and the final bound is an application of Theorem 1.1 (iv), since we substract member by member

$$\frac{1}{4}(n - m)^2 \leq \dim M(L) \leq \frac{1}{2}(n + m - 2)(n - m - 1)$$
$$\frac{1}{4}(u - v)^2 \leq \dim M(L/N) \leq \frac{1}{2}(u + v - 2)(u - v - 1)$$

and find the desired inequality. ■

We want to note that the source of inspiration for the bound of Theorem 1.2 was an analogous condition for groups, proved by Ellis in [13, Proposition 7.2]. Unfortunately, it wasn't possible to use the same methods which work for groups, since the notion of exponent hasn't a perfect analogy in the context of Lie algebras and several techniques cannot be applied directly. This has motivated us to involve the notion of nonabelian exterior product of Lie algebras, which has a very general formulation and applies to wider contexts.

Acknowledgements

The second author thanks IMPA and UFRJ for the project CAPES with ref.no. 061/2013, which is allowing him to improve some of the original ideas illustrated in the present paper.

References

[1] V. Alamian, H. Mohammadzadeh and A.R. Salemkar, Some properties of the Schur multiplier and covers of Lie algebras, *Comm. Algebra* **36** (2008), 697–707.

[2] M. Araskhan, B. Edalatzadeh and A.R. Salemkar, Some inequalities for the dimension of the c-nilpotent multiplier of Lie algebras, *J. Algebra* **322** (2009), 1575–1585.

[3] M. Araskhan and M.R. Rismanchian, Some inequalities for the dimension of the Schur multiplier of a pair of (nilpotent) Lie algebras, *J. Algebra* **352** (2012), 173–179.

[4] P. Batten, *Multipliers and covers of Lie algebras*, North Carolina State University, Phd Thesis, Raylegh (USA), 1993.

[5] P. Batten, K. Moneyhun and E. Stitzinger, On characterizing nilpotent Lie algebras by their multipliers, *Comm. Algebra* **24** (1996), 4319–4330.

[6] L. Bosko, On Schur multipliers of Lie algebras and groups of maximal class, *Int. J. Algebra Comput.* **20** (2010), 807–821.

[7] L. Bosko and E. Stitzinger, *Schur multipliers of nilpotent Lie algebras*, preprint, 2011, available online at http://arxiv.org/abs/1103.1812.

[8] G. Cairns and B. Jessup, New bounds on Betti numbers of nilpotent Lie algebras, *Comm. Algebra* **25** (1997), 415–430.

[9] C.Y. Chao, Uncountably many non–isomorphic nilpotent Lie algebras,*Proc. Amer. Math. Soc.* **13** (1962), 903–906.

[10] D. Degrijse, On a filtration of the second cohomology of nilpotent Lie algebras, *J. Lie Theory* **23** (2013), 551–573.

[11] B. Edalatzadeh, F. Saeedi and A.R. Salemkar, The commutator subalgebra and Schur multiplier of a pair of nilpotent Lie algebras, *J. Lie Theory* **21** (2011), 491–498.

[12] B. Edalatzadeh, H. Mohammadzadeh, A. Salemkar and H. Tavallaee, On the non-abelian tensor product of Lie algebras, *Linear Multilinear Algebra* **58** (2010), 333–341.

[13] G. Ellis, The Schur multiplier of a pair of groups, *Appl. Categ. Structures* **6** (1998), 355–371.

[14] M. Goze and Yu. Khakimyanov, *Nilpotent Lie Algebras*, Kluwer, Dordrecht, 1996.

[15] S. Halperin, *Rational Homotopy and Torus Action*, Aspects of Topology, Ed. I.M. James and E.H. Kronheimer, London Math. Soc. Lecture Notes Ser. 93, 1985, London.

[16] A.W. Knapp, *Lie groups, Lie algebras and cohomology*, Princeton University Press, Princeton, 1988.

[17] E.I. Marshall, The Frattini subalgebra of a Lie algebra, *J. London. Math. Soc.* **42** (1967), 416–422.

[18] P. Niroomand and F.G. Russo, A note on the Schur multiplier of a nilpotent Lie algebra, *Comm. Algebra* **39** (2011), 1293–1297.

[19] P. Niroomand and F.G. Russo, A restriction on the Schur multiplier of a nilpotent Lie algebra, *Elec. J. Linear Algebra* **22** (2011), 1–9.

[20] P. Niroomand and F.G. Russo, An improvement of a bound of Green, *J. Algebra Appl.* **11** No. 6, Paper No. 1250116, (2012), pp. 11.

[21] P. Niroomand, M. Parvizi and F.G. Russo, Some criteria for detecting capable Lie algebras, *J. Algebra* **384** (2013), 36–44.

[22] J. Rotman, *An Introduction to Homological Algebra*, Academic Press, San Diego, 1979.

[23] C. Seeley, Some nilpotent Lie algebras of even dimension, *Bull. Austral. Math. Soc.* **45** (1992), 71–77.

[24] P. Tirao, A refinement of the toral rank conjecture for the 2-step nilpotent Lie algebras, *Proc. Amer. Math. Soc.* **185** (2000), 2875–2878.

[25] B. Yankosky, On the multiplier of a Lie algebra, *J. Lie Theory* **13** (2003), 1–6.

School of Mathematics and Computer Science,
Damghan University,
P.O. Box: 36715-364,
Damghan, Iran.
email:p_niroomand@yahoo.com

Instituto de Matemática, Universidade Federal do Rio de Janeiro
Av. Athos da Silveira Ramos 149
Centro de Tecnologia, Bloco C, Cidade Universitária, Ilha do Fundão
Caixa Postal 68530
21941-909, Rio de Janeiro, Brasil
and
Department of Mathematics and Applied Mathematics
University of Cape Town
Private Bag X1, Rondebosch 7701
South Africa.
email:francescog.russo@yahoo.com