Some remarks concerning pseudocompactness in pointfree topology

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Abstract

Recall that a frame L is pseudocompact if $\mathcal{R}L = \mathcal{R}^*L$, where $\mathcal{R}L$ is the f-ring of real-valued continuous functions on L, and $\mathcal{R}^*L$ its bounded part. Using properties of uniform frames, Walters-Wayland proved that a completely regular frame L is pseudocompact iff the frame homomorphism $\beta L \to L$ is coz-codense. In this note we give a purely ring-theoretic reaffirmation of this characterization by observing that a frame homomorphism $L \to M$ is coz-codense iff the ring homomorphism $\mathcal{R}L \to \mathcal{R}M$ it induces maps non-invertible elements to non-invertible elements, and that L is pseudocompact iff every finitely generated proper ideal of $\mathcal{R}^*L$ is fixed. We also show that if L is not pseudocompact, then $\mathcal{R}^*L$ has a non-maximal free prime ideal – thus generalizing a 1954 result of Gillman and Henriksen.

1 Introduction

In her doctoral thesis [12], Walters-Wayland proves that a completely regular frame L is pseudocompact if and only if the coreflection map $j_L \colon \beta L \to L$ from compact completely regular frames to L is coz-codense, meaning that the only cozero element of βL sent to the top by j_L is the top. This characterization is actually one of several she establishes, and they all use the concept of uniform frames.

In this note we give a self-contained, purely ring-theoretic reaffirmation of this characterization without invoking uniformities. The route, which goes via

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two lemmas of independent interest, is as follows. We show that a frame homomorphism $h\colon L \to M$ is coz-codense if and only if the induced ring homomorphism $\mathcal{R}h\colon \mathcal{R}L \to \mathcal{R}M$ takes non-invertible elements to non-invertible elements (Lemma 3.3), and that pseudocompact frames are precisely those L for which every finitely generated proper ideal of $\mathcal{R}^*L$ is fixed (Lemma 3.1).

If L is not pseudocompact, then $\mathcal{R}^*L$ has a free proper ideal, and hence a free maximal ideal since every free proper ideal is contained in a free maximal ideal. Less obvious to see is that if L is not pseudocompact, then not only does $\mathcal{R}^*L$ have a free maximal ideal, it actually also has a free non-maximal prime ideal (Proposition 3.6). The "pointed" version of this result was proved by Gillman and Henriksen [7] in 1954.

2 Preliminaries

Our references for the general theory of frames are [9] and [11]. A reader who prefers a "covariant" approach to the subject would do well no to overlook [10]. Our notation is standard. All our frames are assumed to be completely regular.

By a *point* of L we mean an element p such that $p < 1$ and $x \wedge y \leq p$ implies $x \leq p$ or $y \leq p$. Points of a frame are also called *prime elements*. The points of any regular frame are precisely those elements which are maximal strictly below the top. We denote the set of all points of L by $\mathrm{Pt}(L)$. A frame *has enough points* if every element is a meet of points above it. Every compact regular frame has enough points if one assumes (as we do throughout) the Prime Ideal Theorem. Frames that have enough points are also said to be *spatial*.

We regard the Stone-Čech compactification of L, denoted βL, as the frame of completely regular ideals of L. We denote the right adjoint of the join map $j_L\colon \beta L \to L$ by r_L and recall that $r_L(a) = \{x \in L \mid x \prec\!\!\prec a\}$.

Regarding the *frame of reals* $\mathfrak{L}(\mathbb{R})$ and the f-ring $\mathcal{R}L$ of *continuous real-valued functions* on L, we use the notation of [2]. See also [1] for other properties of $\mathcal{R}L$. The *bounded part*, in the f-ring sense, of $\mathcal{R}L$ is denoted by $\mathcal{R}^*L$. In the event that $\mathcal{R}^*L = \mathcal{R}L$, the frame L is said to be *pseudocompact*. A crucial link between $\mathcal{R}L$ and L is given by the *cozero map* $\mathrm{coz}\colon \mathcal{R}L \to L$, the properties of which we shall freely use without comment.

Every frame homomorphism $h\colon L \to M$ induces a ring homomorphism $\mathcal{R}h\colon \mathcal{R}L \to \mathcal{R}M$ which takes an element α of $\mathcal{R}L$ to the composite $h\alpha$. Furthermore, $\mathrm{coz}(h\alpha) = h(\mathrm{coz}\,\alpha)$.

A *cozero element* of L is an element of the form $\mathrm{coz}\,\varphi$ for some $\varphi \in \mathcal{R}L$. The *cozero part* of L, denoted $\mathrm{Coz}\,L$, is the regular sub-σ-frame consisting of all the cozero elements of L. It generates L by joins if and only if L is completely regular. General properties of cozero elements and cozero parts of frames can be found in [3]. A frame homomorphism $h\colon L \to M$ is *coz-codense* if, for any $c \in \mathrm{Coz}\,L$, $h(c) = 1$ implies $c = 1$.

3 Pseudocompactness

As indicated in the introduction, we wish to give a purely ring-theoretic proof of Walters-Wayland's characterization of pseudocompact frames in terms of coz-codensity of the map $j_L\colon \beta L \to L$. In [4] the authors observe that $\mathcal{R}^*L \cong \mathcal{R}(\beta L)$. Indeed, the map

$$\phi_L\colon \mathcal{R}(\beta L) \to \mathcal{R}^*L,$$

mapping as $\mathcal{R}(j_L)$, is a ring isomorphism such that $\mathcal{R}(j_L) = i \cdot \phi_L$, for the identical embedding $i\colon \mathcal{R}^*L \to \mathcal{R}L$.

Fixed and free ideals of $\mathcal{R}L$ or $\mathcal{R}^*L$ are defined as in the classical case. An ideal Q of $\mathcal{R}L$ or $\mathcal{R}^*L$ is *fixed* if $\bigvee\{\operatorname{coz}\alpha \mid \alpha \in Q\} < 1$. Otherwise, it is *free*. We recall from [2] that any $\alpha \in \mathcal{R}L$ is a unit in $\mathcal{R}L$ if and only if $\operatorname{coz}\alpha = 1$.

Lemma 3.1. *Every finitely generated proper ideal of $\mathcal{R}L$ is fixed.*

Proof. Let $Q = \langle \alpha_1, \ldots, \alpha_m \rangle$ be a finitely generated proper ideal of $\mathcal{R}L$. Then ${\alpha_1}^2 + \cdots + {\alpha_m}^2 \in Q$, so that $\operatorname{coz}({\alpha_1}^2 + \cdots + {\alpha_m}^2) < 1$ as ${\alpha_1}^2 + \cdots + {\alpha_m}^2$ is not a unit in $\mathcal{R}L$. Thus,

$$\operatorname{coz}\alpha_1 \vee \cdots \vee \operatorname{coz}\alpha_m = \operatorname{coz}({\alpha_1}^2 + \cdots + {\alpha_m}^2) < 1.$$

Now, for any $\gamma \in Q$, there are elements $\tau_1, \ldots, \tau_m$ in $\mathcal{R}L$ such that

$$\gamma = \tau_1\alpha_1 + \cdots + \tau_m\alpha_m.$$

Therefore $\operatorname{coz}\gamma \leq \operatorname{coz}\alpha_1 \vee \cdots \vee \operatorname{coz}\alpha_m$, which implies

$$\bigvee \operatorname{coz}[Q] \leq \operatorname{coz}\alpha_1 \vee \cdots \vee \operatorname{coz}\alpha_m < 1,$$

showing that Q is fixed. ∎

In the proof of one implication in the following lemma we will need to recall that $\mathcal{R}L$ has bounded inversion; meaning that every $\alpha \geq \mathbf{1}$ is invertible. Also, in any f-ring with bounded inversion, $(1 + a^2)^{-1}$ is bounded. Indeed, $0 \leq \frac{1}{(1+a^2)} \leq 1$.

Lemma 3.2. *A completely regular frame L is pseudocompact iff every finitely generated proper ideal of $\mathcal{R}^*L$ is fixed.*

Proof. ($\Rightarrow$): Assume L is pseudocompact, so that $\mathcal{R}L = \mathcal{R}^*L$. Then, by the previous result, every finitely generated proper ideal of $\mathcal{R}^*L$ is fixed.

($\Leftarrow$): If L is not pseudocompact, then $\mathcal{R}L \neq \mathcal{R}^*L$. Let $\alpha \in \mathcal{R}L$ be unbounded. Then $\mathbf{1} + \alpha^2 \notin \mathcal{R}^*L$. Since $\mathbf{1} + \alpha^2 \geq \mathbf{1}$, it is invertible in $\mathcal{R}L$. Its inverse $(\mathbf{1} + \alpha^2)^{-1}$ is in $\mathcal{R}^*L$, but not invertible in $\mathcal{R}^*L$. Thus, the ideal Q of $\mathcal{R}^*L$ generated by $(\mathbf{1} + \alpha^2)^{-1}$ is proper. But it is not fixed because $\bigvee \operatorname{coz}[Q] \geq \operatorname{coz}\left((\mathbf{1} + \alpha^2)^{-1}\right) = 1$. ∎

Next, we express coz-codensity in ring-theoretic terms.

Lemma 3.3. *A frame homomorphism $h\colon L \to M$ is coz-codense iff the induced ring homomorphism $\mathcal{R}h\colon \mathcal{R}L \to \mathcal{R}M$ takes non-units to non-units.*

Proof. Suppose h is coz-codense. Let α be an element of $\mathcal{R}L$ such that $\mathcal{R}h(\alpha)$ is invertible in $\mathcal{R}M$. Then

$$1 = \operatorname{coz}(h\alpha) = h(\operatorname{coz}\alpha),$$

which implies $\operatorname{coz}\alpha = 1$, by coz-codensity, and hence α is invertible in $\mathcal{R}L$.

Conversely, suppose $\mathcal{R}h$ takes non-units to non-units, and let $h(a) = 1$ for some $a \in \operatorname{Coz} L$. Take α in $\mathcal{R}L$ such that $a = \operatorname{coz}\alpha$. Then

$$1 = h(\operatorname{coz}\alpha) = \operatorname{coz}(h\alpha) = \operatorname{coz}\big(\mathcal{R}h(\alpha)\big),$$

so that $\mathcal{R}h(\alpha)$ is invertible. So, by the current hypothesis, α is invertible, and hence $a = \operatorname{coz}\alpha = 1$. Therefore h is coz-codense. ∎

Before stating the main result, let us observe that, for any completely regular frame L, $\mathcal{R}L$ is a ring of quotients of $\mathcal{R}^*L$ relative to the multiplicatively closed set

$$S = \{\alpha \in \mathcal{R}^*L \mid \alpha \text{ is a unit in } \mathcal{R}L\}.$$

In fact, if A is any semi-prime f-ring with bounded inversion, then a straightforward algebraic calculation shows that A is the ring of fractions of its bonded part A^* relative to the set of those elements of A^* which are units in A. Consequently, the expansion, Q^e, of any ideal Q of $\mathcal{R}^*L$ (i.e. the ideal of $\mathcal{R}L$ generated by Q) is given by

$$Q^e = \{\tau\alpha^{-1} \mid \tau \in Q, \alpha \in \mathcal{R}^*L \text{ and } \alpha \text{ is a unit in } \mathcal{R}L\}.$$

Proposition 3.4. *The following are equivalent for a completely regular frame L.*

1. *L is pseudocompact.*
2. *The identical embedding $i\colon \mathcal{R}^*L \to \mathcal{R}L$ takes non-units to non-units.*
3. *$j_L\colon \beta L \to L$ is coz-codense.*
4. *The identical embedding $i\colon \mathcal{R}^*L \to \mathcal{R}L$ preserves properness of ideals.*

Proof. It is immediate that (1) implies (2).

(2) ⇒ (3): Since $\mathcal{R}(j_L) = i \cdot \phi_L$, and ϕ_L is an isomorphism, it follows from (2) that $\mathcal{R}(j_L)$ takes non-units to non-units. Therefore j_L is coz-codense, by Lemma 3.3.

(3) ⇒ (4): Since ϕ_L is an isomorphism, $i = \phi_L^{-1} \cdot \mathcal{R}(j_L)$. Thus, if (3) holds, then $\mathcal{R}(j_L)$ takes non-units to non-units, and hence so does i. Now let Q is a proper ideal of $\mathcal{R}^*L$, and $\tau\alpha^{-1}$ be an arbitrary element of Q^e with $\tau \in Q$, so that τ is not a unit in $\mathcal{R}^*L$ by properness. Since i takes non-units to non-units, we have that τ is not a unit in $\mathcal{R}L$, and hence $\tau\alpha^{-1}$ is not a unit in $\mathcal{R}L$. This shows that Q^e is a proper ideal of $\mathcal{R}L$, as required.

(4) ⇒ (1): We apply Lemma 3.2. Let $Q = \langle \alpha_1, \ldots, \alpha_n \rangle$ be a proper ideal of $\mathcal{R}^*L$. If Q were not fixed, then we would have

$$\bigvee \operatorname{coz}[Q] \geq \operatorname{coz}\alpha_1 \vee \cdots \vee \operatorname{coz}\alpha_n = \operatorname{coz}(\alpha_1^2 + \cdots + \alpha_n^2) = 1,$$

which would imply that the element $\alpha_1^2 + \cdots + \alpha_n^2$ of Q^e is a unit in $\mathcal{R}L$, which would contradict the fact that i preserves properness. ∎

Remark 3.5. Using the Axiom of Choice, condition (4) in the foregoing proposition is equivalent to the statement that the identical embedding $\mathcal{R}^*L \to \mathcal{R}L$ takes maximal ideals to maximal ideals. We thank the referee for this observation.

Finally, we show that if L is not pseudocompact, then not only does $\mathcal{R}^*L$ have a free maximal ideal, it actually also has a free non-maximal prime ideal. We recall from [6] that, for each $1_{\beta L} \neq I \in \beta L$, the proper ideals $\mathbf{M}^I$ and $\mathbf{O}^I$ of $\mathcal{R}L$ are defined by

$$\mathbf{M}^I = \{\alpha \in \mathcal{R}L \mid r_L(\operatorname{coz}\alpha) \subseteq I\} \quad \text{and} \quad \mathbf{O}^I = \{\alpha \in \mathcal{R}L \mid r_L(\operatorname{coz}\alpha) \prec\!\prec I\}.$$

It is shown in [6] that if P is a prime ideal of $\mathcal{R}L$, there is a unique point I of βL such that $\mathbf{O}^I \subseteq P \subseteq \mathbf{M}^I$. It is convenient to introduce the following notation. For any $a < 1$ in L, write

$$\mathbf{M}_a = \{\alpha \in \mathcal{R}L \mid \operatorname{coz}\alpha \leq a\} \quad \text{and} \quad \mathbf{O}_a = \{\alpha \in \mathcal{R}L \mid \operatorname{coz}\alpha \prec\!\prec a\}.$$

Clearly, $\mathbf{M}_a = \mathbf{M}^{r_L(a)}$ and $\mathbf{O}_a = \mathbf{O}^{r_L(a)}$. Thus, if I is a point of βL, then $\mathbf{M}_I$ is the unique maximal ideal of $\mathcal{R}(\beta L)$ containing $\mathbf{O}_I$. In the coming proof we shall need to take cognizance of the fact that if $I \prec J$ in βL, then $\bigvee I \in J$. To see this, take $H \in \beta L$ such that $I \wedge H = 0_{\beta L}$ and $H \vee J = 1_{\beta L}$. Next, pick $x \in H$ and $y \in J$ such that $x \vee y = 1$. Since $\bigvee I \wedge \bigvee H = 0$, it follows that $\bigvee I \wedge x = 0$, and hence $\bigvee I \leq y \in J$.

Proposition 3.6. *If L is not pseudocompact, then $\mathcal{R}^*L$ has a free non-maximal prime ideal.*

Proof. Since L is not pseudocompact, there is an unbounded $\varphi \in \mathcal{R}L$. The element $\alpha = (\mathbf{1} + \varphi^2)^{-1}$ is in $\mathcal{R}^*L$, but is not invertible in $\mathcal{R}^*L$. Therefore $\phi_L^{-1}(\alpha)$ is not invertible in $\mathcal{R}(\beta L)$. Consequently, $\operatorname{coz}(\phi_L^{-1}(\alpha)) \neq 1_{\beta L}$. Since βL has enough points, there is a point I of βL such that $\operatorname{coz}(\phi_L^{-1}(\alpha)) \leq I$. We aim to show that $\mathbf{O}_I \neq \mathbf{M}_I$. Note that, by definition, $\phi_L^{-1}(\alpha) \in \mathbf{M}_I$. If $\phi_L^{-1}(\alpha)$ were in $\mathbf{O}_I$ we would have $\operatorname{coz}(\phi_L^{-1}(\alpha)) \prec\!\prec I$, which would imply $\bigvee \operatorname{coz}(\phi_L^{-1}(\alpha)) \in I$. But

$$\bigvee \operatorname{coz}(\phi_L^{-1}(\alpha)) = j_L\big(\operatorname{coz}(\phi_L^{-1}(\alpha))\big) = \operatorname{coz}(j_L\phi_L^{-1}(\alpha)) = \operatorname{coz}\alpha = 1,$$

since α is invertible in $\mathcal{R}L$. So this would imply $1 \in I$, contrary to the fact that $I \neq 1_{\beta L}$. It follows therefore that $\mathbf{O}_I \neq \mathbf{M}^I$. Now, if φ is an element of $\mathcal{R}(\beta L)$ such that $\varphi^n \in \mathbf{O}_I$ for some $n \in \mathbb{N}$, then $\operatorname{coz}\varphi = \operatorname{coz}(\varphi^n) \prec\!\prec I$, so that $\varphi \in \mathbf{O}_I$. Therefore $\mathbf{O}_I$ is a radical ideal and hence equals the intersection of all prime ideals of $\mathcal{R}(\beta L)$ containing it. But every prime ideals of $\mathcal{R}(\beta L)$ which contains $\mathbf{O}_I$ is contained in $\mathbf{M}_I$. It follows, therefore, that there is a non-maximal prime ideal P in $\mathcal{R}(\beta L)$ such that $\mathbf{O}_I \subseteq P$. Therefore $\phi_L[P]$ is a non-maximal prime ideal of $\mathcal{R}^*L$. We show that this ideal is free. By complete regularity,

$$\begin{aligned} I &= \bigvee_{\beta L}\{\operatorname{coz}\gamma \mid \gamma \in \mathcal{R}(\beta L) \text{ and } \operatorname{coz}\gamma \prec\!\prec I\} \\ &= \bigvee_{\beta L}\{\operatorname{coz}\gamma \mid \gamma \in \mathbf{O}_I\}. \end{aligned}$$

Applying the frame homomorphism j_L, we obtain

$$\begin{aligned} j_L(I) &= \bigvee\{j_L(\operatorname{coz}\gamma) \mid \gamma \in \mathbf{O}_I\} \\ &= \bigvee\{\operatorname{coz}(j_L\gamma) \mid \gamma \in \mathbf{O}_I\} \\ &= \bigvee\{\operatorname{coz}(\phi_L(\gamma)) \mid \gamma \in \mathbf{O}_I\} \\ &\leq \bigvee\{\operatorname{coz}\tau \mid \tau \in \phi_L[\mathbf{O}_I]\}. \end{aligned}$$

Since $\operatorname{coz}(\phi_L^{-1}(\alpha)) \leq I$ and $\bigvee \operatorname{coz}(\phi_L^{-1}(\alpha)) = 1$, it follows that $j_L(I) = 1$, and hence

$$\bigvee\{\operatorname{coz}\tau \mid \tau \in \phi_L[\mathbf{O}_I]\} = 1.$$

Therefore $\phi_L[\mathbf{O}_I]$ is free, and hence $\phi_L[P]$ is free since $\phi_L[\mathbf{O}_I] \subseteq \phi_L[P]$. ∎

As mentioned in the introduction, the classical antecedent of this result appears in [7]. The main thrust of the proposition is that the non-maximal prime ideal we claimed (and proved) to exist was to be free. Without imposing freeness, the result becomes immediate for the following reason. Recall that a *P-frame* is a frame in which every cozero element is complemented. If $\mathcal{R}^*L$ has no non-maximal prime ideal, then every prime ideal of $\mathcal{R}(\beta L)$ is maximal, and hence βL is a P-frame by [5, Proposition 2.9]. But, as observed in [5], every pseudocompact P-frame is finite; therefore βL is finite, so that L is also finite, contradicting the hypothesis that L is not pseudocompact.

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