Property (aw) and perturbations

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Abstract

A bounded linear operator $T \in \mathbf{L}(\mathbb{X})$ acting on a Banach space satisfies property (aw), a variant of Weyl's theorem, if the complement in the spectrum $\sigma(T)$ of the Weyl spectrum $\sigma_w(T)$ is the set of all isolated points of the approximate-point spectrum which are eigenvalues of finite multiplicity. In this article we consider the preservation of property (aw) under a finite rank perturbation commuting with T, whenever T is polaroid, or T has analytical core $K(T - \lambda_0 I) = \{0\}$ for some $\lambda_0 \in \mathbb{C}$. The preservation of property (aw) is also studied under commuting nilpotent or under injective quasi-nilpotent perturbations or under Riesz perturbations. The theory is exemplified in the case of some special classes of operators.

1 Introduction

Throughout this paper, $\mathbb{X}$ denotes an infinite-dimensional complex Banach space, $\mathbf{L}(\mathbb{X})$ the algebra of all bounded linear operators on $\mathbb{X}$. For an operator $T \in \mathbf{L}(\mathbb{X})$ we shall denote by $\alpha(T)$ the dimension of the *kernel* $\ker(T)$, and by $\beta(T)$ the codimension of the *range* $T(\mathbb{X})$. Let

$$\Phi_+(\mathbb{X}) := \{T \in \mathbf{L}(\mathbb{X}) : \alpha(T) < \infty \quad \text{and } T(\mathbb{X}) \text{ is closed}\}$$

be the class of all *upper semi-Fredholm* operators, and let

$$\Phi_-(\mathbb{X}) := \{T \in \mathbf{L}(\mathbb{X}) : \beta(T) < \infty\}$$

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be the class of all *lower semi-Fredholm* operators. The class of all *semi-Fredholm* operators is defined by $\Phi_{\pm}(\mathbb{X}) := \Phi_{+}(\mathbb{X}) \cup \Phi_{-}(\mathbb{X})$, while the class of all *Fredholm* operators is defined by $\Phi(\mathbb{X})) := \Phi_{+}(\mathbb{X}) \cap \Phi_{-}(\mathbb{X})$. If $T \in \Phi_{\pm}(\mathbb{X})$, the *index* of T is defined by

$$ind(T) := \alpha(T) - \beta(T).$$

Recall that a bounded operator T is said *bounded below* if it injective and has closed range. Evidently, if T is bounded below then $T \in \Phi_{+}(\mathbb{X})$ and $ind(T) \leq 0$. Define

$$W_{+}(\mathbb{X}) : = \{T \in \Phi_{+}(\mathbb{X}) : ind(T) \leq 0\},$$

and

$$W_{-}(\mathbb{X}) : = \{T \in \Phi_{-}(\mathbb{X}) : ind(T) \geq 0\}.$$

The set of *Weyl* operators is defined by

$$W(\mathbb{X}) := W_{+}(\mathbb{X}) \cap W_{-}(\mathbb{X}) = \{T \in \Phi(\mathbb{X}) : ind(T) = 0\}.$$

The classes of operators defined above generate the following spectra. Denote by

$$\sigma_a(T) := \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not bounded below}\}$$

the *approximate point spectrum*, and by

$$\sigma_s(T) := \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not surjective}\}$$

the *surjectivity spectrum* of $T \in \mathbf{L}(\mathbb{X})$. The *Weyl spectrum* is defined by

$$\sigma_w(T) := \{\lambda \in \mathbb{C} : T - \lambda \notin W(\mathbb{X})\},$$

the *Weyl essential approximate point spectrum* is defined by

$$\sigma_{uw}(T) := \{\lambda \in \mathbb{C} : T - \lambda \notin W_{+}(\mathbb{X})\},$$

while the *Weyl essential surjectivity spectrum* is defined by

$$\sigma_{lw}(T) := \{\lambda \in \mathbb{C} : T - \lambda \notin W_{-}(\mathbb{X})\},$$

Obviously, $\sigma_w(T) = \sigma_{uw}(T) \cup \sigma_{lw}(T)$ and from basic Fredholm theory we have

$$\sigma_{uw}(T) = \sigma_{ws}(T^*) \qquad \sigma_{ws}(T) = \sigma_{uw}(T^*).$$

Note that $\sigma_{uw}(T)$ is the intersection of all approximate point spectra $\sigma_a(T+K)$ of compact perturbations K of T , while $\sigma_{lw}(T)$ is the intersection of all surjectivity spectra $\sigma_s(T+K)$ of compact perturbations K of T , see, for instance, [1, Theorem 3.65].

Recall that the *ascent*, $a(T)$, of an operator T is the smallest non-negative integer p such that $\ker(T^p) = \ker(T^{p+1})$. If such integer does not exist we put $a(T) = \infty$. Analogously, the *descent*, $d(T)$, of an operator T is the smallest non-negative integer q such that $T^q(\mathbb{X}) = T^{q+1}(\mathbb{X})$, and if such integer does not exist we put $d(T) = \infty$. It is well known that if $a(T)$ and $d(T)$ are both finite then $a(T) = d(T)$ [16, Proposition 1.49]. Moreover, $0 < a(T - \lambda I) = d(T - \lambda I) < \infty$

precisely when λ is a pole of the resolvent of T , see Dowson [16, Theorem 1.54].

The class of all *upper semi-Browder* operators is defined by

$$B_+(\mathbb{X}) := \{T \in \Phi_+(\mathbb{X}) : a(T) < \infty\},$$

while the class of all *lower semi-Browder* operators is defined by

$$B_-(\mathbb{X}) := \{T \in \Phi_+(\mathbb{X}) : d(T) < \infty\}.$$

The class of all *Browder* operators is defined by

$$B(\mathbb{X}) := B_+(\mathbb{X}) \cap B_-(\mathbb{X}) = \{T \in \Phi(\mathbb{X}) : a(T), d(T) < \infty\}.$$

We have

$$B(\mathbb{X}) \subseteq W(\mathbb{X}), \qquad B_+(\mathbb{X}) \subseteq W_+(\mathbb{X}), \qquad B_-(\mathbb{X}) \subseteq W_-(\mathbb{X}),$$

see [1, Theorem 3.4]. The *Browder spectrum* of $T \in \mathbf{L}(\mathbb{X})$ is defined by

$$\sigma_b(T) := \{\lambda \in \mathbb{C} : T - \lambda I \notin B(\mathbb{X})\},$$

the *upper Browder spectrum* is defined by

$$\sigma_{ub}(T) := \{\lambda \in \mathbb{C} : T - \lambda I \notin B_+(\mathbb{X})\},$$

and analogously the *lower Browder spectrum* is defined by

$$\sigma_{lb}(T) := \{\lambda \in \mathbb{C} : T - \lambda I \notin B_-(\mathbb{X})\}.$$

Clearly, $\sigma_b(T) = \sigma_{ub}(T) \cup \sigma_{lb}(T)$ and $\sigma_w(T) \subseteq \sigma_b(T)$.

The *single valued extension property* plays an important role in local spectral theory, see the recent monograph of Laursen and Neumann [23] and Aiena [1]. In this article we shall consider the following local version of this property, which has been studied in recent papers, [4, 22] and previously by Finch [18].

Let $Hol(\sigma(T))$ be the space of all functions that analytic in an open neighborhoods of $\sigma(T)$. Following [18] we say that $T \in \mathbf{L}(\mathbb{X})$ has the single-valued extension property (SVEP) at point $\lambda \in \mathbb{C}$ if for every open neighborhood U_λ of λ, the only analytic function $f : U_\lambda \longrightarrow \mathcal{H}$ which satisfies the equation $(T - \mu)f(\mu) = 0$ is the constant function $f \equiv 0$. It is well-known that $T \in \mathbf{L}(\mathbb{X})$ has SVEP at every point of the resolvent $\rho(T) := \mathbb{C} \setminus \sigma(T)$. Moreover, from the identity Theorem for analytic function it easily follows that $T \in \mathbf{L}(\mathbb{X})$ has SVEP at every point of the boundary $\partial\sigma(T)$ of the spectrum. In particular, T has SVEP at every isolated point of $\sigma(T)$. In [22, Proposition 1.8], Laursen proved that if T is of finite ascent, then T has SVEP.

The basic role of SVEP arises in local spectral theory since all decomposable operators enjoy this property. Recall $T \in \mathbf{L}(\mathbb{X})$ has the *decomposition property* (δ) if $\mathbb{X} = \mathcal{X}_T(\overline{U}) + \mathcal{X}_T(\overline{V})$ for every open cover $\{U, V\}$ of $\mathbb{C}$. Decomposable operators may be defined in several ways for instance as the union of the property (β) and the property (δ), see [23, Theorem 2.5.19] for relevant definitions. Note that the property (β) implies that T has SVEP, while the property (δ) implies SVEP for T^*, see [23, Theorem 2.5.19]. Every *generalized scalar* operator on a Banach space is decomposable, see [23] for relevant definitions and results. In particular, every spectral operators of finite type is decomposable [14, Theorem 3.6]. Also every operator $T \in \mathbf{L}(\mathbb{X})$ with totally disconnected spectrum is decomposable [23, Proposition 1.4.5].

The quasinilpotent part $H_0(T-\lambda I)$ and the analytic core $K(T-\lambda I)$ of $T-\lambda I$ are defined by

$$H_0(T-\lambda I) := \{x \in \mathbb{X} : \lim_{n \longrightarrow \infty} \|(T-\lambda I)^n x\|^{\frac{1}{n}} = 0\}.$$

and

$$K(T-\lambda I) = \{x \in \mathbb{X} : \text{there exists a sequence } \{x_n\} \subset \mathbb{X} \text{ and } \quad \delta > 0$$
$$\text{for which} \quad x = x_0, (T-\lambda I)x_{n+1} = x_n \text{and} \quad \|x_n\| \leq \delta^n \|x\| \text{for all} \quad n = 1, 2, \cdots\}.$$

We note that $H_0(T-\lambda I)$ and $K(T-\lambda I)$ are generally non-closed hyper-invariant subspaces of $T-\lambda I$ such that $(T-\lambda I)^{-p}(0) \subseteq H_0(T-\lambda I)$ for all $p = 0, 1, \cdots$ and $(T-\lambda I)K(T-\lambda I) = K(T-\lambda I)$. Recall that if $\lambda \in iso(\sigma(T))$, then $H_0(T-\lambda I) = \chi_T(\{\lambda\})$, where $\chi_T(\{\lambda\})$ is the global spectral subspace consisting of all $x \in \mathcal{H}$ for which there exists an analytic function $f : \mathbb{C} \setminus \{\lambda\} \longrightarrow \mathbb{X}$ that satisfies $(T-\mu)f(\mu) = x$ for all $\mu \in \mathbb{C} \setminus \{\lambda\}$, see, Duggal [17].

Theorem 1.1. *[3, Theorem 1.3] If $T \in \Phi_{\pm}(\mathbb{X})$ the following statements are equivalent:*

(i) T has SVEP at λ_0;

(ii) $a(T-\lambda_0 I) < \infty$;

(iii) $\sigma_a(T)$ does not cluster at λ_0;

(iv) $H_0(T-\lambda_0 I)$ is finite dimensional.

By duality we have

Theorem 1.2. *If $T \in \Phi_{\pm}(\mathbb{X})$ the following statements are equivalent:*

(i) T^* has SVEP at λ_0;

(ii) $d(T-\lambda_0 I) < \infty$;

(iii) $\sigma_s(T)$ does not cluster at λ_0.

Theorem 1.3. *[4, Theorem 1.3] Suppose that $T-\lambda I \in \Phi_{\pm}(\mathbb{X})$. If T has SVEP at λ then $ind(T-\lambda I) \leq 0$, while if T^* has SVEP at λ then $ind(T-\lambda I) \geq 0$.*

2 Property (aw) and SVEP

Let write *isoK* for the set of all isolated points of $K \subseteq \mathbb{C}$. For a bounded operator $T \in \mathbf{L}(\mathbb{X})$ set

$$\pi_0(T) := \sigma(T) \setminus \sigma_b(T) = \{\lambda \in \sigma(T) : T - \lambda I \in B(\mathbb{X})\}\,.$$

Note that every $\lambda \in \pi_0(T)$ is a pole of the resolvent and hence an isolated point of $\sigma(T)$, see [21, Proposition 50.2]. Moreover, $\pi_0(T) = \pi_0(T^*)$. Define

$$E_0(T) := \{\lambda \in iso\sigma(T) : 0 < \alpha(T - \lambda I) < \infty\}\,.$$

Obviously,

$$\pi_0(T) \subseteq E_0(T) \qquad \text{for every } T \in \mathbf{L}(\mathbb{X}).$$

For a bounded operator $T \in \mathbf{L}(\mathbb{X})$ let us define

$$E_0^a(T) := \{\lambda \in iso\sigma_a(T) : 0 < \alpha(T - \lambda I) < \infty\}\,,$$

and

$$\pi_0^a(T) := \sigma_a(T) \setminus \sigma_{ub}(T) = \{\lambda \in \sigma_a(T) : T - \lambda I \in B_+(\mathbb{X})\}\,.$$

Lemma 2.1. *[4] For every $T \in \mathbf{L}(\mathbb{X})$ we have*

(a) $\pi_0(T) \subseteq \pi_0^a(T) \subseteq E_0^a(T)$ *and*

(b) $E_0(T) \subseteq E_0^a(T)$.

Following Harte and W.Y. Lee [19], we shall say that T satisfies *Browder's theorem* if

$$\sigma_w(T) = \sigma_b(T),$$

while, $T \in \mathbf{L}(\mathbb{X})$ is said to satisfy *a-Browder's theorem* if

$$\sigma_{uw}(T) = \sigma_{ub}(T).$$

Browder's theorem and a-Browder's theorem may be characterized by localized SVEP in the following way:

Lemma 2.2. *[5] If $T \in \mathbf{L}(\mathbb{X})$ the following equivalences hold:*

(i) T satisfies Browder's theorem $\Leftrightarrow$ T has SVEP at every $\lambda \notin \sigma_w(T)$;

(ii) T satisfies a-Browder's theorem $\Leftrightarrow$ T has SVEP at every $\lambda \notin \sigma_{uw}(T)$.
Moreover, the following statements hold:

(iii) If T has SVEP at every $\lambda \notin \sigma_{lw}(T)$ then a-Browder's theorem holds for T^*.

(iv) If T^* has SVEP at every $\lambda \notin \sigma_{uw}(T)$ then a-Browder's theorem holds for T.

Obviously,
a-Browder's theorem holds for $T \Rightarrow$ Browder's theorem holds for T and the converse is not true.

Remark 2.3. The opposite implications of (iii) and (iv) in Theorem 2.2 in general do not hold. In [2] it is given an example of unilateral weighted left shift on $\ell^q(\mathbb{N})$ which shows that these implications cannot be reversed.

By Lemma 2.2 we also have
T or T^* has SVEP $\Rightarrow$ a-Browder's theorem holds for both T, T^*.

Following Coburn [13], we say that Weyl's theorem holds for $T \in \mathbf{L}(\mathbb{X})$ if

$$\Delta(T) := \sigma(T) \setminus \sigma_w(T) = E_0(T).$$

An approximate point version of Weyl's theorem is a-Weyl's theorem: according Rakočević [30] an operator $T \in \mathbf{L}(\mathbb{X})$ is said to satisfy a-Weyl's theorem if

$$\Delta_a(T) := \sigma_a(T) \setminus \sigma_{uw}(T) = E_0^a(T).$$

Since $T - \lambda I \in W_+(\mathbb{X})$ implies that $(T - \lambda I)(\mathbb{X})$ is closed, we can write

$$\Delta_a(T) = \{\lambda \in \mathbb{C} : T - \lambda I \in W_+(\mathbb{X}), 0 < \alpha(T - \lambda I)\}.$$

It should be noted that the set $\Delta_a(T)$ may be empty. This is, for instance, the case of a right shift on $\ell^2(\mathbb{N})$, see [3]. Furthermore,

$$a\text{-Weyl's theorem holds for } T \Rightarrow \text{Weyl's theorem holds for } T,$$

while the converse in general does not hold.

Definition 2.4. A bounded operator $T \in \mathbf{L}(\mathbb{X})$ is said to satisfy property (w) if

$$\Delta_a(T) = \sigma_a(T) \setminus \sigma_{uw}(T) = E_0(T).$$

Definition 2.5. A bounded operator $T \in \mathbf{L}(\mathbb{X})$ is said to satisfy property (aw) if

$$\Delta(T) = \sigma(T) \setminus \sigma_w(T) = E_0^a(T).$$

Following [11], we say that $T \in \mathbf{L}(\mathbb{X})$ satisfies property (ab) if $\Delta(T) = \pi_0^a(T)$. It is shown [11] that an operator $T \in \mathbf{L}(\mathbb{X})$ satisfies property (aw) satisfying property (ab) but the converse is not true in general.

Lemma 2.6. *Let $T \in \mathbf{L}(\mathbb{X})$. Then*

(i) T satisfies property (ab) if and only if T satisfies Browder's theorem and $\pi_0(T) = \pi_0^a(T)$, see [11, Corollary 2.6].

(ii) T satisfies property (aw) if and only if T satisfies property (ab) and $E_0^a(T) = \pi_0^a(T)$, see [11, Theorem 3.6].

Theorem 2.7. *Let $T \in \mathbf{L}(\mathbb{X})$. If T satisfies property (aw) then T satisfies Weyl's theorem.*

Proof. If T satisfies property (aw) then T satisfies Browder's theorem and $\pi_0(T) = E_0^a(T)$. Hence $\Delta(T) = \pi_0(T) = E_0^a(T)$. As $\pi_0(T) \subseteq \pi_0^a(T) \subseteq E_0^a(T)$ is always verified. Therefore, $\Delta(T) = E_0(T)$. ∎

The converse of of Theorem 2.7 is not true in general as shown by the following example.

Example 2.8. Let $R \in \ell^2(\mathbb{N})$ be the unilateral right shift and

$$U(x_1, x_2, \cdots) := (0, x_2, x_3, \cdots) \quad \text{for all } (x_n) \in \ell^2(\mathbb{N}).$$

If $T := R \oplus U$ then $\sigma(T) = \sigma_w(T) = \mathbf{D}(0,1)$, where $\mathbf{D}(0,1)$ is the unit disc of $\mathbb{C}$. So $iso\sigma(T) = E_0(T) = \emptyset$. Moreover, $\sigma_a(T) = \mathbf{C}(0,1) \cup \{0\}$, where $\mathbf{C}(0,1)$ is the unit circle of $\mathbb{C}$, $\sigma_{uw}(T) = \mathbf{D}(0,1)$, so T does not satisfy property (aw), since $\Delta(T) = \emptyset \neq E_0^a(T) = \{0\}$. On the other hand, T satisfies a-Weyl's theorem, since $\Delta_a(T) = E_0^a(T)$ and hence satisfies Weyl's theorem.

Proposition 2.9. *Let $T \in \mathbf{L}(\mathbb{X})$. Then property (aw) holds for T if and only if T satisfies Weyl's theorem and $\pi_0(T) = E_0^a(T)$.*

Proof. If T satisfies property (aw) then it follows from Theorem 2.7 that T satisfies Weyl's theorem and from Lemma 2.6 that $\pi_0(T) = \pi_0^a(T) = E_0^a(T)$. For the converse, assume that T satisfies Weyl's theorem and $\pi_0(T) = E_0^a(T)$. Then T satisfies Browder's theorem and $\pi_0(T) = E_0(T)$. Hence $\Delta(T) = E_0^a(T)$. That is, T satisfies property (aw). ■

Define

$$\Lambda(T) := \{\lambda \in \Delta_a(T) : ind(T - \lambda I) < 0\}. \tag{2.1}$$

Clearly

$$\Delta_a(T) = \Delta(T) \cup \Lambda(T) \quad \text{and} \quad \Lambda(T) \cap \Delta(T) = \emptyset. \tag{2.2}$$

Proposition 2.10. *Suppose that $T \in \mathbf{L}(\mathbb{X})$ is decomposable. Then T satisfies property (aw) if and only if T satisfies Weyl's theorem.*

Proof. If T is decomposable then both T and T^* have SVEP. This, by Theorem 1.3 entails that $T - \lambda I$ has index zero for every $\lambda \in \Delta_a(T) = \Delta(T)$, and hence $\Lambda(T) = \emptyset$. Property (aw) implies Weyl's theorem for every operator $T \in \mathbf{L}(\mathbb{X})$. For the converse, if T satisfies Weyl's theorem then $\Delta(T) = E_0(T)$ and since T^* has SVEP then $E_0(T) = E_0^a(T)$, hence the result. ■

As a consequence of Proposition 2.10, we have that for a bounded operator $T \in \mathbf{L}(\mathbb{X})$ having totally disconnected spectrum then property (aw) and Weyl's theorem are equivalent.

A bounded operator $T \in \mathbf{L}(\mathbb{X})$ is said to have property $H(p)$ if for all $\lambda \in \mathbb{C}$ there exists a $p := p(\lambda)$ such that:

$$H_0(T - \lambda I) = \ker(T - \lambda I)^p. \tag{2.3}$$

Let $f(T)$ be defined by means of the classical functional calculus. In [27] it has been proved that if $T \in \mathbf{L}(\mathbb{X})$ has property $H(p)$ then $f(T)$ and $f(T^*)$ satisfy Weyl's theorem.

Proposition 2.11. *If $T \in \mathbf{L}(\mathbb{X})$ is generalized scalar then property (aw) holds for both T and T^*. In particular, property (aw) holds for every spectral operator of finite type.*

Proof. Every generalized scalar operator T is decomposable and hence also the dual T^* is decomposable, see [23, Theorem 2.5.3]. Moreover, every generalized scalar operator has property $H(p)$ [27, Example 3], so Weyl's theorem holds for both T and T^*. By Proposition 2.10 it then follows that both T and T^* satisfy property (aw). The second statement is clear: every spectral operators of finite type is generalized scalar. ∎

The following example show that property (aw) and property (w) are independent.

Example 2.12. Let T be the hyponormal operator T given by the direct sum of the 1-dimensional zero operator and the unilateral right shift R on $\ell^2(\mathbb{N})$. Then $\sigma(T) = \mathbf{D}(0,1)$, $\mathbf{D}(0,1)$ the closed unit disc in $\mathbb{C}$. Moreover, 0 is an isolated point of $\sigma_a(T) = \mathbf{C}(0,1) \cup \{0\}$, $\mathbf{C}(0,1)$ the unit circle of $\mathbb{C}$, and $0 \in E_0^a(T)$ while $0 \notin \pi_0^a(T) = \emptyset$, since $a(T) = a(R) = \infty$. Hence, by Theorem 2.4 of [4], T does not satisfy a-Weyl's theorem. Now $\pi_0(T) = E_0(T) = \emptyset$, since $\sigma(T)$ has no isolated points, $\pi^a(T) = E_0(T)$. Since every hyponormal operator has SVEP we also know that a-Browder's theorem holds for T, so from Theorem 2.7 of [4] we see that property (w) holds for T. On the other hand, $\sigma_w(T) = \mathbf{D}(0,1)$, then $0 \in E_0^a(T) \neq \Delta(T) = \emptyset$. Therefore, T does not satisfy property (aw). Note that $\Delta(T) = E_0(T) = \emptyset$. That is, T satisfies Weyl's theorem.

The next result shows that property (w) and property (aw) are equivalent in presence of SVEP.

Theorem 2.13. *Let $T \in \mathbf{L}(\mathbb{X})$. Then the following equivalences holds:*

(i) If T^* has SVEP, the property (aw) holds for T if and only if the property (w) holds for T.

(ii) If T has SVEP, the property (aw) holds for T^* if and only if the property (w) holds for T^*.

Proof. (i) The SVEP of T^* implies that $\sigma_a(T) = \sigma(T)$, see [1, Corollary 2.5], $\sigma_{uw}(T) = \sigma_w(T) = \sigma_b(T)$, see [8, Theorem 2.6] so $E_0^a(T) = E_0(T)$, and hence $\Delta_a(T) = \Delta(T)$. Therefore,the property (aw) holds for T if and only if the property (w) holds for T.
(ii) If T has SVEP then $\sigma(T^*) = \sigma(T) = \sigma_s(T) = \sigma_a(T^*)$, see [1, Corollary 2.5], $\sigma_{uw}(T^*) = \sigma_w(T) = \sigma_b(T)$, see [8, Theorem 2.6] and hence $E_0(T^*) = E_0^a(T^*)$. Therefore, $\Delta(T^*) = \Delta_a(T^*)$. Therefore, the property (aw) holds for T^* if and only if the property (w) holds for T^*. ∎

Example 2.8 shows that a-Weyl's theorem does not imply property (aw). But in presence of SVEP a-Weyl's theorem, Weyl's theorem and property (aw) are equivalent as shown by the following result.

Theorem 2.14. *Let $T \in \mathbf{L}(\mathbb{X})$. Then the following equivalences holds:*

(i) If T^* has SVEP, the property (aw) holds for T if and only if Weyl's theorem holds for T , and this is the case if and only if a-Weyl's theorem holds for T.

(ii) If T has SVEP, the property (aw) holds for T^* if and only if Weyl's theorem holds for T^*, and this is the case if and only if a-Weyl's theorem holds for T^*.

Proof. (i) The SVEP of T^* implies that $\sigma_a(T) = \sigma(T)$, see [1, Corollary 2.5], $\sigma_{uw}(T) = \sigma_w(T) = \sigma_b(T)$, see [8, Theorem 2.6] so $E_0^a(T) = E_0(T)$, and hence $\Delta_a(T) = \Delta(T)$. Furthermore, by [1, Corollary 3.53] we also have $\sigma_{ub}(T) = \sigma_w(T)$ from which it follows that $E_0^a(T) = \sigma_a(T) \setminus \sigma_{ub}(T) = \pi_0^a(T)$. Since the SVEP for T^* implies a-Browder's theorem for T we then conclude, by part (ii) of Theorem 2.4 of [4], that a-Weyl's theorem hold s for T. Hence the equivalence follows.
(ii) If T has SVEP then $\sigma(T^*) = \sigma(T) = \sigma_s(T) = \sigma_a(T^*)$, see [1, Corollary 2.5], $\sigma_{uw}(T^*) = \sigma_w(T) = \sigma_b(T)$, see [8, Theorem 2.6] and hence $E_0(T^*) = E_0^a(T^*)$. Therefore, $\Delta(T^*) = \Delta_a(T^*)$. Moreover, by [1, Corollary 3.53] we also have

$$\sigma_w(T^*) = \sigma_w(T) = \sigma_{lb}(T) = \sigma_{ub}(T^*),$$

from which it easily follows that $\pi_0^a(T^*) = E_0^a(T^*)$. The SVEP for T implies that T^* satisfies a-Browder's theorem, so by part (ii) of Theorem 2.4 of [4], a-Weyl's theorem for T^*. Hence the equivalence follows. ■

Corollary 2.15. *If T is generalized scalar then property (aw) holds for both $f(T)$ and $f(T^*)$ for every $f \in Hol(\sigma(T))$.*

Proof. Since T has property $H(p)$ then Weyl's theorem holds for $f(T)$ and $f(T^*)$, see [27, Corollary 3.6]. Moreover, T and T^* being decomposable, both T and T^* have SVEP, hence also $f(T)$ and $f(T^*) = f(T)^*$ have SVEP by Theorem 2.40 of [1]. By Theorem 2.14 it then follows that property (aw) holds for both $f(T)$ and $f(T^*)$. ■

Remark 2.16. Corollary 2.15 applies to a large number of the classes of operators defined in Hilbert spaces. In [27] Oudghiri observed that every sub-scalar operator T (i.e., T is similar to a restriction of a generalized scalar operator to one of its closed invariant subspaces) has property $H(p)$. Consequently, property $H(p)$ is satisfied by p-hyponormal operators and log-hyponormal operators [24, Corollary 2], w-hyponormal operators [25], M-hyponormal operators [23, Proposition 2.4.9], and totally paranormal operators [7]. Also totally $*$-paranormal operators have property $H(1)$ [20].

An operator $T \in \mathbf{L}(\mathbb{X})$ is said to be polaroid if every isolated point of $\sigma(T)$ is a pole of the resolvent operator $(T - \lambda I)^{-1}$, or equivalently $a(T - \lambda I) = d(T - \lambda I) < \infty$, see [21, Proposition 50.2]. An operator $T \in \mathbf{L}(\mathbb{X})$ is said to be a-polaroid if every isolated point of $\sigma_a(T)$ is a pole of the resolvent operator $(T - \lambda I)^{-1}$, or equivalently $a(T - \lambda I) = d(T - \lambda I) < \infty$, see [21, Proposition 50.2]. Clearly,

$$T\ a\text{-polaroid} \Rightarrow T \text{ polaroid}. \tag{2.4}$$

and the opposite implication is not generally true.

Theorem 2.17. *Suppose that T is a-polaroid. Then property* (w) *holds for T if and only if T satisfies property* (aw).

Proof. Note first that if T is a-polaroid then $\pi_0(T) = E_0^a(T)$. In fact, if $\lambda \in E_0^a(T)$ then λ is isolated in $iso\sigma_a(T)$ and hence $a(T - \lambda I) = d(T - \lambda I) < \infty$. Moreover, $\alpha(T - \lambda I) < \infty$, so by Theorem 3.4 of [1] it follows that $\beta(T - \lambda I)$ is also finite, thus $\lambda \in \pi_0(T)$. This shows that $E_0^a(T) \subseteq \pi_0(T)$, and consequently by Lemma 2.1 we have $\pi_0(T) = E_0^a(T)$. Now, if T satisfies property (w) theorem then $\Delta_a(T) = E_0(T)$, and since Weyl's theorem holds for T we also have by Theorem 2.4 of [4] that $\pi_0(T) = E_0(T)$. Hence $\Delta(T) = E_0^a(T)$. Therefore, property (aw) holds for T. Conversely, if T satisfies property (aw) then $\Delta(T) = E_0^a(T)$. Since by Theorem 2.7 T satisfies Weyl's theorem we also have, by Theorem 2.4 of [4], $E_0(T) = \pi_0(T) = E_0^a(T)$. If $\lambda \in \Delta_a(T)$, as T satisfies property (aw) then $\lambda \in E_0(T)$. Since $\Delta(T) \subseteq \Delta_a(T)$ it then follows if $\lambda \in E_0(T) = \Delta(T)$ then $\lambda \in \Delta_a(T)$. So $\Delta_a(T) = E_0(T)$. Therefore, T satisfies property (w). ∎

Recall that a bounded operator $T \in \mathbf{L}(\mathbb{X})$ is said to be isoloid (respectively, a-isoloid) if every isolated point of $\sigma(T)$ (respectively, every isolated point of $\sigma_a(T)$) is an eigenvalue of T. Every a-isoloid operator is isoloid. This is easily seen: if T is a-isoloid and $\lambda \in iso\sigma(T)$ then $\lambda \in \sigma_a(T)$ or $\lambda \notin \sigma_a(T)$. In the first case $T - \lambda I$ is bounded below, in particular upper semi-Fredholm. The SVEP of both T and T^* at λ then implies that $a(T - \lambda I) = d(T - \lambda I) < \infty$, so λ is a pole. Obviously, also in the second case λ is a pole, since by assumption T is a-isoloid.

Theorem 2.18. *Suppose that T is a-polaroid and that* T^* *has SVEP. Then* $f(T)$ *satisfies property* (aw) *for all* $f \in Hol(\sigma(T))$.

Proof. If T is a-polaroid then T is a-isoloid (i.e., every isolated point of $\sigma_a(T)$ is an eigenvalue of T). The SVEP for T^* ensures that the spectral mapping theorem holds for $\sigma_{uw}(T)$, i.e., if $f \in Hol(\sigma(T))$ then $f(\sigma_{uw}(T)) = \sigma_{uw}(f(T))$, [1, Theorem 3.66]. By Theorem 5.4 of [15] then $f(T)$ satisfies a-Weyl's theorem, and since $f(T^*) = f(T)^*$ has SVEP from Theorem 2.14 we conclude that property (aw) holds for $f(T)$. ∎

Theorem 2.19. *Suppose that* $T \in \mathbf{L}(\mathbb{X})$. *Then the following statements hold:*

(i) If T is polaroid and T has SVEP then property (aw) *holds for* T^*.

(ii) If T is polaroid and T^* *has SVEP then property* (aw) *holds for T.*

Proof. (i) By Theorem 2.14 it suffices to show that Weyl's theorem holds for T^*. The SVEP ensures that Browder's theorem holds for T^*. We prove that $\pi_0(T^*) = E_0(T^*)$. Let $\lambda \in E_0(T^*)$ Then $\lambda \in iso\sigma(T^*) = iso\sigma(T)$ and the polaroid assumption implies that λ is a pole of the resolvent, or equivalently $a(T - \lambda I) = d(T - \lambda I) < \infty$. If P denotes the spectral projection associated with $\{\lambda\}$ we have $(T - \lambda I)^p(\mathbb{X}) = \ker(P)$ [1, Theorem 3.74], so $(T - \lambda I)^p(\mathbb{X})$ is closed, and hence also $(T^* - \lambda I)^p(\mathbb{X}^*)$ is closed. Since $\lambda \in E_0(T^*)$ then $\alpha(T^* - \lambda I^*) < \infty$ and this implies $(T^* - \lambda I)^p(\mathbb{X}^*) < \infty$, from which we conclude that $(T^* - \lambda I^*)^p \in \Phi_+(\mathbb{X}^*)$, hence $T^* - \lambda I^* \in \Phi_+(\mathbb{X}^*)$, and consequently $T - \lambda I \in \Phi_-(\mathbb{X})$. Therefore $\beta(T - \lambda I) < \infty$ and since $a(T - \lambda I) = d(T - \lambda I) < \infty$ by Theorem 3.4 of [1]

we then conclude that $\alpha(T-\lambda I)<\infty$. Hence $\lambda\in\pi_0(T)=\pi_0(T^*)$. This proves that $E_0(T^*)\subseteq\pi_0(T^*)$, and since by Lemma 2.1 the opposite inclusion is satisfied by every operator we may conclude that $E_0(T^*)=\pi_0(T^*)$. By Theorem 2.4 of [4] then T^* satisfies Weyl's theorem.
(ii) The SVEP for T^* implies that Browder's theorem holds for T. Again by Theorem 2.14 it suffices to show that T satisfies Weyl's theorem, and hence by Lemma 2.1 and Theorem 2.4 of [4] we need only to prove that $E_0(T)=\pi_0(T)$. Let $\lambda\in E_0(T)$. Then $\lambda\in iso\sigma(T)$ and since T is polaroid then $a(T-\lambda I)=d(T-\lambda I)<\infty$. Since $\alpha(T-\lambda I)<\infty$ we then have $\beta(T-\lambda I)<\infty$ and hence $\lambda\in\pi_0(T)$. Hence $E_0(T)\subseteq\pi_0(T)$ and by Lemma 2.14 we then conclude that $E_0(T)=\pi_0(T)$. ∎

Remark 2.20. Part (i) of Theorem 2.19 shows that the dual T^* of a multiplier $T\in M(A)$ of a commutative semi-simple Banach algebra A has property (aw), since every multiplier $T\in M(A)$ of a commutative semi-simple Banach algebra satisfies Weyl's theorem and is polaroid, see [1, Theorem 4.36].

Theorem 2.21. *Let $T\in\mathbf{L}(\mathbb{X})$ be such that there exists $\lambda_0\in\mathbb{C}$ such that $K(T-\lambda_0 I)=\{0\}$ and $\ker(T-\lambda_0 I)=\{0\}$. Then property (aw) holds for $f(T)$ for all $f\in Hol(\sigma(T))$.*

Proof. We know from [9, Lemma 2.4] that $\sigma_p(T)=\emptyset$, so T has SVEP. We show that also $\sigma_p(f(T))=\emptyset$. Let $\mu\in\sigma(f(T))$ and write $f(\lambda)-\mu=p(\lambda)g(\lambda)$, where g is analytic on an open neighborhood U containing $\sigma(T)$ and without zeros in $\sigma(T)$, p a polynomial of the form

$$p(\lambda)=(\lambda-\lambda_1)^{m_1}(\lambda-\lambda_2)^{m_2}\cdots(\lambda-\lambda_n)^{m_n},$$

with distinct roots $\lambda_1,\lambda_2,\cdots,\lambda_n$ lying in $\sigma(T)$. Then

$$f(T)-\mu I=(T-\lambda_1 I)^{m_1}(T-\lambda_2 I)^{m_2}\cdots(T-\lambda_n I)^{m_n}g(T)$$

Since $g(T)$ is invertible, $\sigma_p(T)=\emptyset$ implies that $\ker(f(T)-\mu I)=\{0\}$ for all $\mu\in\mathbb{C}$, so $\sigma_p(f(T))=\emptyset$. Since T has SVEP then $f(T)$ has SVEP, see Theorem 2.40 of [1], so that a-Browder's theorem holds for $f(T)$ and hence Browder's theorem holds for $f(T)$. To prove that property (aw) holds for $f(T)$, by Lemma 2.6 it then suffices to prove that

$$E_0^a(f(T))=\pi_0(f(T)).$$

Obviously, the condition $\sigma_p(f(T))=\emptyset$ entails that $E_0(f(T))=E_0^a(f(T))=\emptyset$. On the other hand, the inclusion $\pi_0(f(T))\subseteq E_0^a(f(T))$ holds for every operator $T\in\mathbf{L}(\mathbb{X})$, so also $\pi_0(f(T))$ is empty. By Lemma 2.6 it then follows that $f(T)$ satisfies property (aw). ∎

3 Property (aw) under perturbations

In this section we shall give some conditions for which property (aw) is preserved under commuting finite-rank or quasinilpotent perturbations.

As property (w), property (aw) is not preserved under finite rank perturbations (also commuting finite rank perturbations).

Example 3.1. Let $T := Q \oplus I$ defined on $\mathbb{X} \oplus \mathbb{X}$, where Q is an injective quasi-nilpotent operator. It is easily seen that T satisfies a-Weyl's theorem. Define $K := 0 \oplus (-P)$, where P is a finite rank projection. Then $TK = KT$, and since T^* has a finite spectrum then T^* has SVEP, hence $T^* + K^*$ has SVEP, by Lemma 2.8 of [6]. Therefore $\sigma(T+K) = \sigma_a(T+K)$, by Corollary 2.45 of [1]. On the other hand it is easy to see that $0 \in \sigma(T+K) \cap \sigma_w(T+K)$, so $0 \notin \sigma(T+K) \setminus \sigma_w(T+K)$, while $0 \in E_0(T+K) = E_0^a(T+K)$, thus $T+K$ does not verify property (aw).

Theorem 3.2. *Suppose that $T \in \mathbf{L}(\mathbb{X})$ is polaroid and K is a finite rank operator commuting with T.*

(i) If T^* has SVEP then $f(T)+K$ satisfies property (aw) for all $f \in Hol(\sigma(T))$.

(ii) If T has SVEP then $f(T^*)+K^*$ satisfies property (aw) for all $f \in Hol(\sigma(T))$.

Proof. (i) By [1, Corollary 2.45] we have $\sigma_a(T) = \sigma(T)$, so T is a-polaroid and hence a-isoloid. By Theorem 2.18 it then follows that $f(T)$ has property (aw) for all $f \in Hol(\sigma(T))$. Now, by [1, Theorem 2.40] $f(T^*) = f(T)^*$ has SVEP, so that, by Theorem 2.14 a-Weyl's theorem holds for $f(T)$. Since $f(T)$ and K commutes, by Theorem 3.2 of [6] we then obtain that $f(T)+K$ satisfies a-Weyl's theorem. By Lemma 2.8 of [5] $f(T)^* + K^* = (f(T)+K)^*$ has SVEP. This implies that property (aw) and a-Weyl's theorem for $f(T)+K$ are equivalent, again by Theorem 2.14, so the proof is complete.
(ii) The argument is analogous to that of part (i). Just observe that $\sigma_a(T^*) = \sigma(T^*)$ by [1, Corollary 2.45], so that T^* is a-polaroid, hence a-isoloid. Moreover, by Theorem 2.18 it then follows that $f(T^*)$ has property (aw) for all $f \in Hol(\sigma(T))$. By Theorem 2.40 of [1] $f(T)$ has SVEP, so that, so, by Theorem 2.14 a-Weyl's theorem holds for $f(T^*)$. Since $f(T^*)$ and K^* commutes, by Theorem 3.2 of [6] we then obtain that $f(T^*)+K^*$ satisfies a-Weyl's theorem. Again by Lemma 2.8 of [5] $f(T)+K$ has SVEP, so that (aw) and a-Weyl's theorem for $f(T^*)+K^*$ are equivalent, by Theorem 2.14. ∎

The basic role of SVEP arises in local spectral theory since for all decomposable operators both T and T^* have SVEP. Every generalized scalar operator on a Banach space is decomposable (see [23] for relevant definitions and results). In particular, every spectral operators of finite type is decomposable.

Corollary 3.3. *Suppose that $T \in \mathbf{L}(\mathbb{X})$ is generalized scalar and K is a finite rank operator commuting with T. Then property (aw) holds for both $f(T)+K$ and $f(T^*)+K^*$. In particular, this is true for every spectral operator of finite type.*

Proof. Both T and T^* have SVEP. Moreover, every generalized scalar operator T has property $H(p)$ [27, Example 3], so T is polaroid. The second statement is clear: every spectral operators of finite type is generalized scalar. ∎

The next results deal with quasi-nilpotent perturbations. We first recall two well-known results: if Q a quasi-nilpotent operator commuting with $T \in \mathbf{L}(\mathbb{X})$, then

$$\sigma_a(T) = \sigma_a(T+Q) \quad \text{and} \quad \sigma_{uw}(T) = \sigma_{uw}(T+Q). \tag{3.1}$$

Since $\sigma(T+Q) = \sigma(T)$ and $\sigma_b(T+Q) = \sigma_b(T)$ (for the last equality see [32]), we then have $\pi_0(T+Q) = \pi_0(T)$.

Lemma 3.4. *Let $T \in \mathbf{L}(\mathbb{X})$. If $N \in \mathbf{L}(\mathbb{X})$ is a nilpotent operator commuting with T, then $E_0^a(T+N) = E_0^a(T)$.*

Proof. Let $\lambda \in E_0^a(T)$ be arbitrary. There is no loss of generality if we assume that $\lambda = 0$. As N is nilpotent we know that $\sigma_a(T+N) = \sigma_a(T)$, thus $0 \in iso\sigma_a(T+N)$. Let $m \in \mathbb{N}$ be such that $N^m = 0$. If $x \in \ker(T)$, then $(T+N)^m(x) = \sum_{k=0}^{m} C_k^m T^k N^{m-k}(x) = 0$. So $\ker(T) \subset \ker(T+N)^m$. As $0 < \alpha(T) < \infty$, it follows that $0 < \alpha((T+N)^m) < \infty$ and this implies that $0 < \alpha(T+N) < \infty$. Hence $0 \in E_0^a(T+N)$. So $E_0^a(T) \subseteq E_0^a(T+N)$. By symmetry we have $E_0^a(T) = E_0^a(T+N)$. ∎

It is easily seen that property (aw) is transmitted under commuting nilpotent perturbations N.

Theorem 3.5. *If $T \in \mathbf{L}(\mathbb{X})$ satisfies property (aw),$N \in \mathbf{L}(\mathbb{X})$ is a nilpotent operator commuting with T then $T+N$ satisfies property (aw).*

Proof. If T satisfies property (aw) then T satisfies Browder's theorem, so by Lemma 2.6, $E_0^a(T) = \pi_0(T)$. Hence

$$E_0^a(T+N) = E_0^a(T) = \pi_0(T+N) = \pi_0(T).$$

Since $\sigma(T+N) = \sigma(T)$ and $\sigma_w(T+N) = \sigma_w(T)$, we have

$$\sigma(T+N) \setminus \sigma_w(T+N) = \sigma(T) \setminus \sigma_w(T) = E_0^a(T) = E_0^a(T+N).$$

That is, $T+N$ satisfies property (aw). ∎

Generally, property (aw) is not transmitted from T to a quasi-nilpotent perturbation $T+Q$. In fact, if $Q \in \ell^2(\mathbb{N})$ is defined by

$$Q(x_1, x_2, \cdots) = \left(\frac{x_2}{2}, \frac{x_3}{3}, \cdots\right) \quad \text{for all } (x_n) \in \ell^2(\mathbb{N}).$$

Then Q is quasi-nilpotent, $\sigma(Q) = \sigma_w(Q) = \{0\}$ and

$$\{0\} = E_0^a(Q) \neq \sigma(Q) \setminus \sigma_w(Q)$$

Take $T = 0$. Clearly, T satisfies property (aw) but $T+Q = Q$ fails this property. Note that Q is not injective.

Theorem 3.6. *Suppose that for $T \in \mathbf{L}(\mathbb{X})$ there exists an injective quasi-nilpotent Q operator commuting with T. Then both T and $T+Q$ satisfy property (aw), a-Weyl's and Weyl's theorem.*

Proof. We show first a-Weyl's theorem holds for T. It is evident, by Lemma 3.9 of [9], that $E_0^a(T)$ is empty. Suppose that $\sigma_a(T) \setminus \sigma_{uw}(T)$ is not empty and let $\lambda \in \Delta_a(T)$. Since $T - \lambda I \in W_+(\mathbb{X})$ then $\alpha(T-\lambda I) < \infty$ and $T - \lambda I$ has closed range. Since $T - \lambda I$ commutes with Q it then follows, by Lemma 3.9 of [9], that $T - \lambda I$ is injective, so $\lambda \notin \sigma_a(T)$, a contradiction. Therefore, also $\sigma_a(T) \setminus \sigma_{uw}(T)$ is empty. Therefore, a-Weyl's theorem holds for T. To show that property (aw) holds for T. Observe that $\Delta(T) \subseteq \Delta_a(T) = E_0^a(T) = \emptyset$. Hence $\Delta(T) = E_0^a(T) =$

$\emptyset$. That is, property (aw) holds for T.
Analogously, a-Weyl's theorem also holds for $T+Q$, since the operator $T+Q$ commutes with Q. Weyl's theorem is obvious: property (aw), as well as a-Weyl's theorem, entails Weyl's theorem. Property (aw), as well as a-Weyl's theorem and Weyl's theorem, for $T+Q$ is clear, since also $T+Q$ commutes with Q. ■

Theorem 3.7. *Suppose that $iso\sigma_a(T)=\emptyset$. If T satisfies property (aw) and K is a finite rank operator commuting with T , then $T+K$ satisfies property (aw).*

Proof. Since T satisfies Browder's theorem then $T+K$ satisfies Browder's theorem, see [10, Theorem 3.4]. From Lemma 2.6 of [6], we have $iso\sigma_a(T+K)=\emptyset$. Hence $E_0^a(T+K)=\pi_0(T+K)$. Therefore, it follows from Lemma 2.6 that property (aw) holds for $T+K$. ■

From [12], we recall that an operator $R\in\mathbf{L}(\mathbb{X})$ is said to be Riesz if $R-\lambda I$ is Fredholm for every non-zero complex number λ, that is, $\Upsilon(R)$ is quasi-nilpotent in $\mathcal{C}(\mathbb{X})$ where $\mathcal{C}(\mathbb{X}):=\mathbf{L}(\mathbb{X})/\mathbf{K}(\mathbb{X})$ is the Calkin algebra and Υ is the canonical mapping of $\mathbf{L}(\mathbb{X})$ into $\mathcal{C}(\mathbb{X})$. Note that for such operator,$\pi_0(R)=\sigma(R)\setminus\{0\}$, and its restriction to one of its closed subspace is also a Riesz operator, see [12]. The proof of the following result may be found in [32].

Lemma 3.8. *Let $T\in\mathbf{L}(\mathbb{X})$ and R be a Riesz operator commuting with T. Then*

(i) $T\in B_+(\mathbb{X})\Leftrightarrow T+R\in B_+(\mathbb{X})$.

(ii) $T\in B_-(\mathbb{X})\Leftrightarrow T+R\in B_-(\mathbb{X})$.

(iii) $T\in B(\mathbb{X})\Leftrightarrow T+R\in B(\mathbb{X})$.

Lemma 3.9. *[28, Lemma 2.2] Let $T\in\mathbf{L}(\mathbb{X})$ and R be a Riesz operator such that $TR=RT$.*

(i) If T is Fredholm then so is $T+R$ and $ind(T+R)=ind(T)$.

(ii) If T is Weyl then so is $T+R$. In particular $\sigma_w(T+R)=\sigma_w(T)$.

(iii) If T satisfies Browder's theorem then so does $T+R$.

For a bounded operator T on $\mathbb{X}$, we use $E_{0f}^a(T)$ to denote the set of isolated points λ of $\sigma_a(T)$ such that $\ker(T-\lambda I)$ is finite-dimensional. Evidently,

$$\pi_0^a(T)\subseteq E_0^a(T)\subseteq E_{0f}^a(T).$$

Lemma 3.10. *Let T be a bounded operator on $\mathbb{X}$. If R is a Riesz operator that commutes with T, then*

$$E_0^a(T+R)\cap\sigma_a(T)\subseteq iso\sigma_a(T).$$

Proof. Clearly,

$$E_0^a(T+R)\cap\sigma_a(T)\subseteq E_{0f}^a(T+R)\cap\sigma_a(T).$$

and by Proposition 2.4 of [29] the last set contained in $iso\sigma_a(T)$. ■

For a bounded operator T on $\mathbb{X}$, we denote by $E_{0f}(T)$ the set of isolated points λ of $\sigma(T)$ such that $\ker(T-\lambda I)$ is finite-dimensional. Evidently,$E_0(T) \subseteq E_{0f}(T)$.

Lemma 3.11. *Let T be a bounded operator on $\mathbb{X}$. If R is a Riesz operator that commutes with T, then*

$$E_0(T+R) \cap \sigma(T) \subseteq iso\sigma(T).$$

Proof. Clearly,

$$E_0(T+R) \cap \sigma(T) \subseteq E_{0f}(T+R) \cap \sigma(T).$$

and by Lemma 2.3 of [28] the last set contained in $iso\sigma(T)$. ∎

Recall that $T \in \mathbf{L}(\mathbb{X})$ is called finite a-isoloid (resp., finite isoloid) operator if $iso\sigma_a(T) \subseteq \sigma_p(T)$ (resp., $iso\sigma(T) \subseteq \sigma_p(T)$). Clearly, finite a-isoloid implies a-isoloid and finite isoloid, but the converse is not true in general.

Lemma 3.12. *Suppose that $T \in \mathbf{L}(\mathbb{X})$ be finite-isoloid satisfies property (aw) and R is a Riesz operator commuting with T. Then $\pi_0^a(T+R) \subseteq E_0(T+R)$.*

Proof. Let $\lambda \in \pi_0^a(T+R)$ be arbitrary given. Then $\lambda \in iso\sigma_a(T+R)$ and $T+R-\lambda I \in B_+(\mathbb{X})$, so $\alpha(T+R-\lambda I) < \infty$. Since $T+R-\lambda I$ has closed range, the condition $\lambda \in \sigma_a(T+R)$ entails that $\alpha(T+R-\lambda I) > 0$. Therefore, in order to show that $\lambda \in E_0(T+R)$, we need only to prove that λ is an isolated point of $\sigma(T+R)$.

Now, by assumption T satisfies property (aw) so, by Lemma 2.6, $\pi_0^a(T) = E_0(T) = E_0^a(T)$. Moreover, T satisfies Weyl's theorem and hence, by Theorem 2.7 of [28], $T+R$ satisfies Weyl's theorem. So

$$\pi_0(T+R) = E_0(T+R) = \sigma(T+R) \setminus \sigma_b(T+R).$$

Therefore, $T+R-\lambda I$ is Browder, so

$$0 < a(T+R-\lambda I) = d(T+R-\lambda I) < \infty$$

and hence λ is a pole of the resolvent of $T+R$. Consequently, λ an isolated point of $\sigma(T+R)$, as desired. ∎

Theorem 3.13. *Let $T \in \mathbf{L}(\mathbb{X})$ be an isoloid operator satisfying property (aw). If F is an operator that commutes with T and for which there exists a positive integer n such that F^n is finite rank, then $T+F$ satisfies property (aw).*

Proof. First observe that F is a Riesz operator. Since Weyl's theorem holds for $T+F$, by Theorem 2.4 of [28], then $E_0(T+F) = \pi_0(T+F)$. As T satisfies property (aw) then it follows from Lemma 3.12 that $\pi_0^a(T+F) \subseteq E_0(T+F)$. Hence

$$\pi_0^a(T+F) = E_0(T+R) = \Delta(T+F) = \pi_0(T+F) = \pi_0(T) = E_0^a(T) = \Delta(T).$$

To prove property (aw) holds for $T+F$, it suffices to show that $E_0(T+F) = E_0^a(T+F)$. To show this, let $\lambda \in E_0^a(T+F)$. If $T-\lambda I$ is invertible, then $T+F-\lambda I$ is Weyl, and hence $\lambda \in E_0(T+R)$. Suppose that $\lambda \in \sigma(T)$. Then it follows from Lemma 3.11 that $\lambda \in iso\sigma(T)$. Furthermore, since the operator $(T+F-\lambda I)^n|_{\ker(T-\lambda I)} = F^n|_{\ker(T-\lambda I)}$ is both of finite-dimensional range and kernel, we obtain easily that also $\ker(T-\lambda I)$ is finite-dimensional, and therefore that $\lambda \in E_0(T)$, because T is a-isoloid. On the other hand, if T satisfies property (aw), then $E_0^a(T)\cap\sigma_w(T)=\emptyset$. Consequently, $T-\lambda I$ is Weyl and hence so is $T+F-\lambda I$, which implies that $\lambda \in E_0(T+F)$. The other inclusion is trivial. Thus, property (aw) holds for $T+F$. ■

Corollary 3.14. *Let $T \in \mathbf{L}(\mathbb{X})$ be an isoloid operator. If property (aw) holds for T, then it also holds for $T+F$ for every finite rank operator F commuting with T.*

Theorem 3.15. *Let T be a finite-isoloid operator on $\mathbb{X}$ that satisfies property (aw). If R is a Riesz operator that commutes with T, then $T+R$ satisfies property (aw).*

Proof. Suppose that T satisfies property (aw). Then From Theorem 2.7, Theorem 2.7 of [28], and Lemma 3.12, we conclude that

$$\pi_0^a(T+R)=E_0(T+R)=\Delta(T+R)=\pi_0(T+R)=\pi_0(T)=\Delta(T)=E_0^a(T).$$

To prove property (aw) holds for $T+R$, it suffices to show that $E_0(T+R)=E_0^a(T+R)$. Let $\lambda \in E_0^a(T+R)$. If $T-\lambda I$ is invertible, then $T+R-\lambda I \in W(\mathbb{X})$ and hence $\lambda \in E_0(T+R)$. Suppose that $\lambda \in \sigma(T)$. It follows by Lemma 3.11 that λ is an isolated point of $\sigma(T)$, and because T is finite-isoloid, we see that $\lambda \in E_0(T)$. On the other hand, property (aw) holds for T implies that $\sigma_w(T)\cap E_0^a(T)=\emptyset$, therefore $T-\lambda I$ is Weyl and hence so is $T+R-\lambda I$. Thus, $\lambda \in E_0(T+R)$. The other inclusion is trivial, therefore $T+R$ satisfies property (aw). ■

Corollary 3.16. *Let T be an finite-isoloid operator on $\mathbb{X}$ that satisfies property (aw). If K is a compact operator commuting with T, then property (aw) holds for $T+K$.*

Theorem 3.17. *Let T be an operator on $\mathbb{X}$ that satisfies property (aw) and such that $\sigma_p(T)\cap iso\sigma_a(T)\subseteq E_0^a(T)$. If Q is a quasi-nilpotent operator that commutes with T, then $T+Q$ satisfies property (aw).*

Proof. Since $\sigma(T+Q)=\sigma(T)$ and also, by Lemma 2 of [26], $\sigma_w(T+Q)=\sigma_w(T)$, it suffices to show that $E_0^a(T+Q)=E_0^a(T)$. Let $\lambda \in E_0^a(T)=\sigma(T)\setminus\sigma_w(T)$. If $T-\lambda I$ is invertible, then $T-\lambda I \in W(\mathbb{X})$ and so $T+R-\lambda I \in W(\mathbb{X})$. Hence $\lambda \in E_0(T+R)\subseteq E_0^a(T+Q)$. Conversely, suppose $\lambda \in E_0^a(T+Q)$. Since Q is a quasi-nilpotent operator that commutes with T, we obtain that the restriction of $T-\lambda I$ to the finite-dimensional subspace $\ker(T+Q-\lambda I)$ is not invertible, and hence $\ker(T-\lambda I)$ is non-trivial. Therefore, $\lambda \in \sigma_p(T)\cap iso\sigma_a(T)\subseteq E_0^a(T)$, which completes the proof. ■

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