

Grothendieck quantaloids for allegories of enriched categories

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Abstract

For any small involutive quantaloid \mathcal{Q} we define, in terms of symmetric quantaloid-enriched categories, an involutive quantaloid $\text{Rel}(\mathcal{Q})$ of \mathcal{Q} -sheaves and relations, and a category $\text{Sh}(\mathcal{Q})$ of \mathcal{Q} -sheaves and functions; the latter is equivalent to the category of symmetric maps in the former. We prove that $\text{Rel}(\mathcal{Q})$ is the category of relations in a topos if and only if \mathcal{Q} is a modular, locally localic and weakly semi-simple quantaloid; in this case we call \mathcal{Q} a Grothendieck quantaloid. It follows that $\text{Sh}(\mathcal{Q})$ is a Grothendieck topos whenever \mathcal{Q} is a Grothendieck quantaloid. Any locale L is a Grothendieck quantale, and $\text{Sh}(L)$ is the topos of sheaves on L . Any small quantaloid of closed cibles is a Grothendieck quantaloid, and if \mathcal{Q} is the quantaloid of closed cibles in a Grothendieck site (\mathcal{C}, J) then $\text{Sh}(\mathcal{Q})$ is equivalent to the topos $\text{Sh}(\mathcal{C}, J)$. Any inverse quantal frame is a Grothendieck quantale, and if $\mathcal{O}(G)$ is the inverse quantal frame naturally associated with an étale groupoid G then $\text{Sh}(\mathcal{O}(G))$ is the classifying topos of G .

1 Introduction

A topos arising as the category of left adjoints in a locally ordered category, is the subject of P. Freyd and A. Scedrov's [1990] study of *allegories*. More precisely, an allegory \mathcal{A} is a modular locally ordered 2-category whose hom-posets have binary intersections; taking left adjoints (also known as "maps") in an allegory \mathcal{A} thus produces a category $\text{Map}(\mathcal{A})$; and the interesting case is where the latter category is in fact a topos. Thus, in Freyd and Scedrov's own words, allegories

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“are to binary relations between sets as categories are to functions between sets”. In practice, those interesting allegories arise most often as universal constructions on much smaller sub-allegories which are easier to describe explicitly. Freyd and Scedrov [1990] (but see also [Johnstone, 2002, A3]) give several theorems to this effect.

In [1982], R. Walters proved that any small site (\mathcal{C}, J) gives rise to a small quantaloid $\mathcal{R}(\mathcal{C}, J)$ in such a way that the topos $\text{Sh}(\mathcal{C}, J)$ is equivalent to the category of Cauchy-complete symmetric $\mathcal{R}(\mathcal{C}, J)$ -enriched categories and functors. But the latter category is further equivalent to the category of all symmetric $\mathcal{R}(\mathcal{C}, J)$ -categories and left adjoint distributors, and the quantaloid $\text{SymDist}(\mathcal{R}(\mathcal{C}, J))$ of all symmetric $\mathcal{R}(\mathcal{C}, J)$ -enriched categories and all distributors is modular. In other words, the topos $\text{Sh}(\mathcal{C}, J)$ is the category of maps in the allegory of symmetric $\mathcal{R}(\mathcal{C}, J)$ -enriched categories and distributors— which thus qualifies as an “interesting” allegory.

In this paper we shall explain more precisely how “sheaves via quantaloid-enrichment” fit with “toposes via allegories”. To that end, we define in Section 2, for any involutive quantaloid \mathcal{Q} , a new involutive quantaloid $\text{Rel}(\mathcal{Q})$, to be thought of as the locally posetal 2-category of “ \mathcal{Q} -sheaves and relations”, and a new category $\text{Sh}(\mathcal{Q})$, to be thought of as the category of “ \mathcal{Q} -sheaves and functions”. The objects of $\text{Rel}(\mathcal{Q})$, resp. $\text{Sh}(\mathcal{Q})$, are particular symmetric quantaloid-enriched categories, and the morphisms are distributors, resp. functors; the relation between the two is that $\text{Sh}(\mathcal{Q})$ is the category of *symmetric* left adjoints in $\text{Rel}(\mathcal{Q})$. For appropriate \mathcal{Q} , these \mathcal{Q} -sheaves are, among the \mathcal{Q} -orders of [Stubbe, 2005b], precisely the *symmetric* ones.

We show in Section 3 that, if $\mathcal{Q} = \mathcal{R}(\mathcal{C}, J)$, then $\text{Sh}(\mathcal{Q})$ is equivalent to $\text{Sh}(\mathcal{C}, J)$ and $\text{Rel}(\mathcal{Q})$ is equivalent to $\text{Rel}(\text{Sh}(\mathcal{C}, J))$; thus we recover and refine Walters’ [1982] insight. More generally, we prove in Section 4 that $\text{Rel}(\mathcal{Q})$ is equivalent to $\text{Rel}(\mathcal{T})$ for some Grothendieck topos \mathcal{T} (and thus $\text{Sh}(\mathcal{Q})$ is equivalent to \mathcal{T}) if and only if \mathcal{Q} is a modular, locally localic and weakly semi-simple quantaloid; we call these *Grothendieck quantaloids*. In other words, these Grothendieck quantaloids are precisely those for which the \mathcal{Q} -sheaves and relations form an “interesting allegory”. Locales and inverse quantal frames [Resende, 2007, 2012] are examples of Grothendieck quantales. If L is a locale, then $\text{Sh}(L)$ is in fact the topos of sheaves on L . And if $\mathcal{O}(G)$ is the inverse quantal frame associated to an étale groupoid G [Resende, 2007], then it follows from [Heymans and Stubbe, 2009b; Resende, 2012] that $\text{Sh}(\mathcal{O}(G))$ is the classifying topos of that groupoid.

2 Sheaves on an involutive quantaloid

The new notions that we will present at the end of this section draw heavily on the theory of quantaloid-enriched categories. For self-containedness we present some preliminaries in the first couple of subsections. For more details and for the many appropriate historical references we refer to [Stubbe, 2005a; Heymans and Stubbe, 2011].

Enrichment, involution and symmetry

A *quantaloid* \mathcal{Q} is, by definition, a category enriched in the symmetric monoidal closed category Sup of complete lattices and supremum-preserving functions; and a *homomorphism* $F: \mathcal{Q} \rightarrow \mathcal{R}$ of quantaloids is a Sup -enriched functor. An *involution* on a quantaloid \mathcal{Q} is a homomorphism $(-)^{\circ}: \mathcal{Q}^{\text{op}} \rightarrow \mathcal{Q}$ which is the identity on objects and satisfies $f^{\circ\circ} = f$ for any morphism f in \mathcal{Q} . The pair $(\mathcal{Q}, (-)^{\circ})$ is then said to form an *involutive quantaloid*; we shall often simply speak of “an involutive quantaloid \mathcal{Q} ”, leaving the notation for the involution understood. When both \mathcal{Q} and \mathcal{R} are involutive quantaloids, then we say that $F: \mathcal{Q} \rightarrow \mathcal{R}$ is a homomorphism of involutive quantaloids when it is a homomorphism such that $F(f^{\circ}) = (Ff)^{\circ}$.

Whenever a morphism $f: A \rightarrow B$ in a quantaloid (or in a locally ordered category, for that matter) is supposed to be a left adjoint, we write f^* for its right adjoint. A *symmetric left adjoint* in an involutive quantaloid \mathcal{Q} is a left adjoint whose right adjoint is its involute: $f^* = f^{\circ}$. Precisely as we write $\text{Map}(\mathcal{Q})$ for the category of left adjoints in \mathcal{Q} , we write $\text{SymMap}(\mathcal{Q})$ for the category of symmetric left adjoints.

A *category* \mathbb{A} enriched in a quantaloid \mathcal{Q} consists of a set \mathbb{A}_0 of objects, each $x \in \mathbb{A}_0$ having a type $tx \in \mathcal{Q}_0$, and for any $x, y \in \mathbb{A}_0$ there is a hom-arrow $\mathbb{A}(y, x): tx \rightarrow ty$ in \mathcal{Q} , subject to associativity and unit requirements: $\mathbb{A}(z, y) \circ \mathbb{A}(y, x) \leq \mathbb{A}(z, x)$ and $1_{tx} \leq \mathbb{A}(x, x)$ for all $x, y, z \in \mathbb{A}_0$. A *functor* $F: \mathbb{A} \rightarrow \mathbb{B}$ between such \mathcal{Q} -categories is an object-map $x \mapsto Fx$ such that $tx = t(Fx)$ and $\mathbb{A}(y, x) \leq \mathbb{B}(Fy, Fx)$ for all $x, y \in \mathbb{A}$. Such a functor is smaller than a functor $G: \mathbb{A} \rightarrow \mathbb{B}$ if $1_{tx} \leq \mathbb{B}(Fx, Gx)$ for every $x \in \mathbb{A}$. With obvious composition one gets a locally ordered 2-category $\text{Cat}(\mathcal{Q})$ of \mathcal{Q} -categories and functors.

For two objects $x, y \in \mathbb{A}$, the hom-arrows $\mathbb{A}(y, x)$ and $\mathbb{A}(x, y)$ go in opposite directions. Hence, to formulate a notion of “symmetry” for \mathcal{Q} -categories, it is far too strong to require $\mathbb{A}(y, x) = \mathbb{A}(x, y)$. Instead, at least for involutive quantaloids, a \mathcal{Q} -category \mathbb{A} is *symmetric* when $\mathbb{A}(x, y) = \mathbb{A}(y, x)^{\circ}$ for every two objects $x, y \in \mathbb{A}$ [Betti and Walters, 1982]. We shall write $\text{SymCat}(\mathcal{Q})$ for the full sub-2-category of $\text{Cat}(\mathcal{Q})$ determined by the symmetric \mathcal{Q} -categories (in which the local order is in fact symmetric, but not anti-symmetric).

A *distributor* $\Phi: \mathbb{A} \dashv\vdash \mathbb{B}$ between \mathcal{Q} -categories consists of arrows $\Phi(y, x): tx \rightarrow ty$ in \mathcal{Q} , one for each $(x, y) \in \mathbb{A}_0 \times \mathbb{B}_0$, subject to two action requirements: $\mathbb{B}(y', y) \circ \Phi(y, x) \leq \Phi(y', x)$ and $\Phi(y, x) \circ \mathbb{A}(x, x') \leq \Phi(y, x')$ for all $y, y' \in \mathbb{B}_0$ and $x, x' \in \mathbb{A}_0$. The composite of such a distributor with another $\Psi: \mathbb{B} \dashv\vdash \mathbb{C}$ is written as $\Psi \otimes \Phi: \mathbb{A} \dashv\vdash \mathbb{C}$, and its elements are

$$(\Psi \otimes \Phi)(z, x) = \bigvee_{y \in \mathbb{B}_0} \Psi(z, y) \circ \Phi(y, x)$$

for $x \in \mathbb{A}_0$ and $z \in \mathbb{C}_0$. Parallel distributors can be compared elementwise, and in fact one gets a (large) quantaloid $\text{Dist}(\mathcal{Q})$ of \mathcal{Q} -categories and distributors. Each functor $F: \mathbb{A} \rightarrow \mathbb{B}$ determines an adjoint pair of distributors: $\mathbb{B}(-, F-): \mathbb{A} \dashv\vdash \mathbb{B}$, with elements $\mathbb{B}(y, Fx)$ for $(x, y) \in \mathbb{A}_0 \times \mathbb{B}_0$, is left adjoint to $\mathbb{B}(F-, -): \mathbb{B} \dashv\vdash \mathbb{A}$ in the quantaloid $\text{Dist}(\mathcal{Q})$. These distributors are said to be *represented by* F . More

generally, a (necessarily left adjoint) distributor $\Phi: \mathbb{A} \multimap \mathbb{B}$ is *representable* if there exists a (necessarily essentially unique) functor $F: \mathbb{A} \rightarrow \mathbb{B}$ such that $\Phi = \mathbb{B}(-, F-)$. This amounts to a 2-functor

$$\text{Cat}(\mathcal{Q}) \rightarrow \text{Map}(\text{Dist}(\mathcal{Q})): (F: \mathbb{A} \rightarrow \mathbb{B}) \mapsto (\mathbb{B}(-, F-): \mathbb{A} \multimap \mathbb{B}). \quad (1)$$

We write $\text{SymDist}(\mathcal{Q})$ for the full subquantaloid of $\text{Dist}(\mathcal{Q})$ determined by the symmetric \mathcal{Q} -categories. It is easily verified that the involution $f \mapsto f^\circ$ on the base quantaloid \mathcal{Q} extends to the quantaloid $\text{SymDist}(\mathcal{Q})$: explicitly, if $\Phi: \mathbb{A} \multimap \mathbb{B}$ is a distributor between symmetric \mathcal{Q} -categories, then so is $\Phi^\circ: \mathbb{B} \multimap \mathbb{A}$, with elements $\Phi^\circ(a, b) := \Phi(b, a)^\circ$. And if $F: \mathbb{A} \rightarrow \mathbb{B}$ is a functor between symmetric \mathcal{Q} -categories, then the left adjoint distributor represented by F has the particular feature that it is a symmetric left adjoint in $\text{SymDist}(\mathcal{Q})$. That is to say, the functor in (1) restricts to the symmetric situation as

$$\text{SymCat}(\mathcal{Q}) \rightarrow \text{SymMap}(\text{SymDist}(\mathcal{Q})): (F: \mathbb{A} \rightarrow \mathbb{B}) \mapsto (\mathbb{B}(-, F-): \mathbb{A} \multimap \mathbb{B}), \quad (2)$$

obviously giving a commutative diagram

$$\begin{array}{ccc} \text{Cat}(\mathcal{Q}) & \longrightarrow & \text{Map}(\text{Dist}(\mathcal{Q})) \\ \text{incl.} \uparrow & & \uparrow \text{incl.} \\ \text{SymCat}(\mathcal{Q}) & \longrightarrow & \text{SymMap}(\text{SymDist}(\mathcal{Q})) \end{array}$$

The full embedding $\text{SymCat}(\mathcal{Q}) \hookrightarrow \text{Cat}(\mathcal{Q})$ has a right adjoint functor:

$$\text{SymCat}(\mathcal{Q}) \begin{array}{c} \xrightarrow{\text{incl.}} \\ \perp \\ \xleftarrow{(-)_s} \end{array} \text{Cat}(\mathcal{Q}).$$

This *symmetrisation* sends a \mathcal{Q} -category \mathbb{A} to the symmetric \mathcal{Q} -category \mathbb{A}_s whose objects (and types) are those of \mathbb{A} , but for any two objects x, y the hom-arrow is $\mathbb{A}_s(y, x) := \mathbb{A}(y, x) \wedge \mathbb{A}(x, y)^\circ$. A functor $F: \mathbb{A} \rightarrow \mathbb{B}$ is sent to $F_s: \mathbb{A}_s \rightarrow \mathbb{B}_s: a \mapsto Fa$. It is a result of [Heymans and Stubbe, 2011] that the inclusion $\text{SymMap}(\text{SymDist}(\mathcal{Q})) \rightarrow \text{Map}(\text{Dist}(\mathcal{Q}))$ admits a right adjoint that makes the diagram

$$\begin{array}{ccc} \text{Cat}(\mathcal{Q}) & \longrightarrow & \text{Map}(\text{Dist}(\mathcal{Q})) \\ (-)_s \downarrow & & \downarrow \text{dotted} \\ \text{SymCat}(\mathcal{Q}) & \longrightarrow & \text{SymMap}(\text{SymDist}(\mathcal{Q})) \end{array}$$

commute if and only if, for each family $(f_i: X \rightarrow X_i, g_i: X_i \rightarrow X)_{i \in I}$ of morphisms

in \mathcal{Q} ,

$$\left. \begin{array}{l} \forall j, k \in I : f_k \circ g_j \circ f_j \leq f_k \\ \forall j, k \in I : g_j \circ f_j \circ g_k \leq g_k \\ 1_X \leq \bigvee_{i \in I} g_i \circ f_i \end{array} \right\} \implies 1_X \leq \bigvee_{i \in I} (g_i \wedge f_i^\circ) \circ (g_i^\circ \wedge f_i).$$

Such an involutive quantaloid \mathcal{Q} is said to be *Cauchy-bilateral*. We will encounter examples of Cauchy-bilateral quantaloids further on in this paper.

Presheaves, Cauchy-completion and symmetric-completion

A (contravariant) presheaf on \mathbb{A} is a distributor into \mathbb{A} whose domain is a one-object category with an identity hom-arrow. Writing $*_X$ for the one-object \mathcal{Q} -category whose single object $*$ has type $X \in \mathcal{Q}_0$ and whose single hom-arrow is the identity 1_X , a presheaf is then typically written as $\phi : *_X \multimap \mathbb{A}$. The set of presheaves on \mathbb{A} is written $\mathcal{P}(\mathbb{A})$: it is a \mathcal{Q} -category when we define that $t(\phi : *_X \multimap \mathbb{A}) = X$ and $\mathcal{P}(\mathbb{A})(\psi, \phi) = [\psi, \phi]$ (this being a lifting in the quantaloid $\text{Dist}(\mathcal{Q})$, i.e. the value at ϕ of the right adjoint to composition with ψ). The *Yoneda embedding* of \mathbb{A} into $\mathcal{P}(\mathbb{A})$ is the fully faithful functor of \mathcal{Q} -enriched categories $Y_{\mathbb{A}} : \mathbb{A} \rightarrow \mathcal{P}(\mathbb{A})$ that sends $a \in \mathbb{A}$ to the *representable presheaf* $\mathbb{A}(-, a) : *_a \multimap \mathbb{A}$. In fact, this procedure extends to a functor $\mathcal{P} : \text{Cat}(\mathcal{Q}) \rightarrow \text{Cat}(\mathcal{Q})$, which is the *free cocompletion KZ-doctrine* on the category of \mathcal{Q} -categories. (A *covariant presheaf* on \mathbb{A} is a distributor $\phi : \mathbb{A} \multimap *_X$; they are not of much importance in this paper.)

A \mathcal{Q} -category \mathbb{A} is said to be *Cauchy complete* when each left adjoint distributor with codomain \mathbb{A} is represented by a functor [Lawvere, 1973], that is, when for each \mathcal{Q} -category \mathbb{B} the functor in (1) determines an equivalence of ordered sets

$$\text{Cat}(\mathcal{Q})(\mathbb{B}, \mathbb{A}) \simeq \text{Map}(\text{Dist}(\mathcal{Q}))(\mathbb{B}, \mathbb{A}).$$

This clearly implies that the functor in (1) restricts to a biequivalence of locally ordered 2-categories between $\text{Cat}_{\text{cc}}(\mathcal{Q})$, the full subcategory of $\text{Cat}(\mathcal{Q})$ determined by the Cauchy complete \mathcal{Q} -categories, and $\text{Map}(\text{Dist}(\mathcal{Q}))$. Moreover, the full inclusion of $\text{Cat}_{\text{cc}}(\mathcal{Q})$ in $\text{Cat}(\mathcal{Q})$ admits a left adjoint:

$$\text{Cat}_{\text{cc}}(\mathcal{Q}) \begin{array}{c} \xleftarrow{(-)_{\text{cc}}} \\ \xrightarrow{\text{full incl.}} \\ \perp \end{array} \text{Cat}(\mathcal{Q}). \tag{3}$$

That is to say, each \mathcal{Q} -category \mathbb{A} has a *Cauchy completion* \mathbb{A}_{cc} : it is the full subcategory of the presheaf category $\mathcal{P}(\mathbb{A})$ whose objects are the left adjoint presheaves on \mathbb{A} . The Yoneda embedding $Y_{\mathbb{A}} : \mathbb{A} \rightarrow \mathcal{P}(\mathbb{A})$ factors through \mathbb{A}_{cc} , and the distributor induced by $Y_{\mathbb{A}} : \mathbb{A} \rightarrow \mathbb{A}_{\text{cc}}$ turns out to be an isomorphism in $\text{Dist}(\mathcal{Q})$. Therefore the quantaloid $\text{Dist}(\mathcal{Q})$ is equivalent to its full subquantaloid $\text{Dist}_{\text{cc}}(\mathcal{Q})$ whose objects are the Cauchy complete \mathcal{Q} -categories. As a result, there is an equivalence of locally ordered 2-categories

$$\text{Cat}_{\text{cc}}(\mathcal{Q}) \simeq \text{Map}(\text{Dist}_{\text{cc}}(\mathcal{Q})) \simeq \text{Map}(\text{Dist}(\mathcal{Q})). \tag{4}$$

The Cauchy completion can of course be applied to a symmetric \mathcal{Q} -category (assuming that \mathcal{Q} is involutive), but the resulting Cauchy complete category need not be symmetric anymore: the functor $(-)\text{cc}: \text{Cat}(\mathcal{Q}) \rightarrow \text{Cat}(\mathcal{Q})$ does not restrict to $\text{SymCat}(\mathcal{Q})$ in general. However, its very definition suggests the following modification [Heymans and Stubbe, 2011]: a symmetric \mathcal{Q} -category \mathbb{A} is *symmetrically complete* if, for any symmetric \mathcal{Q} -category \mathbb{B} , the functor in (2) determines an equivalence of symmetrically ordered sets

$$\text{SymCat}(\mathcal{Q})(\mathbb{B}, \mathbb{A}) \simeq \text{SymMap}(\text{SymDist}(\mathcal{Q}))(\mathbb{B}, \mathbb{A}).$$

This implies that the functor in (2) restricts to a biequivalence between $\text{SymCat}_{\text{sc}}(\mathcal{Q})$, the full subcategory of $\text{SymCat}(\mathcal{Q})$ of its symmetrically complete objects, and $\text{SymMap}(\text{SymDist}(\mathcal{Q}))$. Moreover, the full inclusion of $\text{SymCat}_{\text{sc}}(\mathcal{Q})$ in $\text{SymCat}(\mathcal{Q})$ admits a left adjoint:

$$\text{SymCat}_{\text{sc}}(\mathcal{Q}) \begin{array}{c} \xleftarrow{(-)_{\text{sc}}} \\ \perp \\ \xrightarrow{\text{full incl.}} \end{array} \text{SymCat}(\mathcal{Q}). \tag{5}$$

Explicitly, for a symmetric \mathcal{Q} -category \mathbb{A} , its *symmetric completion* \mathbb{A}_{sc} is the full subcategory of the Cauchy completion \mathbb{A}_{cc} (and thus also a full subcategory of the presheaf category $\mathcal{P}(\mathbb{A})$) determined by the *symmetric* left adjoint presheaves. For similar reasons as above, there is an equivalence of involutive quantaloids between $\text{SymDist}(\mathcal{Q})$ and its full subquantaloid $\text{SymDist}_{\text{sc}}(\mathcal{Q})$ of symmetrically complete \mathcal{Q} -categories, and therefore also an equivalence of categories

$$\text{SymCat}_{\text{sc}}(\mathcal{Q}) \simeq \text{SymMap}(\text{SymDist}_{\text{sc}}(\mathcal{Q})) \simeq \text{SymMap}(\text{SymDist}(\mathcal{Q})). \tag{6}$$

Importantly, a result of [Heymans and Stubbe, 2011] says that, if \mathcal{Q} is a Cauchy-bilateral quantaloid, then the symmetric-completion and the Cauchy-completion of any symmetric \mathcal{Q} -category coincide, and the symmetrisation of a Cauchy complete \mathcal{Q} -category is symmetrically complete. In fact, there is a distributive law of the monad $(-)\text{cc}: \text{Cat}(\mathcal{Q}) \rightarrow \text{Cat}(\mathcal{Q})$ over the comonad $(-)_s: \text{Cat}(\mathcal{Q}) \rightarrow \text{Cat}(\mathcal{Q})$. This means in particular that there is a commutative diagram of adjunctions as follows:

$$\begin{array}{ccccc} \text{Cat}_{\text{cc}}(\mathcal{Q}) & \xlongequal{\quad} & \text{Map}(\text{Dist}_{\text{cc}}(\mathcal{Q})) & \xlongequal{\quad} & \text{Map}(\text{Dist}(\mathcal{Q})) \\ \downarrow \scriptstyle (-)_s \left(\uparrow \right) \text{incl.} & & \downarrow \scriptstyle (-)_s \left(\uparrow \right) \text{incl.} & & \downarrow \scriptstyle (-)_s \left(\uparrow \right) \text{incl.} \\ \text{SymCat}_{\text{sc}}(\mathcal{Q}) & \xlongequal{\quad} & \text{SymMap}(\text{SymDist}_{\text{sc}}(\mathcal{Q})) & \xlongequal{\quad} & \text{SymMap}(\text{SymDist}(\mathcal{Q})) \end{array}$$

The equal signs in this diagram are the equivalences of (4) and (6); the bottom row is fully included in the top row, and can be obtained from it by ‘symmetrisation’.

Universal constructions

An idempotent in a quantaloid \mathcal{Q} is, of course, an endomorphism $e: A \rightarrow A$ such that $e^2 = e$. Such an idempotent is said to split in \mathcal{Q} when there exists a diagram

$$\begin{array}{ccc}
 qp = e & & 1_B = pq \\
 \curvearrowleft & \xrightarrow{p} & \curvearrowright \\
 A & & B \\
 & \xleftarrow{q} &
 \end{array} \tag{7}$$

in \mathcal{Q} . If \mathcal{E} is a class of idempotents in a quantaloid \mathcal{Q} , then we write $\mathcal{Q}_{\mathcal{E}}$ for the quantaloid obtained by splitting the idempotents in \mathcal{E} . An explicit description goes as follows: the objects of $\mathcal{Q}_{\mathcal{E}}$ are the elements of \mathcal{E} , and $\mathcal{Q}_{\mathcal{E}}(e, f) = \{x: A \rightarrow B \mid f \circ x = x = x \circ e\}$ whenever $e: A \rightarrow A$ and $f: B \rightarrow B$ are in \mathcal{E} . Composition and local suprema in $\mathcal{Q}_{\mathcal{E}}$ are as in \mathcal{Q} , but the identity on an idempotent e is, obviously, $e: e \rightarrow e$ itself. If all identities in \mathcal{Q} are in \mathcal{E} , then there is a fully faithful homomorphism of quantaloids

$$I: \mathcal{Q} \rightarrow \mathcal{Q}_{\mathcal{E}}: (x: A \rightarrow B) \mapsto (x: 1_A \rightarrow 1_B)$$

which is the universal splitting in \mathcal{Q} of idempotents in \mathcal{E} . Spelled out, this means that if $F: \mathcal{Q} \rightarrow \mathcal{R}$ is a homomorphism of quantaloids, and the images of all idempotents in \mathcal{E} split in \mathcal{R} , then there is an essentially unique homomorphism $\bar{F}: \mathcal{Q}_{\mathcal{E}} \rightarrow \mathcal{R}$ such that $\bar{F} \circ I = F$. Moreover, if F is fully faithful then so is \bar{F} .

When \mathcal{Q} is an involutive quantaloid, then we say that an idempotent $e: A \rightarrow A$ in \mathcal{Q} is *symmetric* when $e^\circ = e$. It is straightforward that $\mathcal{Q}_{\mathcal{E}}$ is then involutive too: the involute of $x \in \mathcal{Q}_{\mathcal{E}}(e, f)$ is computed as in \mathcal{Q} , for the symmetry of e and f make sure that $x^\circ \in \mathcal{Q}_{\mathcal{E}}(f, e)$. As before, it is surely the case that, whenever all identities in \mathcal{Q} are in \mathcal{E} , the involutive quantaloid $\mathcal{Q}_{\mathcal{E}}$ has a universal property for the splitting of idempotents. Noting however that $I: \mathcal{Q} \rightarrow \mathcal{Q}_{\mathcal{E}}$ preserves the involution, we can point out a slightly more subtle universal property. Say that the splitting in the diagram in (7) is *symmetric* when $q = p^\circ$. If $F: \mathcal{Q} \rightarrow \mathcal{R}$ is a homomorphism of involutive quantaloids and the images of all idempotents in \mathcal{E} split symmetrically in \mathcal{R} , then there is an essentially unique homomorphism $\bar{F}: \mathcal{Q}_{\mathcal{E}} \rightarrow \mathcal{R}$ of involutive quantaloids such that $\bar{F} \circ I = F$; in other words, if F preserves the involution then so does \bar{F} . And again, if F is fully faithful then so is \bar{F} .

In any quantaloid \mathcal{Q} , products and sums are the same thing, so they are usually referred to as *direct sums*. We write $A = \oplus_{i \in I} A_i$ for the direct sum of a family $(A_i)_i$ of objects of \mathcal{Q} , with injections $s_i: A_i \rightarrow A$ and projections $p_i: A \rightarrow A_i$; in fact, for A to be the direct sum of the $(A_i)_i$, it is a necessary and sufficient condition that $p_i \circ s_j = \delta_{ij}$ and $\bigvee_i s_i \circ p_i = 1_A$. In these equations, $\delta_{ij}: A_j \rightarrow A_i$ is the ‘‘Kronecker delta’’: it is the identity morphism when $i = j$ and the zero morphism otherwise. The universal direct sum completion of a small quantaloid \mathcal{Q} exists, and can explicitly be described as the quantaloid $\text{Matr}(\mathcal{Q})$ of *matrices* over \mathcal{Q} . An object in $\text{Matr}(\mathcal{Q})$ is a \mathcal{Q} -typed set, i.e. a set A together with a type function $t: A \rightarrow \mathcal{Q}_0$, and a morphism between two such \mathcal{Q} -typed sets is a *matrix* $M: A \rightarrow B$, i.e. a family $M(b, a): ta \rightarrow tb$ of morphisms in \mathcal{Q} , one for

each $(a, b) \in A \times B$. Of course, matrices can be composed: for $M: A \rightarrow B$ and $N: B \rightarrow C$ we have $N \circ M: A \rightarrow C$ with elements

$$(N \circ M)(c, a) := \bigvee_{b \in B} N(c, b) \circ M(b, a).$$

The identity on a \mathcal{Q} -typed set A is the matrix $\Delta_A: A \rightarrow A$ all of whose elements are “Kronecker deltas”. With elementwise supremum, this makes $\text{Matr}(\mathcal{Q})$ a quantaloid; and whenever \mathcal{Q} is involutive, so is $\text{Matr}(\mathcal{Q})$ (for elementwise involution). There is a fully faithful homomorphism

$$J: \mathcal{Q} \rightarrow \text{Matr}(\mathcal{Q}): (f: X \rightarrow Y) \mapsto ((f): \{X\} \rightarrow \{Y\})$$

sending a morphism to the matrix between singletons in the obvious way (which preserves the involution on \mathcal{Q} whenever there is one), which is the universal direct sum completion of \mathcal{Q} .

Any \mathcal{Q} -typed set A determines a \mathcal{Q} -category \mathbb{A} by putting $\mathbb{A}_0 = A$ and $\mathbb{A}(a', a) = \Delta_A(a', a)$: this is precisely a *discrete* \mathcal{Q} -category in the sense that the hom-arrow between two different objects is a zero morphism and every endo-hom-arrow is an identity morphism. A matrix between \mathcal{Q} -typed sets is easily seen to be precisely a distributor between discrete \mathcal{Q} -categories, so the quantaloid $\text{Matr}(\mathcal{Q})$ is precisely the full subquantaloid of $\text{Dist}(\mathcal{Q})$ of discrete \mathcal{Q} -categories. A discrete \mathcal{Q} -category is obviously symmetric, so whenever \mathcal{Q} is an involutive quantaloid, $\text{Matr}(\mathcal{Q})$ can also be considered as full involutive subcategory of $\text{SymDist}(\mathcal{Q})$. Furthermore, a monad in $\text{Matr}(\mathcal{Q})$ is exactly a \mathcal{Q} -category, and (assuming that \mathcal{Q} is involutive) a symmetric monad is a symmetric \mathcal{Q} -category. In other words, both $\text{Dist}(\mathcal{Q})$ and $\text{SymDist}(\mathcal{Q})$ can be constructed from $\text{Matr}(\mathcal{Q})$ by splitting a particular class of idempotents:

- $\text{Dist}(\mathcal{Q}) = \text{Matr}(\mathcal{Q})_{\mathcal{E}}$ for \mathcal{E} the class of monads in $\text{Matr}(\mathcal{Q})$,
- $\text{SymDist}(\mathcal{Q}) = \text{Matr}(\mathcal{Q})_{\mathcal{E}}$ for \mathcal{E} the class of symmetric monads in $\text{Matr}(\mathcal{Q})$.

Composing the various universal constructions we thus find how $\text{Dist}(\mathcal{Q})$ and $\text{SymDist}(\mathcal{Q})$ can be considered as completions of \mathcal{Q} itself.

For any involutive quantaloid \mathcal{Q}' it is a matter of fact that the process of splitting *all* monads in \mathcal{Q}' can be broken down in two steps: first split all *symmetric* monads in \mathcal{Q}' , then split all *anti-symmetric* monads in the thusly obtained quantaloid. (A monad $m: X \rightarrow X$ in an involutive quantaloid is said to be anti-symmetric when $m \wedge m^\circ = 1_X$.) Applying this to $\mathcal{Q}' = \text{Matr}(\mathcal{Q})$ for a small involutive quantaloid \mathcal{Q} , this exhibits how $\text{Dist}(\mathcal{Q})$ is also a completion of $\text{SymDist}(\mathcal{Q})$.

All this goes to show that both $\text{Dist}(\mathcal{Q})$ and $\text{SymDist}(\mathcal{Q})$ lead a “double life”. On the one hand, they are concretely constructed quantaloids: their objects are (symmetric) \mathcal{Q} -categories, and their morphisms are distributors. This makes it possible to compute with *individual objects and morphisms* of $\text{Dist}(\mathcal{Q})$ (or $\text{SymDist}(\mathcal{Q})$). But on the other hand, $\text{Dist}(\mathcal{Q})$ and $\text{SymDist}(\mathcal{Q})$ are universal constructions on \mathcal{Q} : first add all direct sums to \mathcal{Q} , then split either all monads or only the symmetric ones. These universal properties thus say something about

the collection of all objects and morphisms of $\text{Dist}(\mathcal{Q})$ or $\text{SymDist}(\mathcal{Q})$. The first approach is clearly rooted in the theory of quantaloid-enriched categories, whereas the second approach is close in spirit to allegory theory. Indeed, quoting P. Johnstone [2002, p. 138], “many allegories of interest may be generated by idempotent-splitting processes from quite small full sub-allegories”. Of course, $\text{Dist}(\mathcal{Q})$ or $\text{SymDist}(\mathcal{Q})$ need not be allegories (neither of them is necessarily modular, see further), but they are both generated by universal processes from a quite small full sub-quantaloid, namely from \mathcal{Q} itself.

Orders and sheaves over a base quantaloid

We now have everything ready to state the central definitions with which we shall work in this paper. First we recall a definition first given in [Stubbe, 2005b]:

Definition 2.1 *Given a small quantaloid \mathcal{Q} and a set \mathcal{E} of idempotents in \mathcal{Q} , we define*

$$\text{Ord}(\mathcal{Q}, \mathcal{E}) := \text{Cat}_{\text{cc}}(\mathcal{Q}_{\mathcal{E}}) \text{ and } \text{Idl}(\mathcal{Q}, \mathcal{E}) := \text{Dist}_{\text{cc}}(\mathcal{Q}_{\mathcal{E}})$$

for, respectively, the locally ordered 2-category of $(\mathcal{Q}, \mathcal{E})$ -orders and order functions, and the quantaloid of $(\mathcal{Q}, \mathcal{E})$ -orders and ideal relations. If \mathcal{E} is taken to be the set of all idempotents in \mathcal{Q} , then we write \mathcal{Q}_{si} instead of $\mathcal{Q}_{\mathcal{E}}$, $\text{Ord}(\mathcal{Q})$ instead of $\text{Ord}(\mathcal{Q}, \mathcal{E})$, and $\text{Idl}(\mathcal{Q})$ instead of $\text{Idl}(\mathcal{Q}, \mathcal{E})$; we then simply speak of \mathcal{Q} -orders (and order functions and ideal relations).

Next we present a new definition, intended as “symmetric” version of the previous definition. Because the term “symmetric \mathcal{Q} -order” is technically inadequate (it suggests a \mathcal{Q} -order with a symmetric hom, *quod non*), and the term “ \mathcal{Q} -set” already means something related-but-different in the literature (see e.g. [Higgs, 1973; Fourman and Scott, 1979; Borceux, 1994; Mulvey and Nawaz, 1995; Gyls, 2001; Johnstone, 2002; and others]), we opt to speak of “ \mathcal{Q} -sheaves”:

Definition 2.2 *Given a small involutive quantaloid \mathcal{Q} and a set \mathcal{E} of symmetric idempotents in \mathcal{Q} , we define*

$$\text{Sh}(\mathcal{Q}, \mathcal{E}) := \text{SymCat}_{\text{sc}}(\mathcal{Q}_{\mathcal{E}}) \text{ and } \text{Rel}(\mathcal{Q}, \mathcal{E}) := \text{SymDist}_{\text{sc}}(\mathcal{Q}_{\mathcal{E}})$$

for, respectively, the category of $(\mathcal{Q}, \mathcal{E})$ -sheaves and functions, and the quantaloid of $(\mathcal{Q}, \mathcal{E})$ -sheaves and relations. If \mathcal{E} is taken to be the set of all symmetric idempotents in \mathcal{Q} , then we write \mathcal{Q}_{ssi} instead of $\mathcal{Q}_{\mathcal{E}}$, $\text{Sh}(\mathcal{Q})$ instead of $\text{Sh}(\mathcal{Q}, \mathcal{E})$, and $\text{Rel}(\mathcal{Q})$ instead of $\text{Rel}(\mathcal{Q}, \mathcal{E})$; we then simply speak of \mathcal{Q} -sheaves (and functions and relations).

We shall explain at the end of Section 3 how, for so-called small quantaloids of closed cribles, the symmetry condition in the above definition is in fact equivalent to an appropriate discreteness condition.

From the general theory on (symmetric) \mathcal{Q} -categories that we explained in the previous subsections, we can now conclude that:

Proposition 2.3 *For any small quantaloid \mathcal{Q} and any set \mathcal{E} of idempotents in \mathcal{Q} , there is a biequivalence of locally ordered 2-categories*

$$\text{Ord}(\mathcal{Q}, \mathcal{E}) \xrightarrow{\sim} \text{Map}(\text{Idl}(\mathcal{Q}, \mathcal{E})): (F: \mathbb{A} \rightarrow \mathbb{B}) \mapsto (\mathbb{B}(-, F-): \mathbb{A} \multimap \mathbb{B}).$$

For any small involutive quantaloid \mathcal{Q} and any set \mathcal{E} of symmetric idempotents in \mathcal{Q} , there is an equivalence of categories

$$\text{Sh}(\mathcal{Q}, \mathcal{E}) \xrightarrow{\sim} \text{SymMap}(\text{Rel}(\mathcal{Q}, \mathcal{E})): (F: \mathbb{A} \rightarrow \mathbb{B}) \mapsto (\mathbb{B}(-, F-): \mathbb{A} \multimap \mathbb{B}).$$

If \mathcal{Q} is an involutive quantaloid and \mathcal{E} a set of symmetric idempotents such that $\mathcal{Q}_{\mathcal{E}}$ is Cauchy-bilateral, then both squares in

$$\begin{array}{ccc} \text{Ord}(\mathcal{Q}, \mathcal{E}) & \xrightarrow{\sim} & \text{Map}(\text{Idl}(\mathcal{Q}, \mathcal{E})) \\ \downarrow \scriptstyle{(-)_s} \uparrow \scriptstyle{\vdash} \text{incl.} & & \downarrow \scriptstyle{(-)_s} \uparrow \scriptstyle{\vdash} \text{incl.} \\ \text{Sh}(\mathcal{Q}, \mathcal{E}) & \xrightarrow{\sim} & \text{SymMap}(\text{Rel}(\mathcal{Q}, \mathcal{E})) \end{array}$$

commute, and the bottom row is obtained by “symmetrising” the top row.

Here is yet another result of the general theory of \mathcal{Q} -categories:

Proposition 2.4 *For any small (resp. involutive) quantaloid \mathcal{Q} and any set \mathcal{E} of (resp. symmetric) idempotents in \mathcal{Q} , there is an equivalence of (resp. involutive) quantaloids*

$$\text{Idl}(\mathcal{Q}, \mathcal{E}) \simeq \text{Dist}(\mathcal{Q}_{\mathcal{E}}), \text{ resp. } \text{Rel}(\mathcal{Q}, \mathcal{E}) \simeq \text{SymDist}(\mathcal{Q}_{\mathcal{E}}).$$

This proposition explains an important subtlety: each $(\mathcal{Q}, \mathcal{E})$ -order (or $(\mathcal{Q}, \mathcal{E})$ -sheaf) is *Morita equivalent* to a (symmetric) $\mathcal{Q}_{\mathcal{E}}$ -category. This fact has often been used (implicitly) to forget about Cauchy completeness altogether: several definitions of “sheaf on an involutive quantaloid” that can be found in the literature, amount (in one form or another) to stating that a sheaf is a symmetric category, and a morphism of sheaves is a left adjoint distributor. (An example that springs to mind, is the formalism of *projection matrices*, on which we shall comment in more detail in Section 4.) However, we have deliberately opted to include the requirement of Cauchy (or symmetric) completeness in the definition of “sheaf” on a quantaloid \mathcal{Q} , for it expresses precisely the “gluing condition” that one expects of such a notion (as well illustrated by [Walters, 1981]). But of course it comes in handy that, modulo Morita equivalence, this completeness can be swiped under the carpet.

The whole of Section 3 is devoted to showing that the topos of sheaves on a site (\mathcal{C}, J) is equivalent to $\text{Sh}(\mathcal{Q})$ when $\mathcal{Q} = \mathcal{R}(\mathcal{C}, J)$ is the small quantaloid of closed cibles in (\mathcal{C}, J) .

3 Sheaves on a site

For any small involutive quantaloid \mathcal{Q} we stated in Definition 2.2 and Proposition 2.3 that

$$\mathrm{Sh}(\mathcal{Q}) := \mathrm{SymCat}_{\mathrm{sc}}(\mathcal{Q}_{\mathrm{ssi}}) \simeq \mathrm{SymMap}(\mathrm{SymDist}(\mathcal{Q}_{\mathrm{ssi}})).$$

Walters [1982] showed that, for a small site (\mathcal{C}, J) ,

$$\mathrm{Sh}(\mathcal{C}, J) \simeq \mathrm{SymCat}_{\mathrm{cc}}(\mathcal{R}(\mathcal{C}, J)) \simeq \mathrm{Map}(\mathrm{SymDist}(\mathcal{R}(\mathcal{C}, J))),$$

where $\mathcal{R}(\mathcal{C}, J)$ is the so-called *small quantaloid of closed cibles* (which Walters originally called the *bicategory of relations*) constructed from (\mathcal{C}, J) . In this section we shall show that sheaves on a small site (\mathcal{C}, J) (in the topos-theoretic sense) correspond with sheaves on the small involutive quantaloid $\mathcal{R}(\mathcal{C}, J)$ (in the sense of our Definition 2.2).

More precisely, we shall prove that, if $\mathcal{Q} = \mathcal{R}(\mathcal{C}, J)$ is a small quantaloid of closed cibles, then for any set \mathcal{E} of symmetric idempotents in \mathcal{Q} containing the identities, $\mathrm{SymDist}(\mathcal{Q})$ and $\mathrm{SymDist}(\mathcal{Q}_{\mathcal{E}})$ are equivalent modular quantaloids; and because each left adjoint in a modular quantaloid is necessarily a symmetric left adjoint, it follows that $\mathrm{Sh}(\mathcal{Q}, \mathcal{E})$ is equivalent to $\mathrm{Map}(\mathrm{SymDist}(\mathcal{Q}))$, which in turn is equivalent to $\mathrm{Sh}(\mathcal{C}, J)$ by Walters' [1982] result. To give our proof, we shall use the axiomatic description of $\mathcal{R}(\mathcal{C}, J)$ due to [Heymans and Stubbe, 2012], for it allows us to prove our claim via elementary computations in involutive quantaloids, much in the line of Freyd and Scedrov's [1990] work on allegories (see also [Johnstone, 2002]). In the next subsection we recall the necessary results from our earlier work.

Axioms for a small quantaloid of closed cibles

First we recall some definitions:

Definition 3.1 *A quantaloid \mathcal{Q} is:*

1. locally localic if, for all objects X and Y , $\mathcal{Q}(X, Y)$ is a locale,
2. map-discrete if, for any left adjoints $f: X \rightarrow Y$ and $g: X \rightarrow Y$ in \mathcal{Q} , $f \leq g$ implies $f = g$,
3. weakly tabular if, for every $q: X \rightarrow Y$ in \mathcal{Q} ,

$$q = \bigvee \left\{ fg^* \mid (f, g): X \rightrightarrows Y \text{ is a span of left adjoints such that } fg^* \leq q \right\},$$

4. map-tabular if for every $q: X \rightarrow Y$ in \mathcal{Q} there is a span $(f, g): X \rightrightarrows Y$ of left adjoints in \mathcal{Q} such that $fg^* = q$ and $f^*f \wedge g^*g = \mathbf{1}_{\mathrm{dom}(f)}$,
5. weakly modular if, for every pair of spans of left adjoints in \mathcal{Q} , say $(f, g): X \rightrightarrows Y$ and $(m, n): X \rightrightarrows Y$, we have $fg^* \wedge mn^* \leq f(g^*n \wedge f^*m)n^*$,

6. tabular if it is involutive and if for every $q: X \rightarrow Y$ in \mathcal{Q} there exists a span $(f, g): X \rightrightarrows Y$ of left adjoints in \mathcal{Q} such that $fg^\circ = q$ and $f^\circ f \wedge g^\circ g = 1_{\text{dom}(f)}$,
7. modular if it is involutive and if for any $f: X \rightarrow Y, g: Y \rightarrow Z$ and $h: X \rightarrow Z$ in \mathcal{Q} we have $gf \wedge h \leq g(f \wedge g^\circ h)$ (or equivalently, $gf \wedge h \leq (g \wedge hf^\circ)f$).

The notions of modularity¹ and tabularity are cited from Freyd and Scedrov [1990] who give them in the context of allegories². Weak modularity, weak tabularity and map-tabularity were introduced in [Heymans and Stubbe, 2012] with the specific aim to axiomatise small quantaloids of closed cibles.

There are many useful relations between several of these notions; we recall some of these in the next lemma.

Lemma 3.2 1. In any modular quantaloid, all left adjoints are symmetric left adjoints.

2. Any modular quantaloid is map-discrete.

3. Any locally localic and modular quantaloid is Cauchy-bilateral.

4. A small quantaloid \mathcal{Q} is weakly tabular if and only if $\text{Dist}(\mathcal{Q})$ is map-tabular.

5. A small quantaloid \mathcal{Q} is locally localic and modular if and only if $\text{Matr}(\mathcal{Q})$ is modular.

The first two statements in the above lemma appear in [Freyd and Scedrov, 1990; Johnstone, 2002], the third is quoted from [Heymans and Stubbe, 2011], and the two other statements come from [Heymans and Stubbe, 2012]. Also the following result appears in the latter reference.

Theorem 3.3 For a small quantaloid \mathcal{Q} , the following conditions are equivalent:

1. \mathcal{Q} is a small quantaloid of closed cibles, i.e. there exists a small site such that $\mathcal{Q} \simeq \mathcal{R}(\mathcal{C}, J)$,

¹In fact, J. Riguet [1948, p. 120] discovered much earlier what he called the *Dedekind formula* (in French, *la relation de Dedekind*) for relations between sets: if $R \subseteq E \times F, S \subseteq F \times G$ and $T \subseteq E \times G$, then $SR \cap T \subseteq (S \cap TR^\circ)(R \cap S^\circ T)$ (where R° is the opposite relation of R , etc.). Whereas it is obvious that the Dedekind formula implies the modular law, it is not difficult to see that the converse holds too: $SR \cap T = (SR \cap T) \cap (SR \cap T) \subseteq (SR \cap T) \cap (S \cap TR^\circ)R \subseteq (S \cap TR^\circ)((S \cap TR^\circ)^\circ(SR \cap T) \cap R) \subseteq (S \cap TR^\circ)(S^\circ T \cap R)$. All this can, of course, be done in any involutive locally ordered 2-category, and indeed Riguet certainly understood that the importance of the Dedekind formula went beyond the calculus of relations: he explains that the term *relation de Dedekind* was deliberately so chosen because “*elle contient comme cas particulier la relation entre idéaux dans un anneau découverte par Dedekind*”.

²Freyd and Scedrov [1990] define an *allegory* \mathcal{A} to be a locally posetal 2-category, equipped with an involution $\mathcal{A} \rightarrow \mathcal{A}^{\text{op}}$: $f \mapsto f^\circ$ (which fixes the objects, reverses the arrows, and preserves the local order), in which the modular law holds. Johnstone [2002] calls an allegory *geometric* when its hom-posets are complete lattices and composition distributes over arbitrary suprema. Thus, a geometric allegory is exactly the same thing as a modular quantaloid.

2. putting, for $X \in \text{Map}(\mathcal{Q})$,

$$J(X) := \left\{ S \text{ is a sieve on } X \mid 1_X = \bigvee_{s \in S} ss^* \right\}$$

defines a Grothendieck topology J on $\text{Map}(\mathcal{Q})$ for which $\mathcal{Q} \cong \mathcal{R}(\text{Map}(\mathcal{Q}), J)$,

3. \mathcal{Q} is locally localic, map-discrete, weakly tabular and weakly modular.

In this case, \mathcal{Q} carries an involution, sending $q: Y \rightarrow X$ to

$$q^\circ := \bigvee \left\{ gf^* \mid (f, g): Y \rightrightarrows X \text{ is a span of left adjoints such that } fg^* \leq q \right\},$$

which makes \mathcal{Q} also modular.

Splitting symmetric idempotents

In this subsection we study the properties of the involutive quantaloid $\text{SymDist}(\mathcal{Q})$ when \mathcal{Q} is a small quantaloid of closed cribles.

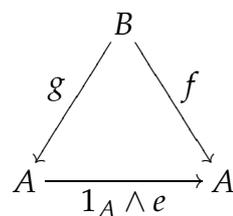
First we point out two useful conditions to determine whether a (small or large) involutive quantaloid \mathcal{Q}' has symmetric splittings for its symmetric idempotents. The first lemma can be found in [Freyd and Scedrov, 1990, 2.162; Johnstone, 2002, Lemma A3.3.3] and the second is a corollary of [Freyd and Scedrov, 1990, 2.166 and 2.169; Johnstone, 2002, A3.3.6 and A3.3.12]. For completeness' sake we shall give proofs here too, specifically adapted to the situation at hand.

Lemma 3.4 *If \mathcal{Q}' is a modular quantaloid, then each splitting of a symmetric idempotent is necessarily a symmetric splitting.*

Proof: First observe that for any $f: X \rightarrow Y$ in a modular \mathcal{Q}' we always have $f \leq ff^\circ f$: because $f = 1_Y f \wedge f \leq (1_Y \wedge ff^\circ) f \leq ff^\circ f$. Now suppose that $e: A \rightarrow A$, $p: A \rightarrow B$ and $q: B \rightarrow A$ satisfy $e = e^2 = e^\circ = qp$ and $pq = 1_B$ in \mathcal{Q}' . Then it follows that $q^\circ = pqq^\circ \leq pp^\circ pqq^\circ = pp^\circ q^\circ = p(qp)^\circ = pe^\circ = pe = p$, and similarly $p^\circ \leq q$. ■

Lemma 3.5 *If \mathcal{Q}' is a modular and tabular quantaloid in which all symmetric monads³ split, then all symmetric idempotents in \mathcal{Q}' have a (necessarily symmetric) splitting.*

Proof: Let $e: A \rightarrow A$ be a symmetric idempotent in \mathcal{Q}' : we shall exhibit a splitting. To that end, first consider a tabulation (f, g) of $e \wedge 1_A$:



³In the context of allegories, Freyd and Scedrov [1990] use the term *equivalence relation* (and Johnstone [2002] speaks simply of an *equivalence*) for what we call a symmetric monad; when it splits, then it does so symmetrically (because an allegory is modular), and they say that the equivalence relation is *effective*. If all equivalence relations in an allegory split, they say that the allegory is effective.

Thus, f and g are left adjoints in \mathcal{Q}' such that $fg^\circ = 1_A \wedge e$ and $g^\circ g \wedge f^\circ f = 1_B$. Because \mathcal{Q}' is modular, we know moreover that $f \dashv f^\circ$ and $g \dashv g^\circ$. It is useful to point out that $g \leq ef$ and $f \leq eg$ follow from these assumptions, and that, in turn, this implies that $g \leq eg$ and $eg = ef$.

Now define $t := (eg)^\circ(eg) = g^\circ eg: B \rightarrow B$. Then clearly $t^\circ = t$ holds; it is furthermore easy to check that $tt = g^\circ egg^\circ eg \leq g^\circ e1_A eg = g^\circ eg = t$; and $t = (eg)^\circ eg \geq g^\circ g \geq 1_B$ follows from inequalities pointed out above. In sum, this says that $t: B \rightarrow B$ is a symmetric monad. By assumption we can split t : there is a diagram

$$\begin{array}{ccc} 1_{B^t} & & t \\ \curvearrowright & \xrightarrow{h^\circ} & \curvearrowright \\ B^t & & B \\ \xleftarrow{h} & & \end{array}$$

such that $t = h^\circ h$ and $hh^\circ = 1_{B^t}$ (where, again by modularity, $h \dashv h^\circ$).

Next, consider the diagram

$$\begin{array}{ccc} t & & e \\ \curvearrowright & \xrightarrow{eg} & \curvearrowright \\ B & & A \\ \xleftarrow{(eg)^\circ} & & \end{array}$$

in which, by definition of t , we have $t = (eg)^\circ(eg)$. Using modularity of \mathcal{Q}' and the tabulation (f, g) of $1_A \wedge e$, we can compute that $e = e1_A e \wedge e \leq e(1_A \wedge e^\circ e e^\circ)e = e(1_A \wedge e)e = e(fg^\circ)e = (ef)(eg)^\circ = (eg)(eg)^\circ$. But $(eg)(eg)^\circ = egg^\circ e \leq ee = e$ follows immediately from $g \dashv g^\circ$, hence we obtain $e = (eg)(eg)^\circ$.

Composing these two diagrams produces a splitting in \mathcal{Q}' for the symmetric idempotent $e: A \rightarrow A$, as required. ■

For any small involutive quantaloid \mathcal{Q} , $\text{SymDist}(\mathcal{Q})$ is a quantaloid in which all symmetric monads split: simply because it is the universal splitting of symmetric monads in $\text{Matr}(\mathcal{Q})$. Below we shall furthermore prove that, whenever \mathcal{Q} is a small quantaloid of closed cribles, $\text{SymDist}(\mathcal{Q})$ is modular and tabular too.

Lemma 3.6 *If \mathcal{Q} is a locally localic quantaloid and \mathcal{E} is a collection of idempotents in \mathcal{Q} , then $\mathcal{Q}_{\mathcal{E}}$ is locally localic too.*

Proof: If $p: L \rightarrow L$ is an idempotent sup-morphism on a complete lattice, then $p(L) \subseteq L$ is a complete lattice too, with the same suprema as in L , but with $p(x) \wedge' p(y) := p(p(x) \wedge p(y))$ as binary infimum and $p(\top)$ as empty infimum (i.e. top element). A simple computation shows that, if L is a locale, then so is $p(L)$. This applies to $e_2 \circ - \circ e_1: \mathcal{Q}(X, Y) \rightarrow \mathcal{Q}(X, Y)$ for idempotents $e_1: X \rightarrow X$ and $e_2: Y \rightarrow Y$ in \mathcal{E} , to show that $\mathcal{Q}_{\mathcal{E}}(e_1, e_2)$ is a locale whenever $\mathcal{Q}(X, Y)$ is; hence $\mathcal{Q}_{\mathcal{E}}$ is locally localic whenever \mathcal{Q} is. ■

Lemma 3.7 *If \mathcal{Q} is a modular quantaloid and \mathcal{E} is a collection of symmetric idempotents in \mathcal{Q} , then $\mathcal{Q}_{\mathcal{E}}$ is modular too.*

Proof: Local suprema, composition and involution in $\mathcal{Q}_{\mathcal{E}}$ are the same as in \mathcal{Q} . As pointed out in the above proof, the infimum of $f, g: e_1 \rightarrow e_2$ in $\mathcal{Q}_{\mathcal{E}}$ is $f \wedge' g :=$

$e_2(f \wedge g)e_1$, but thanks to the modular law it is easily seen that

$$f \wedge g = e_2f \wedge ge_1 \leq e_2(fe_1^\circ \wedge e_2^\circ g)e_1 = e_2(f \wedge g)e_1,$$

whereas $e_2(f \wedge g)e_1 \leq f \wedge g$ is always valid, hence in this case the local binary infima in $\mathcal{Q}_\mathcal{E}$ are the same as in \mathcal{Q} . Thus it follows that $\mathcal{Q}_\mathcal{E}$ is modular whenever \mathcal{Q} is. ■

Proposition 3.8 *If \mathcal{Q} is a small, locally localic, modular quantaloid, then $\text{SymDist}(\mathcal{Q})$ is modular.*

Proof: $\text{Matr}(\mathcal{Q})$ is modular by Lemma 3.2, so $\text{SymDist}(\mathcal{Q}) = (\text{Matr}(\mathcal{Q}))_\mathcal{E}$, with \mathcal{E} the collection of symmetric monads in $\text{Matr}(\mathcal{Q})$, is modular too by Lemma 3.7. ■

Proposition 3.9 *If \mathcal{Q} is a small, weakly tabular, Cauchy-bilateral quantaloid, then $\text{SymDist}(\mathcal{Q})$ is tabular.*

Proof: From [Heymans and Stubbe, 2012, Proposition 3.5] we recall that a small quantaloid \mathcal{Q} is weakly tabular if and only if $\text{Dist}(\mathcal{Q})$ is map-tabular. The proof for the necessity goes as follows: Suppose that $\Phi: \mathbb{A} \rightrightarrows \mathbb{B}$ is a distributor. We can assume without loss of generality that \mathbb{A} and \mathbb{B} are Cauchy complete, because every \mathcal{Q} -category is isomorphic to its Cauchy completion in $\text{Dist}(\mathcal{Q})$. Now define the \mathcal{Q} -category \mathbb{R} to be the full subcategory of $\mathbb{A} \times \mathbb{B}$ whose objects are those $(a, b) \in \mathbb{A} \times \mathbb{B}$ for which $1_{ta} \leq \Phi(a, b)$, and write T (resp. S) for the composition of the inclusion $\mathbb{R} \hookrightarrow \mathbb{A} \times \mathbb{B}$ with the projection of $\mathbb{A} \times \mathbb{B}$ onto \mathbb{A} (resp. onto \mathbb{B}). By construction we then have $\mathbb{B}(S-, S-) \wedge \mathbb{A}(T-, T-) = \mathbb{R}$; and, relying on the weak tabularity of \mathcal{Q} and the Cauchy completeness of \mathbb{A} and \mathbb{B} , a lengthy computation shows that $\Phi = \mathbb{A}(-, T-) \otimes \mathbb{B}(S-, -)$. That is to say, the left adjoints $\mathbb{A}(-, T-): \mathbb{R} \rightrightarrows \mathbb{A}$ and $\mathbb{B}(-, S-): \mathbb{R} \rightrightarrows \mathbb{B}$ in $\text{Dist}(\mathcal{Q})$ provide for a map-tabulation of $\Phi: \mathbb{A} \rightrightarrows \mathbb{B}$.

We now modify this proof to suit our needs. For any $\Phi: \mathbb{B} \rightrightarrows \mathbb{A}$ in $\text{SymDist}(\mathcal{Q})$ we must find $\Sigma: \mathbb{R} \rightrightarrows \mathbb{A}$ and $\Theta: \mathbb{R} \rightrightarrows \mathbb{B}$ in $\text{SymDist}(\mathcal{Q})$ such that $\Sigma \otimes \Theta^\circ = \Phi$ and $\Sigma^\circ \otimes \Sigma \wedge \Theta^\circ \otimes \Theta = \mathbb{R}$. If \mathcal{Q} is Cauchy-bilateral then the Cauchy completion of a symmetric \mathcal{Q} -category is again symmetric, hence any symmetric \mathcal{Q} -category is isomorphic to its Cauchy completion in $\text{SymDist}(\mathcal{Q})$ (and not merely in $\text{Dist}(\mathcal{Q})$). Therefore we may still suppose that \mathbb{A} and \mathbb{B} are Cauchy complete. Referring to the above, the category \mathbb{R} is clearly symmetric whenever \mathbb{A} and \mathbb{B} are, and the left adjoint distributors represented by the functors $S: \mathbb{R} \rightarrow \mathbb{B}$ and $T: \mathbb{R} \rightarrow \mathbb{A}$ are evidently symmetric left adjoints. Thus the result follows. ■

In view of Theorem 3.3 we may now conclude from the above:

Theorem 3.10 *If \mathcal{Q} is a small quantaloid of closed cribles, then $\text{SymDist}(\mathcal{Q})$ is a modular and tabular quantaloid in which all symmetric idempotents split symmetrically.*

Change of base

This subsection is devoted to the proof of the fact that, when \mathcal{Q} is a small quantaloid of closed cribles and \mathcal{E} is a class of symmetric idempotents in \mathcal{Q} containing all identities, then also $\mathcal{Q}_\mathcal{E}$ is a small quantaloid of closed cribles, and the

involutive quantaloids $\text{SymDist}(\mathcal{Q})$ and $\text{SymDist}(\mathcal{Q}_{\mathcal{E}})$ are equivalent. To tackle this problem, we study the “change of base” homomorphism from $\text{SymDist}(\mathcal{Q})$ to $\text{SymDist}(\mathcal{Q}_{\mathcal{E}})$ which is determined by the universal property of splitting symmetric idempotents. Let us first recall the appropriate terminology.

Let \mathcal{Q} and \mathcal{Q}' be small involutive quantaloids and $F: \mathcal{Q} \rightarrow \mathcal{Q}'$ be a homomorphism that preserves the involution. It is easily seen that a symmetric \mathcal{Q} -category \mathbb{A} determines a symmetric \mathcal{Q}' -category $F\mathbb{A}$ by putting:

- objects: $(F\mathbb{A})_0 = \mathbb{A}_0$ with types $t_{F\mathbb{A}}a = F(ta)$ in \mathcal{Q}'_0 ,
- hom-arrows: $(F\mathbb{A})(a', a) = F(\mathbb{A}(a', a))$ for all objects a, a' .

Similarly for distributors, and F so determines a homomorphism $\bar{F}: \text{SymDist}(\mathcal{Q}) \rightarrow \text{SymDist}(\mathcal{Q}')$ of involutive quantaloids that makes the diagram

$$\begin{array}{ccc}
 \mathcal{Q} & \xrightarrow{F} & \mathcal{Q}' \\
 \downarrow I & & \downarrow I' \\
 \text{SymDist}(\mathcal{Q}) & \xrightarrow{\bar{F}} & \text{SymDist}(\mathcal{Q}')
 \end{array} \tag{8}$$

commute: \bar{F} is the *change of base* homomorphism induced by F . (Of course, I denotes the canonical inclusion of \mathcal{Q} in $\text{SymDist}(\mathcal{Q})$, and similarly for I' .)

Now we recall a necessary and sufficient condition for the “change of base” induced by some $F: \mathcal{Q} \rightarrow \mathcal{Q}'$ to be an equivalence. As it is straightforward to verify that $F: \mathcal{Q} \rightarrow \mathcal{Q}'$ is fully faithful if and only if the change of base \bar{F} is fully faithful, we need to take a closer look at the essential surjectivity on objects of \bar{F} .

Lemma 3.11 *Let $F: \mathcal{Q} \rightarrow \mathcal{Q}'$ be a homomorphism of small involutive quantaloids. The change of base $\bar{F}: \text{SymDist}(\mathcal{Q}) \rightarrow \text{SymDist}(\mathcal{Q}')$ is an equivalence of involutive quantaloids if and only if there exists a fully faithful homomorphism of involutive quantaloids $G: \mathcal{Q}' \rightarrow \text{SymDist}(\mathcal{Q})$ making the diagram below essentially commutative:*

$$\begin{array}{ccc}
 \mathcal{Q} & \xrightarrow{F} & \mathcal{Q}' \\
 \downarrow I & & \swarrow G \\
 \text{SymDist}(\mathcal{Q}) & &
 \end{array}$$

Proof : First suppose that \bar{F} is an equivalence. Considering the commutative square in (8), the required fully faithful G is obtained by composing I' with the pseudo-inverse of \bar{F} .

Conversely, suppose that a fully faithful G exists such that $G \circ F \cong I$. Because G and I are fully faithful, so is F , and thus also \bar{F} . Size issues apart, also I and G

induce a change of base, and we end up with an essentially commutative diagram

$$\begin{array}{ccc}
 \text{SymDist}(\mathcal{Q}) & \xrightarrow{\bar{F}} & \text{SymDist}(\mathcal{Q}') \\
 \bar{I} \downarrow & \swarrow \bar{G} & \\
 \text{SymDist}(\text{SymDist}(\mathcal{Q})) & &
 \end{array}$$

The homomorphisms \bar{I} , \bar{F} and \bar{G} are fully faithful, because I , F and G are. If we show that \bar{I} is essentially surjective on objects, then it is an equivalence, and hence so is \bar{F} .

To see that \bar{I} is indeed essentially surjective on objects, one can do as follows. Given \mathbf{C} in $\text{SymDist}(\text{SymDist}(\mathcal{Q}))$, let us explicitly write the hom-arrow from an object $x \in \mathbf{C}$ to an object $y \in \mathbf{C}$ as $\Gamma_{y,x}: \mathbb{A}_x \rightarrow \mathbb{A}_y$; these morphisms in $\text{SymDist}(\mathcal{Q})$ satisfy the conditions that make \mathbf{C} a symmetric category: $\mathbb{A}_x \leq \Gamma_{x,x}$, $\bigvee_{y \in \mathbf{C}} \Gamma_{z,y} \otimes \Gamma_{y,x} \leq \Gamma_{z,x}$ and $\Gamma_{x,y} = \Gamma_{y,x}^\circ$ (for all $x, y \in \mathbf{C}$). With these data, we define a symmetric \mathcal{Q} -category \mathbb{A} as follows:

- objects: $\mathbb{A}_0 := \biguplus_{x \in \mathbf{C}} \mathbb{A}_x$, with inherited types,
- hom-arrows: for $u \in \mathbb{A}_x$ and $v \in \mathbb{A}_y$, $\mathbb{A}(v, u) := \Gamma_{y,x}(v, u)$.

Regarding \mathbb{A} now as an object in $\text{SymDist}(\text{SymDist}(\mathcal{Q}))$, via the change of base \bar{I} , we further define a distributor $\Gamma: \bar{I}(\mathbb{A}) \rightarrow \mathbf{C}$ by:

- distributor-elements: for $u \in \mathbb{A}_x$ and $y \in \mathbf{C}$, $\Gamma(y, u) := \Gamma_{y,x}(-, u)$.

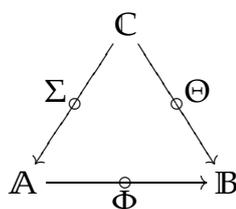
It is then a fact that $\Gamma \otimes \Gamma^\circ = \mathbf{C}$ and $\Gamma^\circ \otimes \Gamma = \bar{I}(\mathbb{A})$. All verifications are long but straightforward computations. ■

In the exact same situation as in the above lemma, we can sometimes say more:

Lemma 3.12 *In the situation of Lemma 3.11, if G is fully faithful and $\text{SymDist}(\mathcal{Q})$ is modular and tabular then \mathcal{Q}' is modular and weakly tabular.*

Proof : Modularity of \mathcal{Q}' follows straightforwardly from the modularity of $\text{SymDist}(\mathcal{Q})$ and the fully faithful homomorphism $\mathcal{Q}' \rightarrow \text{SymDist}(\mathcal{Q})$ of involutive quantaloids.

To deduce the weak tabularity of \mathcal{Q}' from the tabularity and modularity of $\text{SymDist}(\mathcal{Q})$, we first make a helpful observation. Given any $\Phi: \mathbb{A} \rightarrow \mathbb{B}$ in $\text{SymDist}(\mathcal{Q})$, let



be a tabulation; then, in particular, $\Phi = \Theta \otimes \Sigma^\circ$ and $\Sigma \dashv \Sigma^\circ$. Now consider the family

$$\left(\mathbb{C}(-, c) : *_{tc} \dashv \rightarrow \mathbb{C} \right)_{c \in \mathbb{C}}$$

of all representable presheaves on \mathbb{C} , each of which is a left adjoint in $\text{SymDist}(\mathbb{Q})$. Precomposing both $\Sigma : \mathbb{C} \dashv \rightarrow \mathbb{A}$ and $\Theta : \mathbb{C} \dashv \rightarrow \mathbb{B}$ with these thus gives a family, indexed by the $c \in \mathbb{C}$,

$$\begin{array}{ccc} & *_{tc} & \\ \swarrow & & \searrow \\ \Sigma \otimes \mathbb{C}(-, c) = \Sigma(-, c) & & \Theta(-, c) = \Theta \otimes \mathbb{C}(-, c) \\ \swarrow & & \searrow \\ \mathbb{A} & & \mathbb{B} \end{array}$$

of spans of left adjoints in $\text{SymDist}(\mathbb{Q})$, whose domains are in the image of the canonical embedding $\mathbb{Q} \hookrightarrow \text{SymDist}(\mathbb{Q})$, such that

$$\Phi = \bigvee_{c \in \mathbb{C}} \Theta(-, c) \otimes \left(\Sigma(-, c) \right)^*$$

In particular, if $\Phi : \mathbb{A} \dashv \rightarrow \mathbb{B}$ is in the image of $G : \mathbb{Q}' \rightarrow \text{SymDist}(\mathbb{Q})$, then – because the image of $\mathbb{Q} \hookrightarrow \text{SymDist}(\mathbb{Q})$ is contained in the image of G – it admits a weak tabulation by spans of left adjoints in the image of G . By fully faithfulness of G , \mathbb{Q}' is weakly tabular. ■

The above results apply in particular when \mathbb{Q} is a small quantaloid of closed cribles and when we put $\mathbb{Q}' = \mathbb{Q}_{\text{ssi}}$: they show that splitting the symmetric idempotents in a small quantaloid of closed cribles is “harmless” for the theory of sheaves. In fact, instead of splitting *all* symmetric idempotents, we can choose to split only those in a class \mathcal{E} of symmetric idempotents containing all identities.

Theorem 3.13 *If \mathbb{Q} is a small quantaloid of closed cribles and \mathcal{E} is a class of symmetric idempotents in \mathbb{Q} containing all identities, then also $\mathbb{Q}_{\mathcal{E}}$ is a small quantaloid of closed cribles and the inclusion $\mathbb{Q} \hookrightarrow \mathbb{Q}_{\mathcal{E}}$ induces an equivalence $\text{SymDist}(\mathbb{Q}) \simeq \text{SymDist}(\mathbb{Q}_{\mathcal{E}})$ of involutive quantaloids.*

Proof: If \mathbb{Q} is a small quantaloid of closed cribles, then it is locally localic, hence so is $\mathbb{Q}_{\mathcal{E}}$, by Lemma 3.6. The other results follow from the commutative diagram

$$\begin{array}{ccc} \mathbb{Q} & \xrightarrow{\quad} & \mathbb{Q}_{\mathcal{E}} \\ \downarrow & \searrow & \\ \text{SymDist}(\mathbb{Q}) & & \end{array}$$

of fully faithful functors, and the fact that $\text{SymDist}(\mathbb{Q})$ is modular and tabular. ■

Of course, taking \mathcal{E} to be the class of all symmetric idempotents in \mathbb{Q} , we find that \mathbb{Q}_{ssi} is a small quantaloid of closed cribles such that $\text{SymDist}(\mathbb{Q}) \simeq$

$\text{SymDist}(\mathcal{Q}_{\text{ssi}})$. But taking \mathcal{E} to be the class of all symmetric monads in \mathcal{Q} , or the class of all symmetric comonads⁴, produces other important examples.

Walters' theorem revisited

We now have everything ready to make the following extension to the result of [Walters, 1982]. As is customary, we write $\text{Rel}(\mathcal{T})$ for the quantaloid of internal relations in a topos \mathcal{T} . The next theorem excludes all confusion with our earlier notation $\text{Rel}(\mathcal{Q})$.

Theorem 3.14 *For any small site (\mathcal{C}, J) , any small quantaloid $\mathcal{Q} \simeq \mathcal{R}(\mathcal{C}, J)$ and any set \mathcal{E} of symmetric idempotents in \mathcal{Q} containing all identities, we have the following equivalences:*

1. $\text{Sh}(\mathcal{Q}, \mathcal{E}) \simeq \text{SymCat}_{\text{cc}}(\mathcal{Q}) \simeq \text{Sh}(\mathcal{C}, J)$,
2. $\text{Rel}(\mathcal{Q}, \mathcal{E}) \simeq \text{SymDist}(\mathcal{Q}) \simeq \text{Rel}(\text{Sh}(\mathcal{C}, J))$.

Proof: This proof relies on Walters' [1982, p. 101] theorem that the topos $\text{Sh}(\mathcal{C}, J)$ is biequivalent to the bicategory $\text{SymCat}_{\text{cc}}(\mathcal{R}(\mathcal{C}, J))$ (Walters' insistence on the term *biequivalence* stresses the fact that a single morphism in the category $\text{Sh}(\mathcal{C}, J)$ gets identified with an equivalence class of morphisms in the bicategory $\text{SymCat}_{\text{cc}}(\mathcal{R}(\mathcal{C}, J))$ whose homs are symmetric preorders), on Freyd and Scecrow's [1990, 2.148] theorem that any tabular allegory \mathcal{A} is equivalent (as allegory) to the allegory $\text{Rel}(\text{Map}(\mathcal{A}))$ of internal relations in the regular category $\text{Map}(\mathcal{A})$, and on the particular properties of $\text{SymDist}(\mathcal{Q})$, for \mathcal{Q} a small quantaloid of closed cribles, that we summarised in Theorems 3.10 and 3.13.

Because \mathcal{Q} is a small quantaloid of closed relations, so is $\mathcal{Q}_{\mathcal{E}}$ (see Theorem 3.13); one particular consequence is that, for symmetric categories enriched in either quantaloid, the symmetric completion coincides with the Cauchy completion (cf. Lemma 3.2 and Proposition 2.4). Furthermore, again by Theorem 3.13, $\text{SymDist}(\mathcal{Q})$ is equivalent to $\text{SymDist}(\mathcal{Q}_{\mathcal{E}})$. All this justifies the following equivalences of involutive quantaloids:

$$\text{Rel}(\mathcal{Q}, \mathcal{E}) := \text{SymDist}_{\text{sc}}(\mathcal{Q}_{\mathcal{E}}) = \text{SymDist}_{\text{cc}}(\mathcal{Q}_{\mathcal{E}}) \simeq \text{SymDist}(\mathcal{Q}_{\mathcal{E}}) \simeq \text{SymDist}(\mathcal{Q}).$$

By Theorem 3.10 we know that $\text{SymDist}(\mathcal{Q})$ is a modular quantaloid, hence so are its equivalents; all left adjoints in the above involutive quantaloids are therefore symmetric left adjoints, by Lemma 3.2. Taking (symmetric) left adjoints therefore produces the following equivalences of categories (or rather, biequivalences of 2-categories which are locally symmetrically ordered):

$$\begin{aligned} \text{Sh}(\mathcal{Q}, \mathcal{E}) := \text{Cat}_{\text{sc}}(\mathcal{Q}_{\mathcal{E}}) &\simeq \text{SymMap}(\text{SymDist}_{\text{sc}}(\mathcal{Q}_{\mathcal{E}})) \simeq \\ &\text{Map}(\text{SymDist}(\mathcal{Q})) \simeq \text{SymCat}_{\text{cc}}(\mathcal{Q}). \end{aligned}$$

⁴In a modular quantaloid \mathcal{Q} , an arrow $c: A \rightarrow A$ is a symmetric comonad if and only if it satisfies $c \leq 1_A$. Indeed, if $c \leq 1_A$ then $c^\circ \leq 1_A$ too, hence $c \leq cc^\circ c$ (cf. the proof of Lemma 3.4) implies that $c \leq cc$ but also that $c \leq c^\circ$, and by involution also $c^\circ \leq c$. Therefore, particularly in the context of allegories, the term *coreflexive* is often used.

Invoking at this point Walters' theorem, this proves (1). But because the involutive quantaloid $\text{SymDist}(\mathcal{Q})$ is not only modular but also tabular (see again Theorem 3.10), Freyd and Scedrov's theorem proves it to be equivalent to the involutive quantaloid of internal relations in $\text{Map}(\text{SymDist}(\mathcal{Q}))$, which in turn proves (2). ■

The theorem above thus says two things about a small quantaloid of closed cibles \mathcal{Q} and a set \mathcal{E} of symmetric idempotents in \mathcal{Q} containing all identities: firstly, that the category $\text{Sh}(\mathcal{Q}, \mathcal{E}) := \text{SymCat}_{\text{sc}}(\mathcal{Q}_{\mathcal{E}})$ is the category of sheaves on a site; secondly, that this category $\text{Sh}(\mathcal{Q}, \mathcal{E})$ admits, up to equivalence, the simpler description $\text{SymCat}_{\text{cc}}(\mathcal{Q})$. (And similar for $\text{Rel}(\mathcal{Q}, \mathcal{E})$.) Choosing \mathcal{E} to be the set of all symmetric idempotents in \mathcal{Q} , we find:

Corollary 3.15 *If \mathcal{Q} is a small quantaloid of closed cibles, then $\text{Sh}(\mathcal{Q})$ is a Grothendieck topos and $\text{Rel}(\mathcal{Q})$ is its category of relations.*

Symmetric vs. discrete

In this subsection we wish to make a remark on the symmetry axiom that we used in Definition 2.2 of \mathcal{Q} -sheaves. In any locally ordered category \mathcal{K} , an object D is said to be *discrete* when, for any other object $X \in \mathcal{K}$, the order $\mathcal{K}(X, D)$ is symmetric. In [Heymans and Stubbe, 2012] we showed that, for a Cauchy-bilateral quantaloid \mathcal{Q} , every symmetric and Cauchy complete \mathcal{Q} -category is a discrete object of $\text{Cat}_{\text{cc}}(\mathcal{Q})$. In general the converse need not hold, but:

Proposition 3.16 *If \mathcal{Q} is a small quantaloid of closed cibles, then a Cauchy complete \mathcal{Q} -category is discrete in $\text{Cat}_{\text{cc}}(\mathcal{Q})$ if and only if it is symmetric.*

Proof : Suppose that \mathbb{A} is a discrete object in $\text{Cat}_{\text{cc}}(\mathcal{Q})$; we seek to prove that $\mathbb{A}(y, x) = \mathbb{A}(x, y)^\circ$ for any $x, y \in \mathbb{A}$. Relying in particular on the weak tabularity of \mathcal{Q} , it is sufficient to show that, for any span $(f, g): ty \dashrightarrow tx$ of left adjoints in \mathcal{Q} ,

$$fg^* \leq \mathbb{A}(x, y) \iff fg^* \leq \mathbb{A}(y, x)^\circ.$$

But, because \mathbb{A} is Cauchy complete, for any such span (f, g) we can consider the tensors $x \otimes f$ and $y \otimes g$ in \mathbb{A} , and writing $U = \text{dom}(f) = \text{dom}(g)$ we indeed have

$$\begin{aligned} f \circ g^* \leq \mathbb{A}(x, y) &\iff 1_U \leq \mathbb{A}(x \otimes f, y \otimes g) \\ &\iff 1_U \leq \mathbb{A}(y \otimes g, x \otimes f) \\ &\iff g \circ f^* \leq \mathbb{A}(y, x) \\ &\iff f \circ g^* \leq \mathbb{A}(y, x)^\circ \end{aligned}$$

where the second equivalence is due to the discreteness of \mathbb{A} and the last equivalence holds because $f^* = f^\circ$ due to the modularity of \mathcal{Q} . ■

If \mathcal{Q} is a small quantaloid of closed cibles, then so is \mathcal{Q}_{ssi} , and the Cauchy completion and symmetric completion of a symmetric \mathcal{Q}_{ssi} -enriched category coincide. Thus we find:

Corollary 3.17 *If \mathcal{Q} is a small quantaloid of closed cibles, then $\text{Sh}(\mathcal{Q})$ is the full subcategory of discrete objects of $\text{Ord}(\mathcal{Q})$ and $\text{Rel}(\mathcal{Q})$ is the full subquantaloid of discrete objects of $\text{Idl}(\mathcal{Q})$.*

That is to say, whereas we defined the objects of $\text{Sh}(\mathcal{Q})$ as the *symmetric* objects in $\text{Ord}(\mathcal{Q})$, we now find that they are exactly the *discrete* objects.

4 Grothendieck quantaloids and quantales

In the previous section we showed that, for \mathcal{Q} a small quantaloid of closed cibles, $\text{Sh}(\mathcal{Q})$ is a Grothendieck topos and $\text{Rel}(\mathcal{Q})$ is its category of relations. Given this result, it is a natural to ask whether this is the case for other involutive quantaloids too; and if so, for which ones. Precisely, we wish to find necessary and sufficient conditions on \mathcal{Q} for $\text{Rel}(\mathcal{Q})$ to be the category of relations in a topos.

Definition 4.1 *A small involutive quantaloid \mathcal{Q} is called a Grothendieck quantaloid (if \mathcal{Q} has only one object we speak of a Grothendieck quantale) if there exists a topos \mathcal{T} such that there is an equivalence $\text{Rel}(\mathcal{T}) \simeq \text{Rel}(\mathcal{Q})$ of involutive quantaloids.*

A sufficient condition on \mathcal{Q} is being a small quantaloid of closed cibles. On the other hand, the internal relations in a topos form a modular quantaloid, and \mathcal{Q} is a full subquantaloid of $\text{Rel}(\mathcal{Q})$, so a necessary condition will be the modularity of \mathcal{Q} . To establish a precise necessary-and-sufficient condition, we first point out a connection with *projection matrices*.

Definition 4.2 *If \mathcal{Q} is a small involutive quantaloid, then $\text{ProjMatr}(\mathcal{Q}) := \text{Matr}(\mathcal{Q})_{\text{ssi}}$ is the involutive quantaloid of projection matrices⁵.*

Straightforwardly extending the terminology for quantales [Resende, 2012], we shall say that a quantaloid \mathcal{Q} is *stably Gelfand* if it is an involutive quantaloid in which $ff^\circ f \leq f$ implies $f \leq ff^\circ f$ for any morphism $f: X \rightarrow Y$. Any modular quantaloid is trivially stably Gelfand, as seen in the proof of Lemma 3.4.

Lemma 4.3 *1. For a stably Gelfand quantaloid \mathcal{Q} there is an equivalence $\text{ProjMatr}(\mathcal{Q}) \simeq \text{SymDist}(\mathcal{Q}_{\text{ssi}}) \simeq \text{Rel}(\mathcal{Q})$ of involutive quantaloids.*

2. If \mathcal{Q} is a small quantaloid of closed cibles, then $\text{ProjMatr}(\mathcal{Q})$ and $\text{SymDist}(\mathcal{Q})$ are equivalent involutive quantaloids.

3. A small involutive quantaloid \mathcal{Q} is a Grothendieck quantaloid if and only if there exists a topos \mathcal{T} such that there is an equivalence $\text{Rel}(\mathcal{T}) \simeq \text{ProjMatr}(\mathcal{Q})$ of involutive quantaloids.

Proof: (1) Let $P: X \rightarrow X$ be a symmetric idempotent in $\text{Matr}(\mathcal{Q})$; that is to say, X is a \mathcal{Q} -typed set, and P is a collection of \mathcal{Q} -morphisms $P(x', x): tx \rightarrow tx'$, one for each $(x, x') \in X \times X$, such that

$$\bigvee_{x'' \in X} P(x', x'') \circ P(x'', x) = P(x', x) = P(x, x')^\circ \text{ for every } (x, x') \in X.$$

⁵In [Freyd and Scedrov, 1990, 2.226], in the context of allegories rather than quantaloids, this construction is referred to as the *systemic completion*.

From this it is clear that $P(x, x') \circ P(x, x')^\circ \circ P(x, x') \leq P(x, x')$, so that by hypothesis the converse inequality holds too. The computation

$$\begin{aligned} P(x, x') &\leq P(x, x') \circ P(x, x')^\circ \circ P(x, x') \\ &= P(x, x') \circ P(x', x) \circ P(x, x') \\ &\leq P(x, x) \circ P(x, x') \\ &\leq P(x, x'). \end{aligned}$$

then shows that $P(x, x') = P(x, x) \circ P(x, x')$; and similarly for $P(x, x') = P(x, x') \circ P(x', x)$. In other words, each $P(x, x)$ is an object of \mathcal{Q}_{ssi} , and each $P(x, x')$ is a morphism in \mathcal{Q}_{ssi} from $P(x', x')$ to $P(x, x)$. As a consequence, we can define a symmetric \mathcal{Q}_{ssi} -category \mathbb{P} whose \mathcal{Q}_{ssi} -typed object set is X with types $tx := P(x, x)$, and whose hom-arrows are $\mathbb{P}(x, x') := P(x, x')$. Note that the \mathcal{Q}_{ssi} -category \mathbb{P} is *normal* in the sense of [Stubbe, 2005b]: all of its endo-hom-arrows are identities. Furthermore, if both $P: X \rightarrow X$ and $Q: Y \rightarrow Y$ are projection matrices, and $M: P \rightarrow Q$ is a morphism in $\text{ProjMatr}(\mathcal{Q})$, i.e. a matrix $M: X \rightarrow Y$ such that $Q \circ M = M = M \circ P$, then we can define a distributor $\Phi: \mathbb{P} \rightarrow \mathcal{Q}$ with elements $\Phi(y, x) = M(y, x)$. In fact, each distributor between \mathbb{P} and \mathcal{Q} arises in this way. In short, the correspondence $P \mapsto \mathbb{P}$ extends to an equivalence of involutive quantaloids between $\text{ProjMatr}(\mathcal{Q})$ and the full involutive subquantaloid of $\text{SymDist}(\mathcal{Q}_{\text{ssi}})$ of the *normal* symmetric \mathcal{Q}_{ssi} -categories (compare with [Stubbe, 2005b, Lemma 6.1]). But furthermore, a long but straightforward computation shows that each symmetric \mathcal{Q}_{ssi} -category is Morita equivalent with a normal symmetric \mathcal{Q}_{ssi} -category: so $\text{SymDist}(\mathcal{Q}_{\text{ssi}})$ is equivalent to its full involutive subquantaloid of *normal* objects (compare with [Stubbe, 2005b, Lemma 6.2]). Taken together, all this proves that the correspondence $P \mapsto \mathbb{P}$ extends to an equivalence of involutive quantaloids between $\text{ProjMatr}(\mathcal{Q})$ and $\text{SymDist}(\mathcal{Q}_{\text{ssi}})$. Finally, by Proposition 2.4 the latter is furthermore equivalent to $\text{Rel}(\mathcal{Q}) := \text{SymDist}_{\text{sc}}(\mathcal{Q}_{\text{ssi}})$ (as involutive quantaloid).

(2) Holds by Theorem 3.14, taking \mathcal{E} to be the set of all symmetric idempotents in \mathcal{Q} .

(3) If \mathcal{Q} is a Grothendieck quantaloid, then $\text{Rel}(\mathcal{Q}) \simeq \text{Rel}(\mathcal{T})$ for some topos \mathcal{T} , so \mathcal{Q} is modular because it is a full involutive subquantaloid of $\text{Rel}(\mathcal{Q})$. If, on the other hand, we assume that $\text{ProjMatr}(\mathcal{Q}) \simeq \text{Rel}(\mathcal{T})$ for some topos \mathcal{T} , then again \mathcal{Q} is modular, now because it is a full involutive subquantaloid of $\text{ProjMatr}(\mathcal{Q})$. In either case, \mathcal{Q} is certainly stably Gelfand, so $\text{ProjMatr}(\mathcal{Q}) \simeq \text{Rel}(\mathcal{Q})$ by the first statement in this Lemma, which proves $\text{ProjMatr}(\mathcal{Q}) \simeq \text{Rel}(\mathcal{T}) \simeq \text{Rel}(\mathcal{Q})$ in either case. ■

We shall now recall some notions that [Freyd and Scedrov, 1990, 2.216(1), 2.225] introduced in the context of allegories, but that we adopt here for quantaloids. (In fact, the property that we call ‘weak semi-simplicity’ was not given a name by Freyd and Scedrov [1990].)

Definition 4.4 A morphism $q: X \rightarrow Y$ in an involutive quantaloid \mathcal{Q} is:

1. simple if $qq^\circ \leq 1_Y$,
2. semi-simple if there are simple morphisms f and g such that $q = fg^\circ$,

3. weakly semi-simple if $q = \bigvee \{fg^\circ \mid fg^\circ \leq q \text{ with } f \text{ and } g \text{ simple}\}$.

And an involutive quantaloid \mathcal{Q} is (weakly) (semi-)simple if each of its morphisms is.

The next lemma can be found in [Freyd and Scedrov, 1990, 2.16(10)], but we spell out its proof for later reference.

Lemma 4.5 *A modular quantaloid \mathcal{Q} is semi-simple if and only if \mathcal{Q}_{ssi} is tabular.*

Proof: Suppose that \mathcal{Q}_{ssi} is tabular. Given a morphism $q: X \rightarrow Y$ in \mathcal{Q} , it can be included in \mathcal{Q}_{ssi} as $q: 1_X \rightarrow 1_Y$, so let $q = fg^\circ$ be a tabulation in \mathcal{Q}_{ssi} . If \mathcal{Q} is modular then so is \mathcal{Q}_{ssi} (by Lemma 3.7) so every left adjoint is a symmetric left adjoint. From this it is straightforward that both f and g are simple morphisms in \mathcal{Q} such that $q = fg^\circ$.

Conversely, suppose first that \mathcal{Q} is only semi-simple, and let $q \in \mathcal{Q}_{\text{ssi}}(r, p)$; that is to say, $r: X \rightarrow X$ and $p: Y \rightarrow Y$ are symmetric idempotents in \mathcal{Q} , and $q: X \rightarrow Y$ is a morphism in \mathcal{Q} satisfying $pq = q = qr$. Now let $b: Z \rightarrow X$ and $a: Z \rightarrow Y$ be simple morphisms in \mathcal{Q} such that $ab^\circ = q$. It is then straightforward to check that $pa: 1_Z \rightarrow p$ and $rb: 1_Z \rightarrow r$ are simple morphisms in \mathcal{Q}_{ssi} such that $(pa)(rb)^\circ = q$. In other words, \mathcal{Q}_{ssi} is semi-simple whenever \mathcal{Q} is. Now the other hypothesis says that \mathcal{Q} is also modular; by Lemma 3.7 we know that \mathcal{Q}_{ssi} is modular too. It thus remains to prove that \mathcal{Q}_{ssi} is tabular when it is semi-simple and modular. So again, let $q \in \mathcal{Q}_{\text{ssi}}(r, p)$ and suppose now that $x: e \rightarrow p$ and $y: e \rightarrow r$ are simple morphisms in \mathcal{Q}_{ssi} such that $q = xy^\circ$. Simplicity of x and y makes $z := x^\circ x \wedge y^\circ y \in \mathcal{Q}_{\text{ssi}}(e, e)$ a symmetric morphism satisfying $zz \leq z$; it follows from the modular law that it is therefore a symmetric idempotent, and furthermore that $xzy^\circ = xy^\circ$. Choosing a (necessarily symmetric) splitting of z in \mathcal{Q}_{ssi} , say a $w \in \mathcal{Q}_{\text{ssi}}(f, e)$ such that $z = ww^\circ$ and $w^\circ w = f$, it is then tedious but routine to check that $x' := xw$ and $y' := yw$ are left adjoints in \mathcal{Q}_{ssi} that tabulate q . ■

The proof of the previous lemma can be tweaked to obtain the following ‘weak’ variant:

Lemma 4.6 *A modular quantaloid \mathcal{Q} is weakly semi-simple if and only if \mathcal{Q}_{ssi} is weakly tabular.*

Proof: This is a straightforward adaptation of the previous proof: instead of working with single pairs of simple morphisms we must work with families of pairs of simple morphisms.

Suppose that \mathcal{Q}_{ssi} is weakly tabular. A morphism $q: X \rightarrow Y$ in \mathcal{Q} can be viewed as a morphism $q: 1_X \rightarrow 1_Y$ in \mathcal{Q}_{ssi} , so consider its weak tabulation in \mathcal{Q}_{ssi} :

$$q = \bigvee \{fg^* \mid f \text{ and } g \text{ are left adjoints in } \mathcal{Q}_{\text{ssi}} \text{ such that } fg^* \leq q\}.$$

If \mathcal{Q} is modular then it follows (as in the previous proof) that all the f ’s and g ’s in the above expression are simple in \mathcal{Q} and exhibit q ’s weak semi-simplicity.

Conversely, suppose first that \mathcal{Q} is weakly semi-simple. If $q: r \rightarrow p$ is a morphism in \mathcal{Q}_{ssi} (between symmetric idempotents $r: X \rightarrow X$ and $p: Y \rightarrow Y$, say) then at least we know that $q: X \rightarrow Y$ is weakly semi-simple in \mathcal{Q} :

$$q = \{ab^\circ \mid a \text{ and } b \text{ are simple morphisms in } \mathcal{Q} \text{ such that } ab^\circ \leq q\}.$$

As in the previous proof, each such pair (a, b) of simple morphisms in \mathcal{Q} determines a pair (pa, rb) of simple morphisms in \mathcal{Q}_{ssi} , and the lot of them exhibit q 's weak semi-simplicity in \mathcal{Q}_{ssi} . Thus \mathcal{Q}_{ssi} is weakly semi-simple whenever \mathcal{Q} is. Adding the hypothesis that \mathcal{Q} is modular, we must prove that \mathcal{Q}_{ssi} is in fact weakly tabular. So again, let $q: r \rightarrow p$ be a morphism in \mathcal{Q}_{ssi} , and suppose now that

$$q = \bigvee \{xy^\circ \mid x \text{ and } y \text{ are simple morphisms in } \mathcal{Q}_{\text{ssi}} \text{ such that } xy^\circ \leq q\}.$$

Each of the pairs (x, y) of simple morphisms in \mathcal{Q}_{ssi} can be transformed, as in the previous proof, into a pair (x', y') of left adjoint morphisms in \mathcal{Q}_{ssi} , and the lot of them provide for a weak tabulation of q . ■

Much like Theorem 3.3 contains an axiomatic description of small quantaloids of closed cibles, we can now give an axiomatisation of Grothendieck quantaloids. In a sense, this is a refined analysis of the notion of 'weak semi-simplicity'.

Theorem 4.7 *For a small involutive quantaloid \mathcal{Q} , the following are equivalent:*

1. \mathcal{Q} is weakly semi-simple,
2. $\text{Matr}(\mathcal{Q})$ is semi-simple,

If \mathcal{Q} is modular then this is also equivalent to:

3. \mathcal{Q}_{ssi} is weakly tabular.

If \mathcal{Q} is modular and locally localic then this is also equivalent to:

4. \mathcal{Q}_{ssi} is a small quantaloid of closed cibles,
5. $\text{ProjMatr}(\mathcal{Q})$ is tabular,
6. there exists a small site $(\mathcal{C}, \mathcal{J})$ such that $\text{Rel}(\mathcal{Q}) \simeq \text{Rel}(\text{Sh}(\mathcal{C}, \mathcal{J}))$,
7. \mathcal{Q} is a Grothendieck quantaloid.

In fact, the small site $(\mathcal{C}, \mathcal{J})$ of which statement (6) speaks, is the site associated (as in Theorem 3.3) with the small quantaloid of closed cibles \mathcal{Q}_{ssi} of which statement (4) speaks.

Proof: (1 \Rightarrow 2) Let $M: A \rightarrow B$ be a morphism in $\text{Matr}(\mathcal{Q})$: we must find semi-simple matrices $F: C \rightarrow B$ and $G: C \rightarrow A$ such that $M = FG^\circ$. Each element of M , that is, each \mathcal{Q} -arrow $M(b, a): ta \rightarrow tb$, is weakly semi-simple by assumption; thus

$$M(b, a) = \bigvee \{fg^\circ \mid fg^\circ \leq M(b, a) \text{ with } f \text{ and } g \text{ simple}\}. \quad (9)$$

For each $(a, b) \in A \times B$ we define the set

$$C_{(a,b)} = \{(f, g) \mid fg^\circ \leq M(b, a) \text{ with } f \text{ and } g \text{ simple morphisms in } \mathcal{Q}\}$$

and furthermore we define C to be the coproduct of the $C_{(a,b)}$'s. The constant functions $C_{(a,b)} \rightarrow A: (f, g) \mapsto a$ and $C_{(a,b)} \rightarrow B: (f, g) \mapsto b$ therefore uniquely

define functions $\alpha: C \rightarrow A$ and $\beta: C \rightarrow A$; and putting the type of $(f, g) \in C$ to be the domain of f (= the domain of g) makes C an object of $\text{Matr}(\mathcal{Q})$.

With the aid of the identity matrices $\Delta_A: A \rightarrow A$ and $\Delta_B: B \rightarrow B$ we now define two \mathcal{Q} -matrices, $F: C \rightarrow B$ and $G: C \rightarrow A$, to have as elements

$$F(b, (f, g)) = \Delta_B(b, \beta(f, g)) \circ f \quad \text{and} \quad G(a, (f, g)) = \Delta_A(a, \alpha(f, g)) \circ g.$$

Simplicity of all f 's and g 's makes sure that F and G are simple matrices, and the formula $M = FG^\circ$ precisely coincides with (9).

(2 \Rightarrow 1) Any $q: X \rightarrow Y$ in \mathcal{Q} may be viewed as a one-element matrix $(q): \{X\} \rightarrow \{Y\}$ between singletons (with obvious types). By hypothesis there are simple matrices $F: C \rightarrow \{Y\}$ and $G: C \rightarrow \{X\}$ such that $(q) = FG^\circ$. The simplicity of F and G implies that, for each $c \in C$, the morphisms $f_c := F(Y, c): tc \rightarrow Y$ and $g_c := G(c, X): X \rightarrow tc$ are simple morphisms in \mathcal{Q} ; and $(q) = FG^\circ$ expresses precisely that $q = \bigvee_{c \in C} f_c g_c^\circ$, showing q to be weakly semi-simple in \mathcal{Q} .

(1 \Leftrightarrow 3) This is the contents of Lemma 4.6

(3 \Leftrightarrow 4) If \mathcal{Q} is a locally localic and modular quantaloid, then so is \mathcal{Q}_{ssi} (by Lemmas 3.6 and 3.7); in particular, \mathcal{Q}_{ssi} is map-discrete and weakly modular too (see Lemma 3.2). Thus \mathcal{Q}_{ssi} is weakly tabular if and only if it is a small quantaloid of closed cribles (cf. Theorem 3.3).

(2 \Leftrightarrow 5) From Lemmas 3.2 and 3.7 we know that $\text{Matr}(\mathcal{Q})$ is modular. Lemma 4.5 does the rest, since $\text{ProjMatr}(\mathcal{Q}) = (\text{Matr}(\mathcal{Q}))_{\text{ssi}}$.

(4 \Rightarrow 6) If $\mathcal{Q}_{\text{ssi}} \simeq \mathcal{R}(\mathcal{C}, J)$ for some small site (\mathcal{C}, J) , then $\text{Rel}(\mathcal{Q}) \simeq \text{Rel}(\mathcal{Q}_{\text{ssi}})$ is equivalent to $\text{Rel}(\text{Sh}(\mathcal{C}, J))$ by Theorem 3.14.

(6 \Rightarrow 7) Is evident.

(7 \Rightarrow 5) If \mathcal{Q} is a Grothendieck quantaloid, then $\text{ProjMatr}(\mathcal{Q})$ – which by Lemma 4.3 is equivalent to $\text{Rel}(\mathcal{Q})$ – is equivalent to the allegory of internal relations in a topos, so it is most certainly tabular (see e.g. [Freyd and Scedrov, 1990, 2.142]). ■

Quantaloids vs. quantales

The theorem above thus says that a Grothendieck quantaloid is precisely a modular, locally localic and weakly semi-simple quantaloid. There is an easier criterion than weak semi-simplicity when dealing with Grothendieck quantales rather than quantaloids. Using the term *quantal frame* to mean a quantale whose underlying sup-lattice is a locale [Resende, 2007], we can state it as:

Theorem 4.8 *A Grothendieck quantale is a modular quantal frame with a weakly semi-simple top (i.e. $\top = \bigvee \{fg^\circ \mid f, g \text{ simple}\}$).*

Proof: One implication is trivial. For the other, let $q \in \mathcal{Q}$; we must show that it is weakly semi-simple. For any two simple elements of \mathcal{Q} , f and g say, the modular law and the simplicity of f and g allow us to compute that

$$q \wedge fg^\circ \leq f(f^\circ qg \wedge 1)g^\circ \leq (ff^\circ qgg^\circ) \wedge f1g^\circ \leq q \wedge fg^\circ.$$

The element $h := f(f^\circ qg \wedge 1)$ is simple, because it is smaller than the simple element f . In other words, this shows that, for any pair (f, g) of simple elements, there exists a simple element h such that $q \wedge fg^\circ = hg^\circ$. Using the remaining hypotheses, we can thus compute that

$$\begin{aligned} q &= q \wedge \top \\ &= q \wedge \bigvee \{fg^\circ \mid f, g \text{ simple}\} \\ &= \bigvee \{q \wedge fg^\circ \mid f, g \text{ simple}\} \\ &= \bigvee \{hg^\circ \mid hg^\circ \leq q \text{ with } h \text{ and } g \text{ simple}\}, \end{aligned}$$

so Q is indeed weakly semi-simple. ■

As an application of the “change of base” principles that we developed in Section 3, we shall now show how every Grothendieck topos is equivalent to a category of Q -sheaves, with Q a Grothendieck quantale.

First recall that two small quantaloids \mathcal{Q} and \mathcal{R} are said to be *Morita-equivalent* when the (large) quantaloids of modules $[\mathcal{Q}^{\text{op}}, \text{Sup}]$ and $[\mathcal{R}^{\text{op}}, \text{Sup}]$ are equivalent. B. Mesablishvili [2004] proved that for any small quantaloid \mathcal{Q} there is a Morita-equivalent quantale \mathcal{Q}^m ; he uses abstract \mathcal{V} -category theoretic arguments to prove his claim. Unraveling his arguments, we can give an explicit construction of \mathcal{Q}^m : it is $\text{Matr}(\mathcal{Q})(\mathcal{Q}_0, \mathcal{Q}_0)$, the quantale of endo-matrices with elements in \mathcal{Q} on the \mathcal{Q} -typed set of objects of \mathcal{Q} (where, of course, the type of an object $X \in \mathcal{Q}$ is X).

Given a morphism $f: A \rightarrow B$ in a small quantaloid \mathcal{Q} , we shall write $M_f \in \mathcal{Q}^m$ for the matrix all of whose elements are zero, except for the element indexed by $(A, B) \in \mathcal{Q}_0 \times \mathcal{Q}_0$, which is equal to f . The function $f \mapsto M_f$ is easily seen to preserve composition and suprema (but evidently not the identities, so it is not a quantaloid homomorphism). However, if \mathcal{E} is a class of idempotents in \mathcal{Q}^m containing all of $\{M_{1_A} \mid A \in \mathcal{Q}_0\}$, and we split these idempotents in \mathcal{Q}^m , then we obtain a homomorphism

$$M: \mathcal{Q} \rightarrow (\mathcal{Q}^m)_{\mathcal{E}}: (f: A \rightarrow B) \mapsto (M_f: M_{1_A} \rightarrow M_{1_B})$$

which is easily seen to be fully faithful and injective on objects. If \mathcal{Q} is a small involutive quantaloid, then it is straightforward to define an involution on the quantale \mathcal{Q}^m as well, which the function $f \mapsto M_f$ preserves. If all elements of \mathcal{E} are symmetric (which is automatic for the M_{1_A}), then the above homomorphism is not only fully faithful and injective on objects, but also preserves the involution.

Furthermore, \mathcal{Q}^m is, by definition, a full subquantaloid of $\text{Matr}(\mathcal{Q})$, which in turn is a full subquantaloid of $\text{SymDist}(\mathcal{Q})$; let us write the full inclusion as $J: \mathcal{Q}^m \rightarrow \text{SymDist}(\mathcal{Q})$. In case symmetric idempotents split symmetrically in $\text{SymDist}(\mathcal{Q})$, there is a fully faithful homomorphism $J': (\mathcal{Q}^m)_{\mathcal{E}} \rightarrow \text{SymDist}(\mathcal{Q})$ of involutive quantaloids. This is in particular the case when \mathcal{Q} is a small quantaloid of closed cibles, which leads us to:

Proposition 4.9 *If \mathcal{Q} is a small quantaloid of closed cribles, \mathcal{Q}^m is its Morita-equivalent quantale and \mathcal{E} is a class of symmetric idempotents in \mathcal{Q}^m containing all of $\{M_{1_A} \mid A \in \mathcal{Q}_0\}$, then also $(\mathcal{Q}^m)_{\mathcal{E}}$ is a small quantaloid of closed cribles and the inclusion $\mathcal{Q} \hookrightarrow (\mathcal{Q}^m)_{\mathcal{E}}$ induces an equivalence $\text{SymDist}(\mathcal{Q}) \xrightarrow{\sim} \text{SymDist}((\mathcal{Q}^m)_{\mathcal{E}})$ of involutive quantaloids.*

Proof: If \mathcal{Q} is a small quantaloid of closed cribles, then it is in particular locally localic and modular. Hence $\text{Matr}(\mathcal{Q})$ is locally localic, implying that \mathcal{Q}^m is locally localic (as a one-object quantaloid), and therefore also $(\mathcal{Q}^m)_{\mathcal{E}}$ is locally localic. Moreover, it is straightforward to compute that the diagram

$$\begin{array}{ccc}
 \mathcal{Q} & \xrightarrow{M} & (\mathcal{Q}^m)_{\mathcal{E}} \\
 \downarrow I & & \swarrow J' \\
 \text{SymDist}(\mathcal{Q}) & &
 \end{array}$$

of involutive quantaloids and homomorphisms that preserve the involution commutes up to natural isomorphism. Because J' is fully faithful, the results in Lemmas 3.11 and 3.12 apply, and prove the proposition. ■

If \mathcal{Q} is a small quantaloid of closed cribles, then $\text{SymDist}(\mathcal{Q}_{\text{ssi}}) \simeq \text{SymDist}(\mathcal{Q})$ by Proposition 3.13, which is further equivalent to $\text{SymDist}((\mathcal{Q}^m)_{\text{ssi}})$ by Proposition 4.9. This produces the following:

Corollary 4.10 *If \mathcal{Q} is a small quantaloid of closed cribles, then*

1. $\text{Sh}(\mathcal{Q}) \simeq \text{Sh}(\mathcal{Q}^m)$,
2. $\text{Rel}(\mathcal{Q}) \simeq \text{Rel}(\mathcal{Q}^m)$.

This implies that \mathcal{Q}^m is a Grothendieck quantale.

This result says in particular that *any Grothendieck topos can equivalently be described as a category of sets with an equality relation taking truth-values in a Grothendieck quantale.*

Examples

We end this paper with some examples, the first two of which clearly illustrate the difference between ‘small quantaloids of closed cribles’ and ‘Grothendieck quantaloids’.

Example 4.11 (Closed cribles) As remarked before, each small quantaloid \mathcal{Q} of closed cribles is a Grothendieck quantaloid, and $\text{Sh}(\mathcal{Q})$ is equivalent to the topos of sheaves on the site canonically associated with \mathcal{Q} .

Example 4.12 (Locales) A locale (L, \vee, \wedge, \top) with its trivial involution is a Grothendieck quantale, but it is not a small quantaloid of closed cibles (because it is not weakly tabular). Upon splitting the (symmetric) idempotents in L one obtains a small quantaloid of closed cibles; the site associated with the latter (as in Theorem 3.3) is exactly the *canonical site* (L, J) (for which $(x_i)_i \in J(x)$ if and only if $\bigvee_i x_i = x$). Thus $\text{Sh}(L)$ - in the sense of Definition 2.2 - is equivalent to the “usual” topos of sheaves on L .

Our next example is somewhat more involved. First we must recall from [Stubbe, 2005b] that the 2-category $\text{Ord}(\mathcal{Q})$ of Definition 2.1 is equivalent to the 2-category $\text{TRSCat}_{\text{cc}}(\mathcal{Q})$ of “Cauchy complete totally regular \mathcal{Q} -semicategories and totally regular semifunctors”; and in [Heymans and Stubbe, 2009a] it is shown to be further equivalent to the 2-category $\text{Map}(\text{Mod}_{\text{lp}}(\mathcal{Q}))$ of “locally principally generated \mathcal{Q} -modules” and left adjoint module morphisms. It is not difficult to deduce, from the symmetrisation of \mathcal{Q} -orders *qua* \mathcal{Q}_{ssi} -enriched categories that we proposed in Definition 2.2, the appropriate symmetrisations of \mathcal{Q} -semicategories and of \mathcal{Q} -modules, thus producing as many different but equivalent descriptions of \mathcal{Q} -sheaves. In fact, in [Heymans and Stubbe, 2009b] we already studied the symmetric variant of locally principally generated \mathcal{Q} -modules, albeit only for involutive quantales (and not quantaloids): the so-called “locally principally symmetric” objects in $\text{Mod}_{\text{lp}}(\mathcal{Q})$ form the subcategory $\text{Mod}_{\text{lp}, \text{lps}}(\mathcal{Q})$. In [Heymans and Stubbe, 2009b, Example 3.7(4)] we showed that, for *any* involutive quantale \mathcal{Q} , the involutive quantaloid $\text{ProjMatr}(\mathcal{Q})$ is equivalent to the involutive quantaloid $\text{Hilb}(\mathcal{Q})$ of so-called “ \mathcal{Q} -modules with Hilbert structure” (and module morphisms between them). (The proof also appears in [Resende, 2012, Lemma 4.26, Theorem 4.29].) And we furthermore proved that, when \mathcal{Q} is a *modular quantal frame*, then $\text{Hilb}(\mathcal{Q})$ is further equivalent to the involutive quantaloid $\text{Mod}_{\text{lp}, \text{lps}}(\mathcal{Q})$ [Heymans and Stubbe, 2009b, Theorems 3.6 and 4.1]. Theorem 4.7 says in particular that a Grothendieck quantale is necessarily a modular quantal frame, so together with Lemma 4.3 this shows that in this case all of the involutive quantaloids $\text{Rel}(\mathcal{Q})$, $\text{ProjMatr}(\mathcal{Q})$, $\text{Hilb}(\mathcal{Q})$ and $\text{Mod}_{\text{lp}, \text{lps}}(\mathcal{Q})$ are equivalent. Taking left adjoints in either of these therefore produces equivalent Grothendieck toposes

$$\text{Sh}(\mathcal{Q}) \simeq \text{Map}(\text{Rel}(\mathcal{Q})) \simeq \text{Map}(\text{ProjMatr}(\mathcal{Q})) \simeq \text{Map}(\text{Hilb}(\mathcal{Q})) \simeq \text{Map}(\text{Mod}_{\text{lp}, \text{lps}}(\mathcal{Q})).$$

Example 4.13 (Inverse quantal frames) An *inverse quantal frame* \mathcal{Q} is a modular quantal frame such that $\top = \bigvee \{p \in \mathcal{Q} \mid p^\circ p \vee pp^\circ \leq 1\}$. It follows trivially from Theorem 4.8 that inverse quantal frames are Grothendieck quantales. There is a correspondence up to isomorphism between inverse quantal frames and étale groupoids [Resende, 2007]: for every étale groupoid G there is an inverse quantal frame $\mathcal{Q} = \mathcal{O}(G)$ (as locale it is the object of morphisms of G , and its quantale multiplication stems from the composition law in G); and for every inverse quantal frame \mathcal{Q} there is an étale groupoid G such that $\mathcal{Q} \cong \mathcal{O}(G)$. Moreover, [Resende, 2012, p. 62–65] proves that $\text{Map}(\text{Hilb}(\mathcal{O}(G)))$ is equivalent to the classifying topos BG of the étale groupoid G . Consequently,

$$\begin{aligned} \text{Sh}(\mathcal{O}(G)) \simeq \text{Map}(\text{Rel}(\mathcal{O}(G))) \simeq \text{Map}(\text{ProjMatr}(\mathcal{O}(G))) \simeq \\ \text{Map}(\text{Hilb}(\mathcal{O}(G))) \simeq \text{Map}(\text{Mod}_{\text{lp}, \text{lps}}(\mathcal{O}(G))) \end{aligned}$$

are all equivalent descriptions of the topos BG in terms of “sheaves on an involutive quantale”.

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