One-step smoothing inexact Newton method for nonlinear complementarity problem with a P_0 function*

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Abstract

Based on Fischer-Burmeister function, we propose a new smoothing function. Using this function,the existence and continuity of the smooth path for solving the nonlinear complementarity problem with a P_0 function are discussed. Then we present a one-step smoothing inexact Newton method for nonlinear complementarity problem with a P_0 function. The proposed method solves the corresponding linear system approximately in each iteration. Furthermore, we investigate the boundedness of the sequence generated by our algorithm and prove the global convergence and local superlinear convergence under mild conditions.

1 Introduction

Consider the nonlinear complementarity problem (denoted the NCP(F)): Find a vector $x \in \mathbb{R}^n$, such that

$$x \geq 0, \quad F(x) \geq 0, \quad x^T F(x) = 0, \tag{1}$$

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where $F : \mathbb{R}^n \to \mathbb{R}^n$ denotes a continuously differentiable function. In this paper, we suppose that F is a P_0 function, i.e. for all $x, y \in \mathbb{R}^n, x \neq y$, there is an index i such that $x_i \neq y_i$ and $(x_i - y_i)\big[F_i(x) - F_i(y)\big] \geq 0$. Duo to the applications in many fields including operations research and engineering design, a number of well known direct and iterative methods for the solution of NCP(F) exist, for example [1]-[8]. Among them the one-step smoothing Newton method were proposed by [1] for NCP(F) and box constrained variational inequalities. Because of its simplicity and stronger numerical results, this method has recently drawn much interest [1]-[4]. At each iteration, this method solves only one linear system of equations and performs only one line search. It is proved that one-step smoothing Newton method has superlinear (quadratic) convergence under weaker conditions. In these methods, the smoothing functions play an important role. Up to now, many smoothing functions have been proposed [9]-[13].

In this paper, on the one hand, based on the Fischer-Burmeister function, we present a new smoothing function that possesses a property not satisfied by many other function. Using the new smoothing function, the nonlinear complementarity problem can be reformulated as a smooth system of equations. On the other hand, we propose a one-step smoothing inexact Newton method for nonlinear complementarity problem with a P_0 function. Compared to the previous literatures (for example [1]-[4]), in each iteration, our algorithm solves the corresponding linear system approximately. Our method has bounded level set. The existence and continuity of a smooth path for solving NCP(F) with P_0 function are discussed. Furthermore, the global convergence and superlinear convergence are established under mild assumptions.

The following notions will be used throughout this paper. $\mathbb{R}^n$ (respectively, $\mathbb{R}$) denotes the space of n-dimensional real column vectors (respectively, real numbers), $\mathbb{R}^n_+$ denotes the nonnegative orthant of $\mathbb{R}^n$, $\mathbb{R}_+$ (respectively, $\mathbb{R}_{++}$) denotes the the nonnegative (respectively, positive) orthant in $\mathbb{R}$. We define $N = \{1, 2, \cdots, n\}$. For any vector $u \in \mathbb{R}^n$, we denote by $\mathrm{diag}\{u_i : i \in N\}$ the diagonal matrix whose ith diagonal elements is u_i and by $\mathrm{vec}\{u_i : i \in N\}$ the vector u.

2 The smoothing function and its properties

For any $(\mu, a, b) \in \mathbb{R}^3$, we give the following smoothing function

$$\phi(\mu, a, b) = a + b - \sqrt{[a - \mu^2(a-b)]^2 + [b + \mu^2(a-b)]^2 + \mu^2}. \tag{2}$$

Obviously, $\phi(0, a, b)$ is just the Fischer-Burmeister function [13]. The following lemma give two simple properties of the function $\phi(\cdot, \cdot, \cdot)$ defined by (2).

Lemma 1. *Let $(\mu, a, b) \in \mathbb{R}^3$ and $\phi(\mu, a, b)$ defined by (2). Then*

(i) $\phi(0, a, b) = 0 \Leftrightarrow a \geq 0, b \geq 0, ab = 0$.

(ii) *If $\mu > 0$, $\phi(\mu, a, b)$ is continuously differentiable on the whole space $\mathbb{R}^2$.*

For any $(\mu, x) \in \mathbb{R}_{++} \times \mathbb{R}^n$, Let $z = (\mu, x)$ and

$$H(z) = \begin{pmatrix} \mu \\ \Phi(z) \end{pmatrix} \text{ and } \Phi(z) = \begin{pmatrix} \phi(\mu, x_1, F_1(x)) \\ \vdots \\ \phi(\mu, x_n, F_n(x)) \end{pmatrix}. \tag{3}$$

Definition 1. *The mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ is said to be a P-function if there is an index i such that*

$$x_i \neq y_i \text{ and } (x_i - y_i)\big[F_i(x) - F_i(y)\big] > 0, \ \ for\ all\ x, y \in \mathbb{R}^n, x \neq y.$$

Lemma 2. *Suppose that F is a continuous P_0 function. Then $\Phi(0, x)$ is a continuous P_0 function, and $\Phi(\mu, x)$ is a continuous P function in x for each $0 < \mu < 1$.*

Proof. It is obvious that $\Phi(0, x)$ is a continuous P_0 function. Next, we show that $\Phi(\mu, x)$ is a continuous P function. Since F is a P_0 function, for any $x \neq y \in \mathbb{R}^n$, there exists an index $i_0 \in \{i : x_i \neq y_i\}$ such that

$$(x_{i_0} - y_{i_0})(F_{i_0}(x) - F_{i_0}(y)) \geq 0.$$

For simplicity, we assume $x_{i_0} > y_{i_0}$. Then, $F_{i_0}(x) \geq F_{i_0}(y)$. Let

$$A = y_{i_0} - \mu^2(y_{i_0} - F_{i_0}(y)), \quad C = F_{i_0}(y) + \mu^2(y_{i_0} - F_{i_0}(y)),$$
$$B = x_{i_0} - \mu^2(x_{i_0} - F_{i_0}(x)), \quad D = F_{i_0}(x) + \mu^2(x_{i_0} - F_{i_0}(x)).$$

Then,

$$A^2 - B^2 = (A + B)\Big((\mu^2 - 1)(x_{i_0} - y_{i_0}) - \mu^2(F_{i_0}(x) - F_{i_0}(y))\Big)$$

and

$$C^2 - D^2 = (C + D)\Big(- \mu^2(x_{i_0} - y_{i_0}) + (\mu^2 - 1)(F_{i_0}(x) - F_{i_0}(y))\Big).$$

Since $0 < \mu < 1$, we have

$$\Big|\frac{(\mu^2 - 1)(A + B) - \mu^2(C + D)}{\sqrt{A^2 + C^2 + \mu^2} + \sqrt{B^2 + D^2 + \mu^2}}\Big| < 1 \text{ and}$$
$$\Big|\frac{(\mu^2 - 1)(C + D) - \mu^2(A + B)}{\sqrt{A^2 + C^2 + \mu^2} + \sqrt{B^2 + D^2 + \mu^2}}\Big| < 1.$$

Therefore,

$$\begin{aligned}
&\phi(\mu, x_{i_0}, F_{i_0}(x)) - \phi(\mu, y_{i_0}, F_{i_0}(y)) \\
&\quad = x_{i_0} - y_{i_0} + F_{i_0}(x) - F_{i_0}(y) + \frac{A^2 - B^2 + C^2 - D^2}{\sqrt{A^2 + C^2 + \mu^2} + \sqrt{B^2 + D^2 + \mu^2}} \\
&\quad = \Big(1 + \frac{(\mu^2 - 1)(A + B) - \mu^2(C + D)}{\sqrt{A^2 + C^2 + \mu^2} + \sqrt{B^2 + D^2 + \mu^2}}\Big)(x_{i_0} - y_{i_0}) + \\
&\qquad \Big(1 + \frac{(\mu^2 - 1)(C + D) - \mu^2(A + B)}{\sqrt{A^2 + C^2 + \mu^2} + \sqrt{B^2 + D^2 + \mu^2}}\Big)(F_{i_0}(x) - F_{i_0}(y)) \\
&\quad > 0.
\end{aligned}$$

That is, $\Phi(\mu, x)$ is a continuous P function in x for each $0 < \mu < 1$. ∎

Lemma 3. [14] *Let $\varphi(a,b) = a + b - \sqrt{a^2 + b^2 + \varepsilon}$, where $(a,b)^T \in \mathbb{R}^2$ and $\varepsilon > 0$ is a constant. Assume that $\{a_k\}$ and $\{b_k\}$ are two sequences in $\mathbb{R}$ such that either $a_k, b_k \to \infty$, or $a_k, b_k \to -\infty$ $(k \to \infty)$. Then $|\varphi(a_k, b_k)| \to \infty$ $(k \to \infty)$.*

Lemma 4. *Suppose that F is a continuous P_0 function and that H is defined by (3). Then $H(z)$ is coercive in x for each $0 < \mu < 1$, i.e.,*

$$\lim_{\|x\| \to \infty} \|H(z)\| = \infty.$$

Proof. Suppose, to the contrary, that the lemma is not true. Then for some fixed $c > 0$, there exists a sequence $\{x^k\}$ such that

$$\|H(\mu, x^k)\| \leq c, \quad \|x^k\| \to \infty. \tag{4}$$

Since F is a P_0 function, by using lemma 1 in [15] there exists a subsequence, which we write without loss of generality as $\{x^k\}$, and an index i_0 such that either $x^k_{i_0} \to \infty$ and $\{F_{i_0}(x^k)\}$ is bounded from below or $x^k_{i_0} \to -\infty$ and $\{F_{i_0}(x^k)\}$ is bounded from above. Thus, from $x^k_{i_0} \to \infty$ we have

$$x^k_{i_0} - \mu^2(x^k_{i_0} - F_{i_0}(x^k)) = (1 - \mu^2)x^k_{i_0} + \mu^2 F_{i_0}(x^k) \to \infty$$

$$F_{i_0}(x^k) + \mu^2(x^k_{i_0} - F_{i_0}(x^k)) = \mu^2 x^k_{i_0} + (1 - \mu^2)F_{i_0}(x^k) \to \infty.$$

From $x^k_{i_0} \to -\infty$ we have

$$x^k_{i_0} - \mu^2(x^k_{i_0} - F_{i_0}(x^k)) = (1 - \mu^2)x^k_{i_0} + \mu^2 F_{i_0}(x^k) \to -\infty$$

$$F_{i_0}(x^k) + \mu^2(x^k_{i_0} - F_{i_0}(x^k)) = \mu^2 x^k_{i_0} + (1 - \mu^2)F_{i_0}(x^k) \to -\infty.$$

Considering lemma 3, we obtain $|\phi(\mu, x^k_{i_0}, F_{i_0}(x^k))| \to \infty$ $(k \to \infty)$. Since $0 < \mu < 1$, $\|H(\mu, x^k)\| \to \infty$ $(k \to \infty)$, which contradicts (4). ■

Corollary 1. *Suppose that F is a continuous P_0 function. Then $\Phi(\mu, x)$ is coercive in x for $0 < \mu < 1$, i.e. $\lim_{\|x\| \to \infty} \|\Phi(\mu, x)\| = \infty$.*

Lemma 5. [16] *Suppose that $\Psi : \mathbb{R}^l \to \mathbb{R}^m$ is a locally Lipschitz continuous function. Then*

(i) *$\Psi(\cdot)$ has a generalized Jacobian $\partial\Psi(x)$, and $\Psi'(x; h)$, the directional derivative of Ψ at x in the direction h, exists for any $h \in \mathbb{R}^l$ if Ψ is semismooth at x. Also, $\Psi : \mathbb{R}^l \to \mathbb{R}^m$ is semismooth at $x \in \mathbb{R}^l$ if and only if all its component functions are.*

(ii) *$\Psi(\cdot)$ is semismooth at x if and only if for any $V \in \partial\Psi(x + h), h \to 0$,*

$$\|\Psi(x + h) - \Psi(x) - \Psi'(x; h)\| = o(\|h\|).$$

(iii) *$\Psi(\cdot)$ is strongly semismooth at x if and only if for any $V \in \partial\Psi(x + h), h \to 0$,*

$$\|\Psi(x + h) - \Psi(x) - \Psi'(x; h)\| = O(\|h\|^2).$$

Lemma 6. *Let $H : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ and $\Phi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ be defined by (3). Then*
(i) *If F is a P_0 function, then its Jacobian $H'(\mu, x)$ is nonsingular on $\mathbb{R}_{++} \times \mathbb{R}^n$ with $0 < \mu < 1$.*
(ii) *H is locally Lipschitz continuous and semismooth on $\mathbb{R}^{n+1}$. Furthermore, H is strongly semismooth on $\mathbb{R}^{n+1}$ if $F'(x)$ is Lipschitz continuous on $\mathbb{R}^n$.*

Proof. By Theorem 19 in Fischer [17], it is not difficult to see that (ii) holds. Now, we prove (i). For any $0 < \mu < 1$, from the definition of H, we have

$$H'(z) = \begin{bmatrix} 1 & 0 \\ v(z) & U(z) \end{bmatrix},$$

where

$$\begin{aligned} v(z) &= \mathrm{vec}\{\phi'_\mu(\mu, x_i, F_i(x)),\ i \in N\}, \\ U(z) &= D_1(z) + D_2(z)F'(x) \\ &= \mathrm{diag}\{\frac{\partial\phi(\mu, x_i, F_i(x))}{\partial x_i},\ i \in N\} + \mathrm{diag}\{\frac{\partial\phi(\mu, x_i, F_i(x))}{\partial F_i},\ i \in N\}F'(x). \end{aligned}$$

For any $i \in N$, by straightforward calculation, we have

$$\begin{aligned} \phi'_\mu(\mu, x_i, F_i(x)) &= \frac{2\mu(1-2\mu^2)(x_i - F_i(x)) - \mu}{\sqrt{[x_i - \mu^2(x_i - F_i(x))]^2 + [F_i(x) + \mu^2(x_i - F_i(x))]^2 + \mu^2}}, \\ \frac{\partial\phi(\mu, x_i, F_i(x))}{\partial x_i} &= 1 - \frac{(1-2\mu^2+2\mu^4)x_i + 2\mu^2(1-\mu^2)F_i(x)}{\sqrt{[x_i - \mu^2(x_i - F_i(x))]^2 + [F_i(x) + \mu^2(x_i - F_i(x))]^2 + \mu^2}}, \\ \frac{\partial\phi(\mu, x_i, F_i(x))}{\partial F_i} &= 1 - \frac{(1-2\mu^2+2\mu^4)F_i(x) + 2\mu^2(1-\mu^2)x_i}{\sqrt{[x_i - \mu^2(x_i - F_i(x))]^2 + [F_i(x) + \mu^2(x_i - F_i(x))]^2 + \mu^2}}. \end{aligned}$$

Thus, $\left|\frac{\partial\phi(\mu, x_i, F_i(x))}{\partial x_i} - 1\right| < 1$ and $\left|\frac{\partial\phi(\mu, x_i, F_i(x))}{\partial F_i} - 1\right| < 1$ for all $i \in N$. Then, the matrices $D_1(z)$ and $D_2(z)$ are positive. Since F is a P_0 function, $F'(x)$ is a P_0 matrix. Hence, $D_2(z)F'(x)$ is a P_0 matrix. By Theorem 3.3 in [9], the matrix $U(z)$ is nonsingular, which implies that the matrix $H'(z)$ is. ∎

Define the smooth path associated with smoothing function (2) as $\mathscr{P} = \{x \in \mathbb{R}^n : H(\mu, x) = 0,\ 0 < \mu < 1\}$. The following lemma is due to Gowda and Tawhid [15].

Lemma 7. *Let $f(x) : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous P_0 function and $f(x, \varepsilon)$ be a continuous perturbation of f; that is, $f(x, \varepsilon) : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n$ is continuous and $f(x, 0) \equiv f(x)$. Suppose that $f(x, \varepsilon)$ is a continuous P function in x for each $\varepsilon > 0$ and $f(x, \varepsilon)$ is coercive in x for each $\varepsilon > 0$. Then $f(x, \varepsilon) = 0$ will have a unique solution $x(\varepsilon)$ for every $\varepsilon > 0$ and the mapping $\varepsilon \to x(\varepsilon)$ is continuous on $(0, \infty)$.*

In the following, we investigate the existence and continuity of smooth path $\mathscr{P}$ using lemma 2, 7 and corollary 1.

Theorem 1. *Suppose that F is a continuous P_0 function. Then the path $\mathscr{P}$ exists, and every point $x(\mu) \in \mathscr{P}$ is continuous in μ on $(0, 1)$.*

By a simple continuity discussion, it is easy to obtain the following result.

Theorem 2. *Suppose that F is a continuous P_0 function. Let $\{\mu_k\}$ be a sequence of positive values converging to 0 and for any $k \geq 0$, let $x(\mu_k)$ converge to a point x^*, then x^* solves NCP(F).*

3 Algorithm

Now, we give our algorithm formally. Define a function $\rho : \mathbb{R}^{n+1} \to \mathbb{R}_+$ and $r : \mathbb{R}^{n+1} \to \mathbb{R}^n$ by

$$\rho(z) = \frac{\gamma^2}{2}\|H(z)\|^2 \max\{\min\{1, \|H(z)\|\}, \mu_0\},\ r(z) = \frac{\gamma}{2\sqrt{n}}\mu^2 e,$$
$$\gamma \in (0,1), e = (1,1,\cdots,1)^T.$$

Algorithm 3.1

Step 0 Choose $\delta, \sigma, \mu_0 \in (0,1)$ and $\varepsilon > 0$. Let $x^0 \in \mathbb{R}^n$ be an arbitrary point and $z^0 = (\mu_0, x^0)$. Choose $\gamma \in (0,1)$ such that $\gamma\|H(z^0)\| < \mu_0$. Set $k = 0$.

Step 1 If $\|H(z^k)\| \leq \varepsilon$, stop. Otherwise, let $\rho_k = \rho(z^k)$ and $r^k = r(z^k)$.

Step 2 Compute $\triangle z^k = (\triangle \mu_k, \triangle x^k)$ by

$$H(z^k) + H'(z^k)\triangle z^k = \begin{pmatrix} \rho_k \\ r^k \end{pmatrix}. \tag{5}$$

Step 3 Find the smallest $m = 0, 1, 2, \cdots$ (denote m_k to be the smallest number) such that

$$\|H(z^k + \delta^{m_k}\triangle z^k)\| \leq [1 - \sigma(1-\gamma)\delta^{m_k}]\|H(z^k)\|. \tag{6}$$

Let $\alpha_k = \delta^{m_k}$.

Step 4 Set $z^{k+1} = z^k + \alpha_k \triangle z^k$, $k = k+1$ and return to Step 1.

Remark: The main feature of algorithm 3.1 is that we use $(\rho_k, (r^k)^T)^T$ in the perturbed Newton equation (5). Compared to the one-step smoothing Newton method [1]-[4], we introduce r^k to measure the inaccuracy with the equation $H(z^k) + H'(z^k)\triangle z^k = (\rho_k, 0)^T$. It should be noted that the equation (5) can guarantee the sequence $\{\mu_k\}$ to be positive and monotone decreasing.

Theorem 3. *Suppose that F is a continuous P_0 function. Then Algorithm 3.1 is well-defined and generates an infinite sequence $\{z^k = (\mu_k, x^k)\} \subset \mathbb{R}_{++} \times \mathbb{R}^n$ with $0 < \mu_k \leq \mu_0$ and $\mu_k > \gamma^2\|H(z^k)\|^2\mu_0/2$ for all $k \geq 0$.*

Proof. Assume that $0 < \mu_k \leq \mu_0$ and $\mu_k > \gamma^2\|H(z^k)\|^2\mu_0/2$, then by lemma 6 (i), the matrix $H'(z^k)$ is invertible. Hence, step 2 is well-defined at the kth iteration. For any $\alpha \in (0,1]$, define

$$t(\alpha) = H(z^k + \alpha\triangle z^k) - H(z^k) - \alpha H'(z^k)\triangle z^k. \tag{7}$$

From (5), we have $\triangle\mu_k = -\mu_k + \rho_k$. Then, for any $\alpha \in (0,1]$, $\mu_k + \alpha\triangle\mu_k > 0$. Lemma 1 implies that $H(\cdot)$ is continuously differentiable around z^k. Thus,

$\|t(\alpha)\| = o(\alpha)$. By the definition of $\rho(\cdot)$ and $r(\cdot)$ and $\|H(z^k)\| \le \|H(z^0)\|$, we have

$$\rho_k \le \gamma\|H(z^k)\|/2, \quad \|r^k\| \le \gamma\|H(z^k)\|/2. \tag{8}$$

It follows from (7) and (8) that, for all $\alpha \in (0,1]$,

$$\begin{aligned}
\|H(z^k+\alpha\triangle z^k)\| &= \|t(\alpha)+H(z^k)+\alpha H'(z^k)\triangle z^k\| \\
&= \|t(\alpha)+(1-\alpha)H(z^k)+\alpha(\rho_k,(r^k)^T)^T\| \\
&\le \|t(\alpha)\|+(1-\alpha)\|H(z^k)\|+\frac{\alpha\gamma}{2}\|H(z^k)\|+\frac{\alpha\gamma}{2}\|H(z^k)\| \\
&= o(\alpha)+[1-(1-\gamma)\alpha]\|H(z^k)\|,
\end{aligned}$$

which indicates that there exists a constant $\overline{\alpha} \in (0,1]$ such that

$$\|H(z^k+\alpha\triangle z^k)\| \le [1-\sigma(1-\gamma)\alpha]\|H(z^k)\|$$

holds for any $\alpha \in (0,\overline{\alpha}]$. This shows that step 3 is well-defined at the kth iteration. Therefore $\mu_{k+1} = (1-\alpha_k)\mu_k + \alpha_k\rho_k > 0$. Noting $\gamma\|H(z^0)\| < \mu_0$, we obtain $\rho_k < \mu_0$. Since $\mu_k > \gamma^2\|H(z^k)\|^2\mu_0/2$ and $\mu_k \le \mu_0$, we can write

$$\begin{aligned}
&\mu_{k+1}-\gamma^2\|H(z^{k+1})\|^2\mu_0/2 \\
&\qquad = (1-\alpha_k)\mu_k+\alpha_k\rho_k-\gamma^2\|H(z^{k+1})\|^2\mu_0/2 \\
&\qquad > (1-\alpha_k)\frac{\gamma^2}{2}\|H(z^k)\|^2\mu_0+\alpha_k\frac{\gamma^2}{2}\|H(z^k)\|^2\mu_0-\frac{\gamma^2}{2}\|H(z^{k+1})\|^2\mu_0 \\
&\qquad = \frac{\gamma^2\mu_0}{2}(\|H(z^k)\|^2-\|H(z^{k+1})\|^2) > 0,
\end{aligned}$$

and

$$\begin{aligned}
\mu_{k+1} &= (1-\alpha_k)\mu_k+\alpha_k\rho_k \\
&\le (1-\alpha_k)\mu_0+\alpha_k\mu_0 = \mu_0.
\end{aligned}$$

Considering $0 < \mu_0 < 1$, our algorithm is well-defined and generates an infinite sequence $\{z^k = (\mu_k, x^k)\} \subset \mathbb{R}_{++} \times \mathbb{R}^n$ with $0 < \mu_k \le \mu_0$ and $\mu_k > \dfrac{\gamma^2}{2}\|H(z^k)\|^2\mu_0$ for all $k \ge 0$. ∎

4 Convergence analysis

In this section, we consider the global convergence and local superlinear convergence of our algorithm. First, we give a condition to ensure $\{z^k\}$ is bounded.

Assumption 1. *The solution set $S = \{x \in \mathbb{R}^n : x, F(x) \in \mathbb{R}^n_+, x^TF(x) = 0\}$ of (1) is nonempty and bounded.*

Note: Assumption 1 is the weakest condition used in previous literatures to ensure the boundedness of iteration sequence. [18] presented a sufficient condition that assumption 1 holds.

Theorem 4. *Suppose that F is a continuous P_0 function. Let $\{z^k = (\mu_k, x^k)\}$ be the iteration sequence generated by our algorithm. Then*

(i) *The sequences $\{\|H(z^k)\|\}$ and $\{\mu_k\}$ tend to zero.*

(ii) *If assumption 1 is satisfied, $\{z^k\}$ is bounded and hence it has at least one accumulation point $z^* = (\mu_*, x^*)$ with $H(z^*) = 0$ and $x^* \in S$.*

Proof. (i) From our algorithm, we know $\{\|H(z^k)\|\}$ is monotonically decreasing and bounded, and hence has limitation, denoted by θ_*. Suppose that $\theta_* > 0$. By theorem 3, there exists a subset $K \subset N$ such that $\lim_{k\in K, k\to\infty} \mu_k = \mu_*$ and $0 < \gamma^2\theta_*^2\mu_0/2 \le \mu_* \le \mu_0$. Thus, there exists a constant $\varepsilon_0 > 0$ satisfying $\varepsilon_0 \le \mu_k \le \mu_0, \ \forall\, k \in K$. Then, $\{z^k\}_{k\in K}$ is bounded by lemma 4. Let $z^* = (\mu_*, x^*)$ be an accumulation point of $\{z^k\}_{k\in K}$. Without loss of generality, we assume that $\lim_{k\in K, k\to\infty} z^k = z^*$. So $\lim_{k\in K, k\to\infty} \alpha_k = 0$ from (6). Thus, the stepsize $\widetilde{\alpha} = \frac{\alpha_k}{\delta}$ does not satisfy the line search criterion in Step 3 for any sufficiently large k, $k \in K$, i.e., the following inequality holds:

$$\|H(z^k + \widetilde{\alpha}\triangle z^k)\| > \left[1 - \sigma(1-\gamma)\widetilde{\alpha}\right]\|H(z^k)\|$$

for any sufficiently large k, $k \in K$, which implies that

$$(\|H(z^k + \widetilde{\alpha}\triangle z^k)\| - \|H(z^k)\|)/\widetilde{\alpha} > -\sigma(1-\gamma)\|H(z^k)\|.$$

From $\mu_* \ne 0$, we know that $H(\cdot)$ is continuously differentiable at z^*. Letting $k \to \infty, k \in K$, then the above inequality gives

$$\frac{1}{\|H(z^*)\|} H(z^*)^T H'(z^*)\triangle z^* \ge -\sigma(1-\gamma)\|H(z^*)\|. \tag{9}$$

Additionally, by taking the limit on (5), we get

$$H(z^*)^T H'(z^*)\triangle z^* = -\|H(z^*)\|^2 + H(z^*)^T \begin{pmatrix} \rho_* \\ r^* \end{pmatrix}. \tag{10}$$

Combining (9) with (10) we have

$$\|H(z^*)\|(\gamma/2\|H(z^*)\| + \gamma/2\|H(z^*)\|) \ge (1 - \sigma(1-\gamma))\|H(z^*)\|^2. \tag{11}$$

This yields $(1-\sigma)(1-\gamma) \le 0$, which contradicts the fact that $\sigma, \gamma \in (0,1)$. Thus, $H(z^*) = 0$. Then, there exists a index k_0 such that $\|H(z^k)\| \le \mu_0$, for all $k > k_0$. In this case, $\rho_k = \gamma^2\|H(z^k)\|^2\mu_0/2$. Therefore, applying $\mu_k > \gamma^2\|H(z^k)\|^2\mu_0/2$, for all $k > k_0$,

$$\begin{aligned}\mu_{k+1} &= (1-\alpha_k)\mu_k + \alpha_k\rho_k \\ &< \mu_k - \alpha_k\gamma^2\|H(z^k)\|^2\mu_0/2 + \alpha_k\gamma^2\|H(z^k)\|^2\mu_0/2 = \mu_k,\end{aligned}$$

which indicates $\{\mu_k\}$ is convergent. From the above analysis, we obtain $\lim_{k\to\infty} \mu_k = \mu_* = 0$.

(ii) It follows from (i), assumption 1 and Theorem 3.1 in [18] that $\{x^k\}$ is bounded and hence $\{z^k\}$ is. Thus, $\{z^k\}$ has at least one accumulation point $z^* = (\mu_*, x^*)$. By (i), we have $H(z^*) = 0$ and $x^* \in S$. ∎

Similarly to the proofs of Theorem 8 in [1] and Theorem 3.7 in [3], we obtain the local superlinear convergence of algorithm 3.1 as follows.

Theorem 5. *Suppose that F is a continuous P_0 function, assumption 1 is satisfied and $z^* = (\mu_*, x^*)$ is an accumulation point of the sequence $\{z^k\}$ generated by algorithm 3.1. If all $V \in \partial H(z^*)$ are invertible, then*

(i) $\alpha_k = 1$, *for all z^k sufficiently close to z^*;*

(ii) *the whole sequence $\{z^k\}$ converges to z^*, i.e., $\lim\limits_{k\to\infty} z^k = z^*$;*

(iii) *$\{z^k\}$ converges to z^* superlinearly, i.e., $\|z^{k+1} - z^*\| = o(\|z^k - z^*\|)$. Moreover, $\mu_{k+1} = o(\mu_k)$.*

Proof. By Theorem 4, we have $\mu_* = 0$ and $H(z^*) = 0$. From all $V \in \partial H(z^*)$ are invertible and Proposition 3.1 in [16], for sufficiently large k, there exists some constant $\beta > 0$ such that $\|H'(z^k)^{-1}\| \leq \beta$. It follows from Lemma 6 (ii) that $H(\cdot)$ is semismooth at z^*. Hence, for all z^k sufficiently close to z^*, we have

$$\|H(z^k) - H(z^*) - H'(z^k)(z^k - z^*)\| = o(\|z^k - z^*\|). \tag{12}$$

On the other hand, Lemma 6 (ii) implies that, for all z^k sufficiently close to z^*, $\|H(z^k)\| = O(\|z^k - z^*\|)$. Then

$$\begin{aligned}\|z^k + \triangle z^k - z^*\| &= \left\|H'(z^k)^{-1}\left[H'(z^k)(z^k - z^*) - H(z^k) + H(z^*) + \begin{pmatrix} \rho_k \\ r^k \end{pmatrix}\right]\right\| \\ &\leq \beta\left[\|H(z^k) - H(z^*) - H'(z^k)(z^k - z^*)\| + \gamma\|H(z^k)\|^2\right] \\ &= o(\|z^k - z^*\|). \end{aligned} \tag{13}$$

Denote $z^{k+1} = z^k + \triangle z^k$. Then

$$\|z^{k+1} - z^k\| = \left\|H'(z^k)^{-1}\left[-H(z^k) + \begin{pmatrix} \rho_k \\ r^k \end{pmatrix}\right]\right\| \leq \beta(1+\gamma)\|H(z^k)\|.$$

We get from (13) that for any $\varepsilon_1 \in (0,1)$, $\|z^{k+1} - z^*\| \leq \varepsilon_1\|z^k - z^*\|$. Thus,

$$\begin{aligned}\|z^k - z^*\| &\leq \|z^{k+1} - z^k\| + \|z^{k+1} - z^*\| \\ &\leq \beta(1+\gamma)\|H(z^k)\| + \varepsilon_1\|z^k - z^*\|,\end{aligned}$$

Consequently, $\|z^k - z^*\| \leq [\beta(1+\gamma)/(1-\varepsilon_1)]\|H(z^k) - H(z^*)\|$, which is equivalent to $\|z^k - z^*\| = O(\|H(z^k) - H(z^*)\|)$. Then, because $H(\cdot)$ is semismooth at z^*, for all z^k sufficiently close to z^*, we have

$$\|H(z^k + \triangle z^k)\| = O(\|z^k + \triangle z^k - z^*\|) = o(\|z^k - z^*\|) = o(\|H(z^k)\|). \tag{14}$$

From theorem 4, $\lim\limits_{k\to\infty} \|H(z^k)\| = 0$. Hence, (14) implies that when z^k sufficiently close to z^*, $\alpha_k = 1$ satisfies the line search in Step 3, which proves (i). Then, (i), together with (13), proves (ii) and

$$\|z^{k+1} - z^*\| = o(\|z^k - z^*\|). \tag{15}$$

Obviously, from Step 2, (i) and (ii), for all sufficiently large k, $\mu_{k+1} = \frac{\gamma^2}{2}\|H(z^k)\|^2\mu_0$. Hence, (14) shows that

$$\frac{\mu_{k+1}}{\mu_k} = \frac{\|H(z^k)\|^2}{\|H(z^{k-1})\|^2} = \frac{o(\|H(z^{k-1})\|^2)}{\|H(z^{k-1})\|^2} = o(1).$$

This completes our proof. ■

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