One Dimensional p-Laplacian with a Concentrated Nonlinear Source

Yuanyuan Ke Jingxue Yin* Shuhe Wang

Abstract

We establish the existence of local solutions for the Cauchy problem for one-dimensional p-Laplacian equation with a concentrated nonlinear source.

1 Introduction

Consider the Cauchy problem for one-dimensional p-Laplacian with a concentrated nonlinear source

$$\frac{\partial u}{\partial t} - D(|Du|^{p-2}Du) = \delta(x)f(u), \qquad (x,t) \in Q_T, \tag{1.1}$$

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}, \tag{1.2}$$

where $\delta(x)$ is the Dirac measure, $p > 2$, $D = \dfrac{\partial}{\partial x}$, $Q_T = \mathbb{R} \times (0,T)$.

During the past years, there are rich references concerning the partial differential equations with measure data as the sources, see for example [1]–[5]. In particular, the problems with the source of the form $\delta(x)f(u)$ have been considered by some authors in recent years, see for instance [6]–[10]. If $p = 2$, Olmstead and Roberts [6] studied the initial boundary value problem, which is motivated by the model that a combustible medium be ignited by using a heated wire or a pair of small electrodes to supply a large amount of energy to a very confined area. The authors discussed a possible blow-up solution of the problem by using Green's function and analyzing its corresponding nonlinear Volterra equation of

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the second kind at the site of the concentrated source. For the porous medium equation with this kind of source, Yin et al [10] studied the existence of generalized solutions for the Cauchy problem based on some a priori estimates on solutions. When the nonlinear source term in Eq. (1.1) is replaced by Dirac measure $\delta(x)$, the existence of generalized solutions was obtained by Li et al [11] for $p > 2$; while when the nonlinear source term is replaced by a bounded Borel measure, in [12] the authors considered the existence of solutions and gave summability results for the gradients of solutions for the case $p > 1$. And also there are a large amount of papers devoted to this type of degenerate parabolic problems when the nonlinear source is of the form $f(u)$, such as [13], in which the authors studied the local existence of solutions by the regularized methods.

The purpose of this paper is to investigate the local existence of generalized solutions for the problem (1.1)–(1.2). Due to the degeneracy, we have to consider solutions in some weak sense, namely

Definition 1.1. *A nonnegative function $u : Q_T \to [0, +\infty)$ is called a generalized solution of the problem (1.1)–(1.2) in Q_T, if $u \in C\big((\mathbb{R}\backslash\{0\}) \times (0,T)\big) \cap L^\infty\big(0,T; L^\infty_{loc}(\mathbb{R})\big) \cap BV_x(Q_T)$, and u satisfies*

$$\int_{\mathbb{R}} u_0(x)\varphi(x,0)dx + \iint_{Q_T} \Big(u\varphi_t - |Du|^{p-2}DuD\varphi\Big)dxdt + \int_0^T \frac{1}{2}\Big(f(u^l(0,t)) + f(u^r(0,t))\Big)\varphi(0,t)dt = 0,$$

for any $\varphi \in C^\infty(Q_T)$, which vanishes for large $|x|$ and $t = T$, where $BV_x(Q_T)$ is a subset of $L^1_{loc}(Q_T)$, in which the derivatives in x of each function are Radon measures in Q_T, for fixed $t \in (0,T)$, $u^l(0,t)$ and $u^r(0,t)$ are the left limit and the right limit of u at $x = 0$.

The main result of the paper is the following theorem.

Theorem 1.1. *Let $f(s)$ be nonnegative, bounded and continuously differentiable, and $u_0(x)$ be nonnegative and Hölder continuous with compact support. Assume that $u_0(x) \in W^{1,p}(\mathbb{R})$. Then there exists a finite $T^* > 0$, such that the Cauchy problem (1.1)–(1.2) admits a generalized solution in Q_{T^*}.*

Since $p > 2$, the appearance of the diffusion term makes the theory of Green's function used in [6] inapplicable. In addition, by noticing that both $\delta(x)$ and $f(u)$ appear in the source term, some estimates we needed cannot be obtained by using the methods in [11]–[13]. In our paper, just as done in [10], we should first make an approximation of the Dirac measure. Owing to the degeneracy of Eq. (1.1), we should also use parabolic regularization to approach the equation. Then we establish the existence of local solutions by some suitable estimates on the approximate solutions.

2 Proof of the Main Result

In this section, we give the proof of our main result. In order to prove the existence of generalized solutions, we first consider the following approximate problem

$$\frac{\partial u}{\partial t} - D\left(\left(|Du|^2 + \frac{1}{n}\right)^{(p-2)/2} Du\right) = \delta_\varepsilon(x) f(u), \qquad (x,t) \in Q_{R,T}, \tag{2.1}$$
$$u(x,0) = u_0(x), \qquad x \in (-R,R), \tag{2.2}$$
$$u(\pm R, t) = 0, \qquad t \in (0,T), \tag{2.3}$$

where $Q_{R,T} = (-R,R) \times (0,T)$, R is a properly large positive constant, and

$$\delta_\varepsilon(x) = \frac{1}{\varepsilon} j(\frac{x}{\varepsilon}), \qquad 0 < \varepsilon < 1,$$
$$j(x) = \begin{cases} \frac{1}{A} e^{1/(|x|^2-1)}, & |x| < 1, \\ 0, & |x| \geq 1, \end{cases}$$
$$A = \int_{-1}^{1} e^{1/(|x|^2-1)} dx.$$

Obviously, we have

$$\int_{\mathbb{R}} \delta_\varepsilon(x) dx = 1, \qquad 0 \leq \delta_\varepsilon(x) \in C_0^\infty(\mathbb{R}),$$
$$\text{supp}\delta_\varepsilon(x) = \{x \in \mathbb{R}, |x| \leq \varepsilon\},$$

and

$$\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}} \delta_\varepsilon(x) \phi(x) dx = \phi(0), \qquad \forall \phi \in C(\mathbb{R}).$$

By classical theories for parabolic equations, the problem (2.1)–(2.3) admits a unique nonnegative solution $u_{\varepsilon,R,n}$.

Next we make some estimates on $u_{\varepsilon,R,n}$.

Lemma 2.1. *There exists $T' \in [0,T]$, such that*

$$\|u_{\varepsilon,R,n}\|_{L^\infty(Q_{R,T'})} \leq C_1,$$

where C_1 is a positive constant depending on ε and T'.

Proof. Let $w(t)$ be the solution of the ordinary differential equation

$$\frac{dw}{dt} = M_\varepsilon f(w), \tag{2.4}$$
$$w(0) = \|u_0(x)\|_{L^\infty(\mathbb{R})}, \tag{2.5}$$

where $M_\varepsilon = \max_{x \in \mathbb{R}} \delta_\varepsilon(x)$. Then there exists $T_0 \in (0,T)$ such that the problem (2.4)–(2.5) has a solution $w(t)$ on $[0,T_0]$ and T_0 depends only on $\|u_0(x)\|_{L^\infty(\mathbb{R})}$, see [14].

Set $\phi = u_{\varepsilon,R,n} - w$. Then ϕ satisfies

$$\begin{aligned}&\frac{\partial\phi}{\partial t} - D\left(\left(|D\phi|^2 + \frac{1}{n}\right)^{(p-2)/2} D\phi\right)\\ =&\delta_\varepsilon(x)f(u_{\varepsilon,R,n}) - M_\varepsilon f(w)\\ \le&M_\varepsilon\big(f(u_{\varepsilon,R,n}) - f(w)\big)\\ =&M_\varepsilon(u_{\varepsilon,R,n} - w)\int_0^1 f'\big(\theta u_{\varepsilon,R,n} + (1-\theta)w\big)d\theta\\ =&M_\varepsilon C_{\varepsilon,R,n}(x,t)(u_{\varepsilon,R,n} - w)\\ =&M_\varepsilon C_{\varepsilon,R,n}(x,t)\phi.\end{aligned}$$

Thus ϕ satisfies

$$\frac{\partial\phi}{\partial t} - D\left(\left(|D\phi|^2 + \frac{1}{n}\right)^{(p-2)/2} D\phi\right) - M_\varepsilon C_{\varepsilon,R,n}\phi \le 0,$$

and $\phi(\pm R, t) \le 0$ on $t \in [0, T_0]$, $\phi(x, 0) \le 0$ on $x \in (-R, R)$. By the maximum principle, $\phi \le 0$ on Q_{R,T_0}. Hence, there exists $T' \in (0, T_0)$, such that

$$\|u_{\varepsilon,R,n}\|_{L^\infty(Q_{R,T'})} \le \max_{t\in(0,T')} w(t).$$

Taking $T' = T_0/2$ and $C_1 = w(T')$, we obtain

$$\|u_{\varepsilon,R,n}\|_{L^\infty(Q_{R,T'})} \le C_1.$$

The proof is complete. ∎

Lemma 2.2. *There exists a positive constant C_2 depending on ε and T', such that*

$$\iint_{Q_{R,T'}} |Du_{\varepsilon,R,n}|^p dxdt \le C_2.$$

Proof. Multiplying (2.1) by $u_{\varepsilon,R,n}$, and integrating it over $Q_{R,T'}$, we get

$$\begin{aligned}&\frac{1}{2}\int_{-R}^{R} u^2_{\varepsilon,R,n}(x,T')dx - \frac{1}{2}\int_{-R}^{R} u_0(x)dx\\ &+\iint_{Q_{R,T'}}\left(|Du_{\varepsilon,R,n}|^2 + \frac{1}{n}\right)^{(p-2)/2}|Du_{\varepsilon,R,n}|^2 dxdt\\ =&\iint_{Q_{R,T'}}\delta_\varepsilon(x)f(u_{\varepsilon,R,n})u_{\varepsilon,R,n}dxdt.\end{aligned}$$

By virtue of Lemma 2.1, we further obtain

$$\begin{aligned}&\iint_{Q_{R,T'}}|Du_{\varepsilon,R,n}|^p dxdt\\ \le&\iint_{Q_{R,T'}}\left(|Du_{\varepsilon,R,n}|^2 + \frac{1}{n}\right)^{(p-2)/2}|Du_{\varepsilon,R,n}|^2 dxdt\\ \le&\frac{1}{2}\int_{-R}^{R} u_0(x)dx - \frac{1}{2}\int_{-R}^{R} u^2_{\varepsilon,R,n}(x,T')dx\\ &+C_1\iint_{Q_{R,T'}}\delta_\varepsilon(x)f(u_{\varepsilon,R,n})dxdt.\end{aligned}$$

According to the assumption on f and u_0, there exists a positive constant C_2, such that

$$\iint_{Q_{R,T'}} |Du_{\varepsilon,R,n}|^p dxdt \leq C_2.$$

The proof is complete. ∎

Lemma 2.3. *There exists a positive constant C_3 depending on ε and T', such that*

$$\iint_{Q_{R,T'}} \left|\frac{\partial u_{\varepsilon,R,n}}{\partial t}\right|^2 dxdt \leq C_3.$$

Proof. Multiplying (2.1) by $\frac{\partial u_{\varepsilon,R,n}}{\partial t}$ and integrating on $Q_{R,T'}$, we have

$$\begin{aligned}
&\iint_{Q_{R,T'}} \left|\frac{\partial u_{\varepsilon,R,n}}{\partial t}\right|^2 dxdt \\
&+ \iint_{Q_{R,T'}} \left(|Du_{\varepsilon,R,n}|^2 + \frac{1}{n}\right)^{(p-2)/2} Du_{\varepsilon,R,n} D\left(\frac{\partial u_{\varepsilon,R,n}}{\partial t}\right) dxdt \\
= &\iint_{Q_{R,T'}} \delta_\varepsilon(x) f(u_{\varepsilon,R,n}) \frac{\partial u_{\varepsilon,R,n}}{\partial t} dxdt.
\end{aligned}$$

Since

$$\begin{aligned}
&\iint_{Q_{R,T'}} \left(|Du_{\varepsilon,R,n}|^2 + \frac{1}{n}\right)^{(p-2)/2} Du_{\varepsilon,R,n} D\left(\frac{\partial u_{\varepsilon,R,n}}{\partial t}\right) dxdt \\
=&\frac{1}{2}\int_{-R}^{R}\int_0^{T'} \frac{\partial}{\partial t}\left(\int_0^{|Du_{\varepsilon,R,n}|^2} \left(s+\frac{1}{n}\right)^{(p-2)/2} ds\right) dtdx \\
=&\frac{1}{2}\int_{-R}^{R}\left(\int_0^{|Du_{\varepsilon,R,n}(x,T')|^2} \left(s+\frac{1}{n}\right)^{(p-2)/2} ds\right) dx \\
&-\frac{1}{2}\int_{-R}^{R}\left(\int_0^{|Du_0(x)|^2} \left(s+\frac{1}{n}\right)^{(p-2)/2} ds\right) dx \\
=&\frac{1}{p}\int_{-R}^{R}\left(|Du_{\varepsilon,R,n}(x,T')|^2 + \frac{1}{n}\right)^{p/2} dx \\
&-\frac{1}{p}\int_{-R}^{R}\left(|Du_0(x)|^2 + \frac{1}{n}\right)^{p/2} dx,
\end{aligned}$$

and

$$\begin{aligned}
&\iint_{Q_{R,T'}} \delta_\varepsilon(x) f(u_{\varepsilon,R,n}) \frac{\partial u_{\varepsilon,R,n}}{\partial t} dxdt \\
\leq &C_\varepsilon \iint_{Q_{R,T'}} \left|f(u_{\varepsilon,R,n})\right|^2 dxdt + \frac{1}{2}\iint_{Q_{R,T'}} \left|\frac{\partial u_{\varepsilon,R,n}}{\partial t}\right|^2 dxdt,
\end{aligned}$$

it follows that

$$\begin{aligned}&\iint_{Q_{R,T'}}\left|\frac{\partial u_{\varepsilon,R,n}}{\partial t}\right|^2dxdt\\ \leq&\frac{2}{p}\int_{-R}^{R}\left(|Du_0(x)|^2+\frac{1}{n}\right)^{p/2}dx+2C_\varepsilon\iint_{Q_{R,T'}}\left|f(u_{\varepsilon,R,n})\right|^2dxdt.\end{aligned}$$

According to Lemma 2.1–2.2, there exists a positive constant C_3, such that

$$\iint_{Q_{R,T'}}\left|\frac{\partial u_{\varepsilon,R,n}}{\partial t}\right|^2dxdt\leq C_3.$$

The proof is complete. ∎

Utilizing the results in [15], we can easily obtain the uniform Hölder norm estimates on $u_{\varepsilon,R,n}$. From Lemma 2.1–2.3, we conclude that there exists a subsequence of $u_{\varepsilon,R,n}$, denoted by $u_{\varepsilon,R,n}$ itself, and a function $u_{\varepsilon,R}$ in $Q_{R,T'}$, such that

$$\begin{aligned}&u_{\varepsilon,R,n}\to u_{\varepsilon,R}, &&\text{a.e. in } Q_{R,T'},\\ &\frac{\partial u_{\varepsilon,R,n}}{\partial t}\rightharpoonup\frac{\partial u_{\varepsilon,R}}{\partial t}, &&\text{in } L^2(Q_{R,T'}),\\ &\left(|Du_{\varepsilon,R,n}|^2+\frac{1}{n}\right)^{(p-2)/2}Du_{\varepsilon,R,n}\rightharpoonup\zeta, &&\text{in } L^{p/(p-1)}(Q_{R,T'}).\end{aligned}$$

Using the methods in [16], we can get

$$\iint_{Q_{R,T'}}|Du_{\varepsilon,R}|^{p-2}Du_{\varepsilon,R}D\varphi dxdt=\iint_{Q_{R,T'}}\zeta D\varphi dxdt,$$

where $\varphi\in C^\infty(\overline{Q}_{R,T'})$, which vanishes for $t=T', x=\pm R$.

Consider the following problem

$$\frac{\partial u_\varepsilon}{\partial t}-D\big(|Du_\varepsilon|^{p-2}Du_\varepsilon\big)=\delta_\varepsilon(x)f(u_\varepsilon(x,t)),\qquad (x,t)\in Q_{R,T'},\tag{2.6}$$

$$u_\varepsilon(x,0)=u_0(x),\qquad x\in(-R,R),\tag{2.7}$$

$$u_\varepsilon(\pm R,t)=0,\qquad t\in(0,T').\tag{2.8}$$

Due to the degeneracy of Eq. (2.6), we should give the definition of generalized solutions of the problem (2.6)–(2.8).

Definition 2.1. *A nonnegative function $u_\varepsilon:Q_{R,T'}\to[0,+\infty)$ is called a generalized solution of problem (2.6)–(2.8) on $Q_{R,T'}$, if $u_\varepsilon\in C(0,T';L^\infty(-R,R))\cap L^p(0,T';W_0^{1,p}(-R,R))$, $\frac{\partial u_\varepsilon}{\partial t}\in L^2(0,T';L^2(-R,R))$, and for all $0<t_1<t_2<T'$, u_ε satisfies*

$$\begin{aligned}&\int_{-R}^{R}u_\varepsilon(x,t_2)\varphi(x,t_2)dx-\int_{-R}^{R}u_\varepsilon(x,t_1)\varphi(x,t_1)dx=\int_{t_1}^{t_2}\int_{-R}^{R}u_\varepsilon\varphi_tdxdt\\ &-\int_{t_1}^{t_2}\int_{-R}^{R}|Du_\varepsilon|^{p-2}Du_\varepsilon D\varphi dxdt+\int_{t_1}^{t_2}\int_{-R}^{R}\delta_\varepsilon(x)f(u_\varepsilon(x,t))\varphi(x,t)dxdt,\end{aligned}\tag{2.9}$$

and

$$\lim_{t\to 0^+}\int_{-R}^{R} u_\varepsilon(x,t)\psi(x)dx = \int_{-R}^{R} u_0(x)\psi(x)dx, \tag{2.10}$$

where $\varphi \in C^\infty(\overline{Q}_{R,T'})$*, which vanishes for* $t = T', x = \pm R$*, and* $\psi(x) \in C_0^\infty(-R,R)$.

Similarly, to define a lower(super) solution $\underline{u}_\varepsilon(x,t)(\overline{u}_\varepsilon(x,t))$, we need only to ask $\varphi(x,t) \geq 0, \psi(x) \geq 0, \underline{u}_\varepsilon(\pm R,t) \leq 0 (\overline{u}_\varepsilon(\pm R,t) \geq 0)$ in $(0,T')$ and the equality in (2.9) and (2.10) is replaced by $\leq$ $(\geq)$.

Utilizing the methods in [15], we can derive the following result.

Lemma 2.4. *Suppose that* $\underline{u}_\varepsilon$ *and* $\overline{u}_\varepsilon$ *are lower and super solutions of (2.6)–(2.8) respectively, then* $\underline{u}_\varepsilon \leq \overline{u}_\varepsilon$ *a.e. in* $Q_{R,T'}$.

By using a standard limiting process and Lemma 2.4, it is easy to see that $u_{\varepsilon,R}$ is a unique generalized solution of the problem (2.6)–(2.8).

Lemma 2.5. *There exist positive constants* λ_1 *and* λ_2 *depending on* R*, such that*

$$u_{\varepsilon,R}(x,t) \leq \lambda_1 - \int_{-R}^{x}\left(\lambda_2\int_{\mathbb{R}} H(s-y)\delta_\varepsilon(y)dy\right)^{1/(p-1)} ds, \qquad \forall (x,t) \in Q_{R,T'},$$

where $H(x)$ *is the Heaviside function.*

Proof. Let

$$\overline{u}_{\varepsilon,R}(x,t) = \lambda_1 - \int_{-R}^{x}\left(\lambda_2\int_{\mathbb{R}} H(s-y)\delta_\varepsilon(y)dy\right)^{1/(p-1)} ds, \qquad (x,t) \in Q_{R,T'},$$

where λ_1 and λ_2 are two positive constants to be determined. Then

$$\int_{t_1}^{t_2}\int_{-R}^{R}\frac{\partial \overline{u}_{\varepsilon,R}}{\partial t}\varphi dxdt - \int_{t_1}^{t_2}\int_{-R}^{R} D\big(|D\overline{u}_{\varepsilon,R}|^{p-2}D\overline{u}_{\varepsilon,R}\big)\varphi dxdt = \int_{t_1}^{t_2}\int_{-R}^{R}\lambda_2\delta_\varepsilon(x)\varphi dxdt,$$

where $0 < t_1 < t_2 < T'$, $\varphi(x,t) \geq 0$, $\varphi \in C^\infty(\overline{Q}_{R,T'})$, which vanishes for $t = T', x = \pm R$.

It is easy to see that

$$\int_{-R}^{x}\left(\lambda_2\int_{\mathbb{R}} H(s-y)\delta_\varepsilon(y)dy\right)^{1/(p-1)} ds = 0, \qquad \text{for } x \leq -\varepsilon,$$

and for $x > -\varepsilon$, we have

$$\begin{aligned}
&\int_{-R}^{x}\left(\lambda_2\int_{\mathbb{R}} H(s-y)\delta_\varepsilon(y)dy\right)^{1/(p-1)} ds \\
&= \int_{-\varepsilon}^{x}\left(\lambda_2\int_{\mathbb{R}} H(s-y)\delta_\varepsilon(y)dy\right)^{1/(p-1)} ds \\
&\leq \int_{-\varepsilon}^{x}\left(\lambda_2\int_{\mathbb{R}} \delta_\varepsilon(y)dy\right)^{1/(p-1)} ds \\
&\leq \lambda_2^{1/(p-1)}(|x|+1).
\end{aligned}$$

According to the assumption on f, there exists a positive constant M_0, such that

$$f(s) \le M_0, \qquad \forall s \in \mathbb{R}.$$

Choosing a positive constant $\lambda_2 \ge M_0$ and $\lambda_1 - \lambda_2^{1/(p-1)}(R+1) \ge \|u_0(x)\|_{L^\infty(\mathbb{R})}$. Thus we can obtain

$$\int_{t_1}^{t_2}\int_{-R}^{R} \lambda_2 \delta_\varepsilon(x)\varphi dxdt \ge \int_{t_1}^{t_2}\int_{-R}^{R} \delta_\varepsilon(x) f(\overline{u}_{\varepsilon,R})\varphi dxdt, \qquad \forall (x,t) \in (-R,R)\times(t_1,t_2).$$

Noticing that

$$u_0(x) \le \|u_0(x)\|_{L^\infty(\mathbb{R})} \le \lambda_1 - \lambda_2^{1/(p-1)}(R+1) \le \overline{u}_{\varepsilon,R}(x,0), \qquad \forall x \in (-R,R),$$

and

$$\overline{u}_{\varepsilon,R}(\pm R, t) \ge 0, \qquad \forall t \in (0,T'),$$

thus $\overline{u}_{\varepsilon,R}$ is a super solution of the problem (2.6)–(2.8). By Lemma 2.4, we get

$$u_{\varepsilon,R}(x,t) \le \lambda_1 - \int_{-R}^{x}\left(\lambda_2 \int_{\mathbb{R}} H(s-y)\delta_\varepsilon(y)dy\right)^{1/(p-1)} ds, \qquad \forall (x,t) \in Q_{R,T'}.$$

The proof is complete. ∎

Now, we prove the property of finite propagation of disturbances of $u_{\varepsilon,R}$.

Lemma 2.6. *Let* $\operatorname{supp} u_0 \subset I$, $r_1 < 0 < r_2$, *where* $I = [r_1, r_2] \subset (-R,R)$ *is a closed set in* $\mathbb{R}$. *Then there exists* $0 < T^* < T'$, *such that*

$$\operatorname{supp} u_{\varepsilon,R}(\cdot,t) \subset [R_1,R_2], \qquad a.e.\ t \in (0,T^*),$$

where

$$R_1 = r_1 - C_4 T^{*\mu}, \qquad R_2 = r_2 + C_5 T^{*\mu},$$

with the positive constants C_4, C_5, μ *depending on* p *and* R.

Proof. Set

$$g_m(y) = \iint_{Q_{R,t}} (x-y)_+^m |Du_{\varepsilon,R}|^p dxd\tau, \qquad m = 1,2,\cdots, y \ge r_2, 0 < t \le T'.$$

Using the techniques in [11], we can obtain

$$g_1'(y) \le -Ct^{-\lambda/(\theta+1)}[g_1(y)]^{1/(\theta+1)},$$

where

$$\lambda = \frac{1-a}{\sigma(2p+1)}, \quad \theta = \frac{p\gamma}{\sigma(2p+1)^2} - \frac{1}{2p+1},$$

with the positive constants

$$a = \frac{\dfrac{1}{2} + \dfrac{1}{p+2} - \dfrac{1}{p}}{\dfrac{1}{2} + \dfrac{2}{p+2} - \dfrac{1}{p}}, \quad \sigma = 1 - \frac{p+1}{2p+1}\gamma, \quad \gamma = a + (1-a)\frac{p}{2}.$$

If $g_1(r_2) = 0$, then $Du_{\varepsilon,R}(x,t) = 0$ for $x \in [r_2, R]$, and hence from the boundary value condition, we see that $u_{\varepsilon,R}(x,t) = 0$ for $x \in [r_2, R]$, i.e., $\operatorname{supp} u_{\varepsilon,R}(\cdot,t) \subset [-R, r_2]$. If $g_1(r_2) \neq 0$, then there exists an interval (r_2, R^*), such that $g_1(y) > 0$ in (r_2, R^*), but $g_1(R^*) = 0$. Therefore, for $y \in (r_2, R^*)$, we have

$$\left(g_1(y)^{\theta/(\theta+1)}\right)' = \frac{\theta}{\theta+1}\frac{g_1'(y)}{g_1(y)^{1/(\theta+1)}} \leq -Ct^{-\lambda/(\theta+1)}.$$

Integrating the above inequality on (r_2, R^*), we obtain

$$g_1(R^*)^{\theta/(\theta+1)} - g_1(r_2)^{\theta/(\theta+1)} \leq -Ct^{-\lambda/(\theta+1)}(R^* - r_2).$$

Therefore

$$R^* \leq r_2 + Ct^{\lambda/(\theta+1)} g_1(r_2)^{\theta/(\theta+1)}.$$

According to Lemma 2.2, we have

$$\iint_{Q_{R,t}} |Du_{\varepsilon,R,n}|^p dxdt \leq \frac{1}{2}\int_{-R}^{R} u_0(x)dx + \iint_{Q_{R,t}} \delta_\varepsilon(x) f(u_{\varepsilon,R,n}) u_{\varepsilon,R,n} dxdt.$$

Letting $n \to \infty$ and using Lemma 2.5, we further obtain

$$\begin{aligned}
&\iint_{Q_{R,t}} |Du_{\varepsilon,R}|^p dxdt \\
&\leq \frac{1}{2}\int_{-R}^{R} u_0(x)dx + \iint_{Q_{R,t}} \delta_\varepsilon(x) f(u_{\varepsilon,R}) u_{\varepsilon,R} dxdt \\
&\leq \frac{1}{2}\int_{-R}^{R} u_0(x)dx + \lambda_1 M_0 \iint_{Q_{R,t}} \delta_\varepsilon(x) dxdt \\
&\leq M_1 + M_2 t,
\end{aligned}$$

which implies that

$$g_1(r_2) = \iint_{Q_{R,t}} (x - r_2)_+ |Du_{\varepsilon,R}|^p dxd\tau \leq (R - r_2)(M_1 + M_2 t),$$

and hence

$$R^* \leq r_2 + (M_3 + M_4 t) t^{\lambda/(\theta+1)}.$$

Obviously, there exists $0 < T_1^* < T'$, such that

$$r_2 + (M_3 + M_4 T_1^*) T_1^{*\lambda/(\theta+1)} = r_2 + C_5 T_1^{*\mu} < R,$$

which implies

$$\operatorname{supp} u_{\varepsilon,R}(\cdot,t) \subset [-R, r_2 + C_5 T_1^{*\mu}], \qquad \text{a.e. } t \in (0, T_1^*).$$

Similarly, there exists $0 < T_2^* < T'$, such that

$$\operatorname{supp} u_{\varepsilon,R}(\cdot,t) \subset [r_1 - C_4 T_2^{*\mu}, R], \qquad \text{a.e. } t \in (0, T_2^*).$$

Taking $T^* = \min\{T_1^*, T_2^*\}$, we obtain

$$\operatorname{supp} u_{\varepsilon,R}(\cdot,t) \subset [r_1 - C_4 T^{*\mu}, r_2 + C_5 T^{*\mu}] = [R_1, R_2], \qquad \text{a.e. } t \in (0, T^*).$$

The proof is complete. ∎

Considering the Cauchy problem

$$\frac{\partial u_\varepsilon}{\partial t} - D\big(|Du_\varepsilon|^{p-2} Du_\varepsilon\big) = \delta_\varepsilon(x) f(u_\varepsilon(x,t)), \qquad (x,t) \in Q_{T^*}, \tag{2.11}$$

$$u_\varepsilon(x,0) = u_0(x), \qquad x \in \mathbb{R}, \tag{2.12}$$

we can also give the definition of generalized solution of the problem (2.11)–(2.12).

Definition 2.2. *A nonnegative function $u_\varepsilon : Q_{T^*} \to [0,+\infty)$ is called a generalized solution of the problem (2.11)–(2.12) in Q_{T^*}, if $u_\varepsilon \in C(0,T^*; L^\infty(\mathbb{R})) \cap L^p(0,T^*; W^{1,p}(\mathbb{R}))$, $\frac{\partial u_\varepsilon}{\partial t} \in L^2(0,T^*; L^2(\mathbb{R}))$, and u_ε satisfies*

$$\int_{\mathbb{R}} u_0(x)\varphi(x,0)dx + \iint_{Q_{T^*}} \Big(u_\varepsilon \varphi_t - |Du_\varepsilon|^{p-2} Du_\varepsilon D\varphi\Big) dxdt + \iint_{Q_{T^*}} \delta_\varepsilon(x) f(u_\varepsilon(x,t))\varphi(x,t) dxdt = 0,$$

for any $\varphi \in C^\infty(Q_{T^*})$, which vanishes for large $|x|$ and $t = T^*$.

Let

$$u_\varepsilon(x,t) = \begin{cases} u_{\varepsilon,R}(x,t), & x \in [-R,R], \\ 0, & x \in \mathbb{R}\backslash[-R,R]. \end{cases}$$

Then by Lemma 2.6, we can see that $u_\varepsilon(x,t)$ is a generalized solution of problem (2.11)–(2.12) on Q_{T^*}.

We want to use Moser iteration to estimate the local boundedness on u_ε. Firstly, we need the following result.

Lemma 2.7. *Let $x_0 \neq 0$, $(x_0 - 2\rho, x_0 + 2\rho) \subset \mathbb{R}\backslash(-\varepsilon,\varepsilon)$. For any $\beta > 0$, it has*

$$\int_0^{T^*} \int_{x_0-\rho}^{x_0+\rho} u_\varepsilon^{\beta+2(p-1)} dxdt \le C_6,$$

where C_6 is a positive constant depending on β, p, ρ, T^* and $\|u_0(x)\|_{L^\infty(\mathbb{R})}$.

Proof. Let $\eta(x) \in C_0^\infty(\mathbb{R})$, and

$$0 \le \eta(x) \le 1; \qquad \eta(x) = 1, \quad \forall x \in (x_0 - \rho, x_0 + \rho);$$

$$\eta(x) = 0, \quad \forall x \in \mathbb{R}\backslash(x_0 - 2\rho, x_0 + 2\rho); \qquad |\eta'(x)| \le \frac{C}{\rho}.$$

Multiplying (2.1) by $\varphi = \eta^{2q} u_\varepsilon^\beta$ and integrating on $(x_0 - 2\rho, x_0 + 2\rho) \times (0, t)$, it is easy to obtain

$$\int_0^t \int_{x_0-2\rho}^{x_0+2\rho} \frac{\partial u_\varepsilon}{\partial \sigma} \left(\eta^{2q} u_\varepsilon^\beta\right) dx d\sigma + \int_0^t \int_{x_0-2\rho}^{x_0+2\rho} |Du_\varepsilon|^{p-2} Du_\varepsilon D\left(\eta^{2q} u_\varepsilon^\beta\right) dx d\sigma = 0,$$

where $0 < t \leq T^*$, $q > p/2$ is a constant to be determined. Then

$$\begin{aligned}
&\int_{x_0-2\rho}^{x_0+2\rho} \frac{1}{\beta+1} \eta^{2q} u_\varepsilon^{\beta+1} dx + \int_0^t \int_{x_0-2\rho}^{x_0+2\rho} \beta \eta^{2q} u_\varepsilon^{\beta-1} |Du_\varepsilon|^p dx d\sigma \\
\leq & C \int_0^{T^*} \int_{x_0-2\rho}^{x_0+2\rho} \eta^q \left|(\eta^q)'\right| \cdot u_\varepsilon^\beta \cdot |Du_\varepsilon|^{p-1} dx dt + \int_{x_0-2\rho}^{x_0+2\rho} \frac{1}{\beta+1} \eta^{2q} \left(u_0(x)\right)^{\beta+1} dx.
\end{aligned}$$

Taking the supremum with respect to t and utilizing Young's inequality, we have

$$\begin{aligned}
&\sup_{t\in(0,T^*)} \int_{x_0-2\rho}^{x_0+2\rho} \frac{1}{\beta+1} \eta^{2q} u_\varepsilon^{\beta+1} dx + \int_0^{T^*} \int_{x_0-2\rho}^{x_0+2\rho} \beta \eta^{2q} u_\varepsilon^{\beta-1} |Du_\varepsilon|^p dx dt \\
\leq & C \int_0^{T^*} \int_{x_0-2\rho}^{x_0+2\rho} \eta^q \left|(\eta^q)'\right| \cdot u_\varepsilon^\beta \cdot |Du_\varepsilon|^{p-1} dx dt + M_5 \\
\leq & C \int_0^{T^*} \int_{x_0-2\rho}^{x_0+2\rho} \eta^{2q-p} |\eta'|^p u_\varepsilon^{\beta+p-1} dx dt + \frac{\beta}{2} \int_0^{T^*} \int_{x_0-2\rho}^{x_0+2\rho} \eta^{2q} u_\varepsilon^{\beta-1} |Du_\varepsilon|^p dx dt + M_5,
\end{aligned}$$

where M_5 is a positive constant depending on β, ρ and $\|u_0(x)\|_{L^\infty(\mathbb{R})}$. Hence

$$\begin{aligned}
&\sup_{t\in(0,T^*)} \int_{x_0-2\rho}^{x_0+2\rho} \eta^{2q} u_\varepsilon^{\beta+1} dx + \int_0^{T^*} \int_{x_0-2\rho}^{x_0+2\rho} \eta^{2q} u_\varepsilon^{\beta-1} |Du_\varepsilon|^p dx dt \\
\leq & C \int_0^{T^*} \int_{x_0-2\rho}^{x_0+2\rho} \eta^{2q-p} |\eta'|^p u_\varepsilon^{\beta+p-1} dx dt + M_6.
\end{aligned} \tag{2.13}$$

For any $t \in (0, T^*)$, using Hölder's inequality and Lemma 2.5, we have

$$\begin{aligned}
&\int_{x_0-2\rho}^{x_0+2\rho} \eta^{2q} u_\varepsilon^{\beta+2(p-1)} dx \\
\leq & \left(\int_{x_0-2\rho}^{x_0+2\rho} u_\varepsilon^p dx\right)^{(p-1)/p} \left(\int_{x_0-2\rho}^{x_0+2\rho} \left(\eta^{2q} u_\varepsilon^{\beta+p-1}\right)^p dx\right)^{1/p} \\
\leq & C \left(\int_{x_0-2\rho}^{x_0+2\rho} \left(\eta^{2q/p} u_\varepsilon^{(\beta+p-1)/p}\right)^{p^2} dx\right)^{\frac{1}{p}} \\
\leq & C \Big(\sup_{x\in(x_0-2\rho,x_0+2\rho)} \left(\eta^{2q/p} u_\varepsilon^{(\beta+p-1)/p}\right)\Big)^p.
\end{aligned}$$

Utilizing the embedding theorem, it has

$$\sup_{x\in(x_0-2\rho,x_0+2\rho)} \left(\eta^{2q/p} u_\varepsilon^{(\beta+p-1)/p}\right) \leq C \left(\int_{x_0-2\rho}^{x_0+2\rho} \left|D\left(\eta^{2q/p} u_\varepsilon^{(\beta+p-1)/p}\right)\right|^p dx\right)^{1/p}.$$

Thus

$$\int_{x_0-2\rho}^{x_0+2\rho} \eta^{2q} u_\varepsilon^{\beta+2(p-1)} dx \le C \int_{x_0-2\rho}^{x_0+2\rho} \left| D\left(\eta^{2q/p} u_\varepsilon^{(\beta+p-1)/p}\right)\right|^p dx.$$

Integrating the above inequality with respect to t on $(0, T^*)$, and utilizing (2.13), we have

$$\begin{aligned}&\int_0^{T^*}\int_{x_0-2\rho}^{x_0+2\rho} \eta^{2q} u_\varepsilon^{\beta+2(p-1)} dxdt\\ \le & C\int_0^{T^*}\int_{x_0-2\rho}^{x_0+2\rho} \left(\eta^{2q} u_\varepsilon^{\beta-1}|Du_\varepsilon|^p + \eta^{2q-p}|\eta'|^p u_\varepsilon^{\beta+p-1}\right) dxdt\\ \le & C\int_0^{T^*}\int_{x_0-2\rho}^{x_0+2\rho} \eta^{2q-p}|\eta'|^p u_\varepsilon^{\beta+p-1} dxdt + M_7.\end{aligned}$$

Applying Young's inequality, it follows that

$$\begin{aligned}&C\int_0^{T^*}\int_{x_0-2\rho}^{x_0+2\rho} \eta^{2q-p}|\eta'|^p u_\varepsilon^{\beta+p-1} dxdt\\ \le & \frac{1}{2}\int_0^{T^*}\int_{x_0-2\rho}^{x_0+2\rho} \eta^{(2q-p)\frac{\beta+2(p-1)}{\beta+p-1}} u_\varepsilon^{\beta+2(p-1)} dxdt + C\int_0^{T^*}\int_{x_0-2\rho}^{x_0+2\rho} |\eta'|^{p\cdot\frac{\beta+2(p-1)}{p-1}} dxdt.\end{aligned}$$

Let $(2q-p)\dfrac{\beta+2(p-1)}{\beta+p-1} = 2q$. Then $q = \dfrac{p[\beta+2(p-1)]}{2(p-1)} > \dfrac{p}{2}$. Hence, there exists a positive constant C_6, such that

$$\int_0^{T^*}\int_{x_0-2\rho}^{x_0+2\rho} \eta^{2q} u_\varepsilon^{\beta+2(p-1)} dxdt \le C_6.$$

Therefore

$$\int_0^{T^*}\int_{x_0-\rho}^{x_0+\rho} u_\varepsilon^{\beta+2(p-1)} dxdt \le C_6.$$

The proof is complete. ■

Next, we consider the local boundedness of u_ε by using the technique of Moser's iteration.

Lemma 2.8. *For $0 < \tau < 1$, there exists a positive constant C_τ depending on τ, p, ρ, T^* and $\|u_0(x)\|_{L^\infty(\mathbb{R})}$, such that*

$$u_\varepsilon(x,t) \le C_\tau, \qquad \forall (x,t) \in Q_{\tau\rho},$$

where $Q_\rho = Q_\rho(x_0, t_0) = (x_0-\rho, x_0+\rho) \times (t_0-\rho^p, t_0+\rho^p)$, $0 < t_0 < T^*$, and $(t_0-\rho^p, t_0+\rho^p) \subset (0, T^*)$.

Proof. For any $\tau\rho \le h < h' \le \rho$, take $\eta(x) \in C_0^\infty(\mathbb{R})$, and

$$\begin{aligned}&0 \le \eta(x) \le 1; \qquad \eta(x) = 1, \quad \forall x \in (x_0-h, x_0+h);\\ &\eta(x) = 0, \quad \forall x \in \mathbb{R}\backslash(x_0-h', x_0+h'); \qquad |\eta'(x)| \le \frac{C}{h'-h}.\end{aligned}$$

And for any $s \in (t_0 - h^p, t_0 + h^p)$, we choose a function $\xi(t) \in C^\infty(-\infty, s]$, such that

$$\xi(t) = 1, \quad \forall t \in [t_0 - h^p, s]; \qquad \xi(t) = 0, \quad \forall t \in (-\infty, t_0 - h'^p];$$
$$0 \le \xi(t) \le 1, \quad 0 \le \xi'(t) \le \frac{C}{(h'-h)^p}, \quad \forall t \le s;$$

and extend it to be zero for $t > s$. Let $v_\varepsilon(x,t) = \max\{u_\varepsilon(x,t), 1\}$. Multiplying (2.11) by $\varphi = \xi^2 \eta^p v_\varepsilon^\gamma$ and integrating on Q_ρ, we can obtain

$$\iint_{Q_\rho} \frac{\partial v_\varepsilon}{\partial t}\left(\xi^2 \eta^p v_\varepsilon^\gamma\right) dxdt + \iint_{Q_\rho} |Dv_\varepsilon|^{p-2} Dv_\varepsilon D(\xi^2 \eta^p v_\varepsilon^\gamma) dxdt = 0.$$

Thus

$$\begin{aligned}
&\frac{1}{\gamma+1}\iint_{Q^s_{h'}} \frac{\partial}{\partial t}\left(\xi^2 \eta^p v_\varepsilon^{\gamma+1}\right) dxdt - \frac{2}{\gamma+1}\iint_{Q^s_{h'}} \xi\xi' \eta^p v_\varepsilon^{\gamma+1} dxdt \\
&+ \gamma \iint_{Q^s_{h'}} \xi^2 \eta^p v_\varepsilon^{\gamma-1} |Dv_\varepsilon|^p dxdt + p\iint_{Q^s_{h'}} \xi^2 \eta^{p-1} \eta' v_\varepsilon^\gamma |Dv_\varepsilon|^{p-2} Dv_\varepsilon dxdt = 0,
\end{aligned}$$

where $Q^s_{h'} = (x_0 - h', x_0 + h') \times (t_0 - h'^p, s)$.

Note that for $t = t_0 - h'^2$, it has $\xi(t) = 0$. Hence

$$\iint_{Q^s_{h'}} \frac{\partial}{\partial t}\left(\xi^2 \eta^p v_\varepsilon^{\gamma+1}\right) dxdt = \int_{x_0-h'}^{x_0+h'} \xi^2 \eta^p v_\varepsilon^{\gamma+1}\Big|_{t=s} dx.$$

Using Young's inequality, we have

$$\begin{aligned}
&p\iint_{Q^s_{h'}} \xi^2 \eta^{p-1} \eta' v_\varepsilon^\gamma |Dv_\varepsilon|^{p-2} Dv_\varepsilon dxdt \\
\le& \frac{1}{2}\iint_{Q^s_{h'}} \xi^2 \eta^p v_\varepsilon^{\gamma-1} |Dv_\varepsilon|^p dxdt + C\iint_{Q^s_{h'}} \xi^2 |\eta'|^p v_\varepsilon^{\gamma+p-1} dxdt.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\frac{1}{\gamma+1}\int_{x_0-h'}^{x_0+h'} \xi^2 \eta^p v_\varepsilon^{\gamma+1}\Big|_{t=s} dx + \gamma\iint_{Q^s_{h'}} \xi^2 \eta^p v_\varepsilon^{\gamma-1} |Dv_\varepsilon|^p dxdt \\
\le& \frac{1}{2}\iint_{Q^s_{h'}} \xi^2 \eta^p v_\varepsilon^{\gamma-1} |Dv_\varepsilon|^p dxdt + C\iint_{Q^s_{h'}} \xi^2 |\eta'|^p v_\varepsilon^{\gamma+p-1} dxdt \\
&+ \frac{2}{\gamma+1}\iint_{Q^s_{h'}} \xi\xi' \eta^p v_\varepsilon^{\gamma+1} dxdt,
\end{aligned}$$

it follows that

$$\begin{aligned}
&\sup_{t \in (t_0-h^p, t_0+h^p)} \int_{x_0-h'}^{x_0+h'} \eta^p v_\varepsilon^{\gamma+1} dx + \int_{t_0-h^p}^{t_0+h^p}\int_{x_0-h'}^{x_0+h'} \eta^p v_\varepsilon^{\gamma-1} |Dv_\varepsilon|^p dxdt \\
\le& \frac{C}{(h'-h)^p}\iint_{Q_{h'}} v_\varepsilon^{\gamma+p-1} dxdt. \qquad (2.14)
\end{aligned}$$

Let $\chi(t)$ be the characteristic function of the interval $[t_0 - h^p, t_0 + h^p]$. Utilizing the embedding theorem, we have

$$
\begin{aligned}
&\left(\frac{1}{\rho^{p+1}}\iint_{Q_\rho}\left|\chi(t)\eta^{p/2}v_\varepsilon^{(\gamma+1)/2}\right|^{2q}dxdt\right)^{1/q}\\
\le&\frac{C}{\rho}\left(\sup_{t_0-\rho^p\le t\le t_0+\rho^p}\int_{x_0-\rho}^{x_0+\rho}\left|\chi(t)\eta^{p/2}v_\varepsilon^{(\gamma+1)/2}\right|^2dx+\iint_{Q_\rho}\left|\chi(t)D\left(\eta^{p/2}v_\varepsilon^{(\gamma+1)/2}\right)\right|^2dxdt\right)\\
\le&\frac{C}{\rho}\left(\sup_{t_0-h^p\le t\le t_0+h^p}\int_{x_0-h'}^{x_0+h'}\eta^p v_\varepsilon^{\gamma+1}dx\right.\\
&\left.+\int_{t_0-h^p}^{t_0+h^p}\int_{x_0-h'}^{x_0+h'}\left|\eta^{p/2}D\left(v_\varepsilon^{(\gamma+1)/2}\right)+\frac{p}{2}\eta^{(p-2)/2}\eta' v_\varepsilon^{(\gamma+1)/2}\right|^2dxdt\right)\\
\le&\frac{C}{\rho}\left(\sup_{t_0-h^p\le t\le t_0+h^p}\int_{x_0-h'}^{x_0+h'}\eta^p v_\varepsilon^{\gamma+1}dx+\frac{(\gamma+1)^2}{2}\int_{t_0-h^p}^{t_0+h^p}\int_{x_0-h'}^{x_0+h'}\eta^p v_\varepsilon^{\gamma-1}|Dv_\varepsilon|^2dxdt\right.\\
&\left.+\frac{p^2}{2}\int_{t_0-h^p}^{t_0+h^p}\int_{x_0-h'}^{x_0+h'}\eta^{p-2}|\eta'|^2v_\varepsilon^{\gamma+1}dxdt\right)\\
\le&\frac{C}{\rho}\left(\sup_{t_0-h^p\le t\le t_0+h^p}\int_{x_0-h'}^{x_0+h'}\eta^p v_\varepsilon^{\gamma+1}dx+M_8\int_{t_0-h^p}^{t_0+h^p}\int_{x_0-h'}^{x_0+h'}\eta^p v_\varepsilon^{\gamma-1}dxdt\right.\\
&+M_9\int_{t_0-h^p}^{t_0+h^p}\int_{x_0-h'}^{x_0+h'}\eta^p v_\varepsilon^{\gamma-1}|Dv_\varepsilon|^pdxdt\\
&+M_{10}\int_{t_0-h^p}^{t_0+h^p}\int_{x_0-h'}^{x_0+h'}\eta^{p-2}v_\varepsilon^{\gamma+1}dxdt\\
&\left.+M_{11}\int_{t_0-h^p}^{t_0+h^p}\int_{x_0-h'}^{x_0+h'}\eta^{p-2}|\eta'|^pv_\varepsilon^{\gamma+1}dxdt\right).
\end{aligned}
$$

Using (2.14), it gives

$$
\begin{aligned}
&\left(\frac{1}{\rho^{p+1}}\iint_{Q_h}\left|v_\varepsilon^{(\gamma+1)/2}\right|^{2q}dxdt\right)^{1/q}\\
\le&\left(\frac{1}{\rho^{p+1}}\iint_{Q_\rho}\left|\chi(t)\eta^{p/2}v_\varepsilon^{(\gamma+1)/2}\right|^{2q}\right)^{1/q}\\
\le&\frac{C}{\rho(h'-h)^p}\iint_{Q_{h'}}v_\varepsilon^{\gamma+p-1}dxdt.
\end{aligned}
$$

In fact, since the dimension is 1, according to the embedding theorem, we have $q=\dfrac{5}{3}$. Hence

$$
\frac{1}{\rho^{p+1}}\iint_{Q_h}v_\varepsilon^{\frac{5}{3}(\gamma+1)}dxdt\le\left[\frac{C}{\rho(h'-h)^p}\iint_{Q_{h'}}v_\varepsilon^{\gamma+p-1}dxdt\right]^{5/3}.
$$

Let

$$h_k = \tau\rho\left(1 + \frac{1-\tau}{\tau 2^{k-1}}\right), \quad h = h_{k+1}, \quad h' = h_k;$$
$$\mu = \frac{5}{3}, \quad \mu^k = \gamma - \frac{3}{2}p + 4.$$

Then

$$\frac{1}{\rho^{p+1}} \iint_{Q_{h_{k+1}}} v_\varepsilon^{\mu^{k+1}+\frac{5}{2}(p-2)} dxdt \le \left[\frac{C}{\rho^{p+1}} \iint_{Q_{h_k}} v_\varepsilon^{\mu^k+\frac{5}{2}(p-2)} dxdt\right]^{\mu},$$

namely,

$$\left[\frac{1}{\rho^{p+1}} \iint_{Q_{h_{k+1}}} v_\varepsilon^{\mu^{k+1}+\frac{5}{2}(p-2)} dxdt\right]^{1/\mu^{k+1}} \le C\left[\frac{1}{\rho^{p+1}} \iint_{Q_{h_k}} v_\varepsilon^{\mu^k+\frac{5}{2}(p-2)} dxdt\right]^{1/\mu^k}.$$

Utilizing Moser iteration, we have

$$\sup_{Q_{\tau\rho}} v_\varepsilon \le C\left[\frac{1}{\rho^{p+1}} \iint_{Q_\rho} v_\varepsilon^{\mu+\frac{5}{2}(p-2)} dxdt\right]^{1/\mu}. \tag{2.15}$$

According to Lemma 2.7 and the definition of v_ε, for $p > \dfrac{8}{3}$, we can choose $\beta > 0$, such that

$$\beta + 2(p-1) = \mu + \frac{5}{2}(p-2),$$

namely, $\beta = \dfrac{1}{2}p - \dfrac{4}{3}$. Then there exists a constant M_{12}, such that

$$\iint_{Q_\rho} v_\varepsilon^{\mu+\frac{5}{2}(p-2)} dxdt \le M_{12}.$$

For $2 < p \le \dfrac{8}{3}$ and $\beta > 0$, it has $\beta + 2(p-1) > \mu + \dfrac{5}{2}(p-2)$. Then there exists a constant M_{13}, such that

$$\iint_{Q_\rho} v_\varepsilon^{\mu+\frac{5}{2}(p-2)} dxdt \le C \iint_{Q_\rho} v_\varepsilon^{\beta+2(p-1)} dxdt \le M_{13}.$$

According to (2.15), there exists a constant C_τ, such that

$$u_\varepsilon \le C_\tau, \qquad \forall (x,t) \in Q_{\tau\rho}.$$

The proof is complete. ∎

Utilizing the results in [15], we can obtain the Hölder estimate on u_ε.

Lemma 2.9. *For $0 < \theta < \tau$, there exist positive constants λ and $\sigma \in (0,1)$, such that*

$$\left|u_\varepsilon(x_1,t_1) - u_\varepsilon(x_2,t_2)\right| \le C_7\left(|x_1-x_2| + \lambda|t_1-t_2|^{1/p}\right)^\sigma, \quad \forall (x_i,t_i) \in Q_{\theta\rho},\ i = 1,2,$$

where C_7 is a positive constant depending on p, θ, τ, ρ, T^* and C_τ.

In addition, it is easy to obtain the following estimate.

Lemma 2.10. *There exists a positive constant C_8 independent of ε, such that*

$$\iint_{Q_{\tau\rho}} |Du_\varepsilon|^p dxdt \le C_8.$$

Now, we are in the position to prove the main result.

Proof of Theorem 1.1. According to Lemma 2.8–2.10, there are a subsequence of $\{u_\varepsilon\}$ (without loss of generality, we denote it by $\{u_\varepsilon\}$ itself) and a function u, such that for any compact set $K \subset (\mathbb{R}\backslash\{0\}) \times (0, T^*)$, we have

$$\begin{aligned} u_\varepsilon &\to u, && \text{a.e. in } K, \\ \big(|Du_\varepsilon|\big)^{p-2} Du_\varepsilon &\rightharpoonup \omega, && \text{in } L^{p/(p-1)}(K), \end{aligned}$$

and we can also prove that the following equality

$$\iint_K |Du|^{p-2} DuD\varphi dxdt = \iint_K \omega D\varphi dxdt$$

holds for any $\varphi \in C^\infty(Q_{T^*})$, which vanishes for large $|x|$ and $t = T^*$.

In addition, according to Lemma 2.5, Lemma 2.8–2.10, we have

$$u \in L^\infty\big(0, T^*; L^\infty_{loc}(\mathbb{R})\big) \cap C\big((\mathbb{R}\backslash\{0\}) \times (0, T^*)\big) \cap BV_x(Q_{T^*}).$$

Then for almost all $t \in (0, T^*)$, $u^l(0, t)$ and $u^r(0, t)$ both exist.

Now we show that u satisfies the integral equation in Definition 1.1. For any $\varphi(x, t)$ given as before, we have

$$\begin{aligned} \int_{\mathbb{R}} u_0(x)\varphi(x, 0)dx + \iint_{Q_{T^*}} u_\varepsilon \varphi_t dxdt - \iint_{Q_{T^*}} |Du_\varepsilon|^{p-2} Du_\varepsilon D\varphi dxdt \\ + \iint_{Q_{T^*}} \delta_\varepsilon(x) f(u_\varepsilon(x, t))\varphi(x, t)dxdt = 0. \end{aligned} \tag{2.16}$$

The second term of (2.16) can be rewritten as

$$\iint_{Q_{T^*}} u_\varepsilon \varphi_t dxdt = \int_0^{T^*} \int_{\mathbb{R}\backslash(-\tau,\tau)} u_\varepsilon \varphi_t dxdt + \int_0^{T^*} \int_{-\tau}^{\tau} u_\varepsilon \varphi_t dxdt, \tag{2.17}$$

where $\tau > 0$. According to Lemma 2.5, we have

$$\int_0^{T^*} \int_{-\tau}^{\tau} u_\varepsilon \varphi_t dxdt \le \int_0^{T^*} \int_{-\tau}^{\tau} u_\varepsilon |\varphi_t| dxdt \le \int_0^{T^*} \int_{-\tau}^{\tau} \lambda_1 |\varphi_t| dxdt \le C\tau\lambda_1 T^*.$$

Thus letting $\varepsilon \to 0^+, \tau \to 0^+$ in (2.17), we obtain

$$\iint_{Q_{T^*}} u_\varepsilon \varphi_t dxdt \to \iint_{Q_{T^*}} u\varphi_t dxdt.$$

Similarly, we can prove that

$$\lim_{\varepsilon \to 0^+} \iint_{Q_{T^*}} |Du_\varepsilon|^{p-2} Du_\varepsilon D\varphi dxdt = \iint_{Q_{T^*}} |Du|^{p-2} DuD\varphi dxdt.$$

In addition,

$$\begin{aligned}
&\iint_{Q_{T^*}} \varphi(x,t)\delta_\varepsilon(x)f(u_\varepsilon(x,t))dxdt \\
=&\int_0^{T^*}\int_{-\varepsilon}^{\varepsilon} \varphi(x,t)\delta_\varepsilon(x)f(u_\varepsilon(x,t))dxdt \\
=&\int_0^{T^*}\int_{-\varepsilon}^{0} \varphi(x,t)\delta_\varepsilon(x)f(u_\varepsilon(x,t))dxdt+\int_0^{T^*}\int_{0}^{\varepsilon} \varphi(x,t)\delta_\varepsilon(x)f(u_\varepsilon(x,t))dxdt.
\end{aligned}$$

Next we show the following equality holds

$$\lim_{\varepsilon\to 0^+}\int_{-\varepsilon}^{0} \varphi(x,t)\delta_\varepsilon(x)f(u_\varepsilon(x,t))dx=\frac{1}{2}f(u^l(0,t))\varphi(0,t), \tag{2.18}$$

for any $t\in(0,T^*)$. In fact,

$$\begin{aligned}
&\left|\int_{-\varepsilon}^{0}\delta_\varepsilon(x)\varphi(x,t)f(u_\varepsilon(x,t))dx-\frac{1}{2}f(u^l(0,t))\varphi(0,t)\right| \\
\le&\left|\int_{-\varepsilon}^{0}\delta_\varepsilon(x)\Big(\varphi(x,t)f(u_\varepsilon(x,t))-\varphi(0,t)f(u^l(0,t))\Big)dx\right| \\
&+\left|\left(\int_{-\varepsilon}^{0}\delta_\varepsilon(x)dx-\frac{1}{2}\right)\varphi(0,t)f(u^l(0,t))\right| \\
\le&\,\omega(\varepsilon)\int_{-\varepsilon}^{0}\delta_\varepsilon(x)dx+\varphi(0,t)f(u^l(0,t))\left|\int_{-\varepsilon}^{0}\delta_\varepsilon(x)dx-\frac{1}{2}\right|,
\end{aligned}$$

where $\omega(\varepsilon)=\sup\limits_{-\varepsilon\le x\le 0}\big|\varphi(x,t)f(u_\varepsilon(x,t))-\varphi(0,t)f(u^l(0,t))\big|$, and

$$\lim_{\varepsilon\to 0^+}\omega(\varepsilon)=0,\qquad \lim_{\varepsilon\to 0^+}\int_{-\varepsilon}^{0}\delta_\varepsilon(x)dx=\frac{1}{2},$$

we see that (2.18) holds. We can use the similar method to prove that

$$\lim_{\varepsilon\to 0^+}\int_{0}^{\varepsilon}\delta_\varepsilon(x)\varphi(x,t)f(u_\varepsilon(x,t))dx=\frac{1}{2}f(u^r(0,t))\varphi(0,t).$$

Hence we obtain

$$\begin{aligned}
&\lim_{\varepsilon\to 0^+}\iint_{Q_{T^*}}\varphi(x,t)\delta_\varepsilon(x)f(u_\varepsilon(x,t))dxdt \\
=&\int_0^{T^*}\frac{1}{2}\Big(f(u^l(0,t))+f(u^r(0,t))\Big)\varphi(0,t)dt.
\end{aligned}\tag{2.19}$$

Thus letting $\varepsilon\to 0^+$ in (2.16), we get

$$\begin{aligned}
&\int_{\mathbb{R}}u_0(x)\varphi(x,0)dx+\iint_{Q_{T^*}}(u\varphi_t-|Du|^{p-2}DuD\varphi)dxdt \\
&\qquad+\int_0^{T^*}\frac{1}{2}\Big(f(u^l(0,t))+f(u^r(0,t))\Big)\varphi(0,t)dt=0.
\end{aligned}$$

The proof of Theorem 1.1 is complete. ∎

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School of Information, Renmin University of China,
Beijing 100872, P.R. China

School of Mathematical Sciences, South China Normal University,
Guangzhou 510631, P.R. China
Email: yjx@scnu.edu.cn

Department of Mathematics, Jilin University,
Changchun 130012, P.R. China
Email: wangshuhe0821@yahoo.com.cn