Ergodic characterizations of character amenability and contractibility of Banach algebras

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Abstract

For a nonzero character ϕ on a Banach algebra $\mathfrak{A}$, we investigate some relations between ϕ-amenability of $\mathfrak{A}$ and ergodic theory. As the main result, we give a characterization for ϕ-amenability of $\mathfrak{A}$ in terms of antirepresentations of $\mathfrak{A}$ on a Banach space.

1 Introduction

Throughout, let $\mathfrak{A}$ denote a Banach algebra and let $\phi \in \sigma(\mathfrak{A})$, the set of all nonzero characters from $\mathfrak{A}$ onto $\mathbb{C}$. The notion of ϕ-amenability of $\mathfrak{A}$ was introduced and studied by Kaniuth, Lau and Pym [4]; see also Hu, Monfared and Traynor [3] and Kaniuth, Lau and Pym [5].

This is a considerable generalization of the concept of left amenability for a Lau algebra; that is, a Banach algebra $\mathfrak{L}$ which is the predual of a W^*-algebra $\mathfrak{M}$ such that the identity u of $\mathfrak{M}$ is a character on $\mathfrak{L}$; the class of Lau algebras was introduced and studied by Lau [8] in 1983 which he called F-algebras. Later on, in his useful monograph, Pier [16] introduced the name "Lau algebra". Several authors have studied and investigated the concept of left amenability of Lau algebras; see for example Lau [9], Lau and Wong [10], Mohammadzadeh and the first author [11] and the first author [13, 14].

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Moreover, the concept of ϕ-contractibility was introduced and studied by Hu, Monfared and Traynor [3]; see also [1] and [15].

Our aim in this paper is to give some characterizations of ϕ-amenability and ϕ-contractibility in terms of ϕ-ergodic antirepresentations of $\mathfrak{A}$. In order to establish these results, we introduce ϕ-ergodic antirepresentations of $\mathfrak{A}$ and give a version of Mean Ergodic Theorem for $\mathfrak{A}$.

2 Ergodic antirepresentations

For a nonzero character ϕ on a Banach algebra $\mathfrak{A}$, consider the semigroup

$$S_1(\mathfrak{A},\phi) = \{a \in \mathfrak{A} : \ \phi(a) = 1\}.$$

Let $\mathfrak{X}$ be a Banach space. For the Banach space $\mathcal{B}(\mathfrak{X})$ of all bounded operators on $\mathfrak{X}$, recall that the *strong operator topology* on $\mathcal{B}(\mathfrak{X})$ is the locally convex topology determined by the family $\{\mathcal{P}_\xi : \xi \in \mathfrak{X}\}$ of seminorms on $\mathcal{B}(\mathfrak{X})$, where

$$\mathcal{P}_\xi(A) = \| A(\xi) \|$$

for all $\xi \in \mathfrak{X}$ and $A \in \mathcal{B}(\mathfrak{X})$.

An *antirepresentation* T of $\mathfrak{A}$ on $\mathfrak{X}$ is a norm continuous, linear map $T : a \mapsto T_a$ from $\mathfrak{A}$ into $\mathcal{B}(\mathfrak{X})$ such that

$$T_{ab} = T_b T_a$$

for all $a, b \in \mathfrak{A}$. In this case, we put

$$\begin{aligned}
K(T,\phi) &= \cap\{ \text{ kernel } (T_a - I) : a \in S_1(\mathfrak{A},\phi)\},\\
R(T,\phi) &= \text{The closure of the span of} \cup \{ \text{ range } (T_a - I) : a \in S_1(\mathfrak{A},\phi)\},\\
\Sigma(T,\phi) &= K(T,\phi) + R(T,\phi), \ \text{ and}\\
C_\xi(T,\phi) &= \text{The closure of } \{T_a(\xi) : a \in S_1(\mathfrak{A},\phi)\},
\end{aligned}$$

for all $\xi \in \mathfrak{X}$. Note that $K(T,\phi)$ and $R(T,\phi)$ are closed subspaces of $\mathfrak{X}$.

We say that an antirepresentation T of $\mathfrak{A}$ on a Banach space $\mathfrak{X}$ is *ϕ-ergodic* if there is a bounded net $(E_\gamma)_{\gamma\in\Gamma}$ in $\mathcal{B}(\mathfrak{X})$ such that
($\mathcal{E}_\ell$) $E_\gamma(T_a - I) \to 0$ in the strong operator topology for all $a \in S_1(\mathfrak{A},\phi)$,
($\mathcal{E}_c$) $E_\gamma(\xi) \in C_\xi(T,\phi)$ for all $\xi \in \mathfrak{X}$ and $\gamma \in \Gamma$.

In order to demonstrate broader interest on the subject, let us point out that the sets $\ker(T - I)$ and $\overline{(T - I)(\mathfrak{A})}$ are considered in when T is a power bounded operator on a commutative Banach algebra $\mathfrak{A}$ in the interesting recent papers [6, 7] by Kaniuth, Lau and Ülger.

We commence with the following version of the Mean Ergodic Theorem for an arbitrary Banach algebras.

Theorem 2.1. *Let $\mathfrak{A}$ be a Banach algebra with $\phi \in \sigma(\mathfrak{A})$ and let T be a ϕ-ergodic antirepresentation of $\mathfrak{A}$ on a Banach space $\mathfrak{X}$ with $(E_\gamma)_{\gamma\in\Gamma}$ satisfying $(\mathcal{E}_\ell)$ and $(\mathcal{E}_c)$. Then the following statements hold.*

(a) $E_\gamma(\xi_K) = \xi_K$ *for all* $\xi_K \in K(T,\phi)$ *and* $\gamma \in \Gamma$ *and* $E_\gamma(\xi_R) \to 0$ *for all* $\xi_R \in R(T,\phi)$. *In particular, for each* $\xi \in \Sigma(T,\phi)$, *the net* $(E_\gamma(\xi))_{\gamma\in\Gamma}$ *is norm convergent to an element of* $K(T,\phi) \cap C_\xi(T,\phi)$.

(b) $\Sigma(T,\phi) = K(T,\phi) \oplus R(T,\phi)$.

(c) $T_a(\Sigma(T,\phi)) \subseteq \Sigma(T,\phi)$ *for all* $a \in S_1(\mathfrak{A},\phi)$.

(d) $C_\xi(T,\phi) \subseteq \Sigma(T,\phi)$ *for all* $\xi \in \Sigma(T,\phi)$.

(e) $E_\gamma(\Sigma(T,\phi)) \subseteq \Sigma(T,\phi)$ *for all* $\gamma \in \Gamma$.

(f) *If* $P : \Sigma(T,\phi) \to K(T,\phi)$ *is the projection associated with the direct sum*

$$\Sigma(T,\phi) = K(T,\phi) \oplus R(T,\phi),$$

then $E_\gamma(\xi) \to P(\xi)$ *and* $K(T,\phi) \cap C_\xi(T,\phi) = \{P(\xi)\}$ *for all* $\xi \in \Sigma(T,\phi)$.

Proof. (a). First, note that $C_{\xi_K}(T,\phi) = \{\xi_K\}$ for all $\xi_K \in K(T,\phi)$ and so

$$E_\gamma(\xi_K) = \xi_K$$

for all $\gamma \in \Gamma$ by $(\mathcal{E}_c)$. It follows from $(\mathcal{E}_\ell)$ that $E_\gamma(\eta) \to 0$ for all

$$\eta \in \cup\{\text{ range } (T_a - I) : a \in S_1(\mathfrak{A},\phi)\}.$$

This together with the fact that (E_γ) is bounded imply that $E_\gamma(\xi_R) \to 0$ for all $\xi_R \in R(T,\phi)$.

(b). $K(T,\phi) \cap R(T,\phi) = \{0\}$ by (a), and hence

$$\Sigma(T,\phi) = K(T,\phi) \oplus R(T,\phi).$$

(c). If $a, b \in S_1(\mathfrak{A},\phi)$, then $ba \in S_1(\mathfrak{A},\phi)$ and so for each $\xi \in \mathfrak{X}$,

$$T_a(T_b - I)(\xi) = (T_{ba} - I)(\xi) - (T_a - I)(\xi) \in R(T,\phi).$$

This shows that $T_a(R(T,\phi)) \subseteq R(T,\phi)$ whence

$$T_a(\Sigma(T,\phi)) \subseteq \Sigma(T,\phi).$$

(d). Fix $\xi \in \Sigma(T,\phi)$. To prove $C_\xi(T,\phi) \subseteq \Sigma(T,\phi)$, let $\eta \in C_\xi(T,\phi)$. Then there is a net (a_δ) in $S_1(\mathfrak{A},\phi)$ such that $T_{a_\delta}(\xi) \to \eta$. Write

$$\xi = \xi_K + \xi_R,$$

where $\xi_K \in K(T,\phi)$ and $\xi_R \in R(T,\phi)$. Since $T_{a_\delta}(R(T,\phi)) \subseteq R(T,\phi)$ and $R(T,\phi)$ is closed, it follows that

$$\eta - \xi_K = \lim_\delta\ T_{a_\delta}(\xi - \xi_K) = \lim_\delta\ T_{a_\delta}(\xi_R) \in R(T,\phi)$$

So, we have shown that $C_\xi(T,\phi) \subseteq \Sigma(T,\phi)$ for all $\xi \in \Sigma(T,\phi)$.

(e). The inclusion $E_\gamma(\Sigma(T,\phi)) \subseteq \Sigma(T,\phi)$ follows from the part (d) and $(\mathcal{E}_c)$.

(f). The first assertion follows from (a). For the second, fix $\xi \in \Sigma(T,\phi)$ and note that

$$P(\xi) \in K(T,\phi) \cap C_\xi(T,\phi)$$

by (a). To prove the converse inclusion, let $\eta \in K(T,\phi) \cap C_\xi(T,\phi)$. Then

$$T_{a_\delta}(\xi) \to \eta$$

for some net (a_δ) in $S_1(\mathfrak{A},\phi)$, and therefore

$$\eta - \xi = \lim_\delta (T_{a_\delta} - I)(\xi) \in R(T,\phi).$$

Consequently, $P(\eta - \xi) = 0$ and so $\eta = P(\xi)$ as required. ■

For $\phi \in \sigma(\mathfrak{A})$, we say that an antirepresentation T of $\mathfrak{A}$ on a Banach space $\mathfrak{X}$ is *two-sided ϕ-ergodic* if there is a bounded net $(E_\gamma)_{\gamma\in\Gamma}$ in $\mathcal{B}(\mathfrak{X})$ satisfying $(\mathcal{E}_\ell)$, $(\mathcal{E}_c)$, and

$(\mathcal{E}_r)$ $\quad (T_a - I)E_\gamma \to 0$ in the strong operator topology for all $a \in S_1(\mathfrak{A},\phi)$.

Theorem 2.1 does not give the closedness of $\Sigma(T,\phi)$ in $\mathfrak{X}$. The following result shows that $\Sigma(T,\phi)$ is closed in $\mathfrak{X}$ if in addition $(\mathcal{E}_r)$ holds.

Proposition 2.2. *Let $\mathfrak{A}$ be a Banach algebra with $\phi \in \sigma(\mathfrak{A})$ and let T be a two-sided ϕ-ergodic antirepresentation of $\mathfrak{A}$ on a Banach space $\mathfrak{X}$ with $(E_\gamma)_{\gamma\in\Gamma}$ satisfying $(\mathcal{E}_\ell)$, $(\mathcal{E}_c)$, and $(\mathcal{E}_r)$. Then $\Sigma(T,\phi)$ is the closed subspace of $\mathfrak{X}$ consisting of all $\xi \in \mathfrak{X}$ such that $(E_\delta(\xi))$ is weakly convergent in $\mathfrak{X}$ for some subnet (E_δ) of (E_γ).*

Proof. Using Theorem 2.1(a), $(E_\gamma(\xi))$ is norm (and hence weakly) convergent to an element of $K(T,\phi) \cap C_\xi(T,\phi)$ for all $\xi \in \Sigma(T,\phi)$.

Now, let $\xi \in \mathfrak{X}$ and suppose that $(E_\delta(\xi))$ is weakly convergent to $\eta \in \mathfrak{X}$ for some subnet (E_δ) of (E_γ). We must show that

$$\xi \in \Sigma(T,\phi).$$

Note that $\eta \in C_\xi(T,\phi)$ because $C_\xi(T,\phi)$ is convex in $\mathfrak{X}$ and $E_\delta(\xi) \in C_\xi(T,\phi)$ for all δ. Also, by $(\mathcal{E}_r)$, for each $a \in S_1(\mathfrak{A},\phi)$ and $f \in \mathfrak{X}^*$ we have

$$\begin{aligned} f(T_a(\eta)) &= \lim_\delta (f \circ T_a)(E_\delta(\xi)) \\ &= \lim_\delta [(f \circ (T_a - I))(E_\delta(\xi)) + f(E_\delta(\xi))] \\ &= f(\eta), \end{aligned}$$

and so $T_a(\eta) = \eta$ whence

$$\eta \in K(T,\phi) \cap C_\xi(T,\phi).$$

It follows that $\xi = \eta + (\xi - \eta) \in K(T,\phi) + R(T,\phi) = \Sigma(T,\phi)$. ■

As a consequence of the above Proposition, we have the following result.

Corollary 2.3. *Let T, $\mathfrak{X}$, and (E_γ) be as in the above proposition. Then $\Sigma(T,\phi) = \mathfrak{X}$ if $C_\xi(T,\phi)$ is weakly compact for all $\xi \in \mathfrak{X}$.*

3 Ergodic characterization of ϕ-amenability

Let $\mathfrak{A}$ be a Banach algebra and let $\phi \in \sigma(\mathfrak{A})$. The Banach algebra $\mathfrak{A}$ is called ϕ-amenable if there exists a bounded linear functional F on $\mathfrak{A}^*$ satisfying

$$F(\phi) = 1 \quad \text{and} \quad F(f \cdot a) = \phi(a)F(f)$$

for all $a \in \mathfrak{A}$ and $f \in \mathfrak{A}^*$; here $f \cdot a \in \mathfrak{A}^*$ is defined by $(f \cdot a)(b) = f(ab)$ for all $b \in \mathfrak{A}$. Any such F is called a ϕ-mean; see also [12].

Recall that a *ϕ-approximate diagonal* for $\mathfrak{A}$ is a net $(\mathbf{m}_\gamma)$ in $\mathfrak{A}\widehat{\otimes}\mathfrak{A}$ such that

$$\phi(\pi(\mathbf{m}_\gamma)) = 1 \quad \text{and} \quad \|a \cdot \mathbf{m}_\gamma - \phi(a)\mathbf{m}_\gamma\| \to 0$$

for all $a \in \mathfrak{A}$, where π denotes the product morphism from $\mathfrak{A}\widehat{\otimes}\mathfrak{A}$ into $\mathfrak{A}$ given by

$$\pi(a \otimes b) = ab$$

for all $a, b \in \mathfrak{A}$. The notion of ϕ-approximate diagonal was introduced and studied by Hu, Monfared and Traynor [3]. They show that $\mathfrak{A}$ has a ϕ-mean if and only if it has a bounded ϕ-approximate diagonal.

For a Banach algebra $\mathfrak{A}$, let Λ denote the antirepresentation of $\mathfrak{A}$ on $\mathfrak{A}^*$ defined by

$$\Lambda_a(f) = f \cdot a$$

for all $a \in \mathfrak{A}$ and $f \in \mathfrak{A}^*$. We now are ready to give a characterization of ϕ-amenability of $\mathfrak{A}$.

Theorem 3.1. *Let $\mathfrak{A}$ be a Banach algebra and let $\phi \in \sigma(\mathfrak{A})$. Then the following assertions are equivalent.*

(a) $\mathfrak{A}$ *is ϕ-amenable.*

(b) *Each antirepresentation of* $\mathfrak{A}$ *is ϕ-ergodic.*

(c) *The antirepresentation Λ is ϕ-ergodic.*

Proof. (a)$\Rightarrow$(b). Let T be an antirepresentation of $\mathfrak{A}$ on a Banach space $\mathfrak{X}$. Since $\mathfrak{A}$ is ϕ-amenable, it has a bounded ϕ-approximate diagonal $(\mathbf{m}_\gamma)_{\gamma\in\Gamma}$. Thus

$$\pi(\mathbf{m}_\gamma) \in S_1(\mathfrak{A}, \phi) \quad \text{and} \quad \|a\pi(\mathbf{m}_\gamma) - \pi(\mathbf{m}_\gamma)\| \to 0$$

for all $a \in S_1(\mathfrak{A}, \phi)$. So, if we put

$$E_\gamma = T_{\pi(\mathbf{m}_\gamma)}$$

for all $\gamma \in \Gamma$, then (E_γ) is bounded in $B(\mathfrak{X})$ and

$$\begin{aligned} \| E_\gamma(T_a - I) \| &= \| T_{a\pi(\mathbf{m}_\gamma)-\pi(\mathbf{m}_\gamma)} \| \\ &\leq \| T \| \, \| a\pi(\mathbf{m}_\gamma) - \pi(\mathbf{m}_\gamma) \| \to 0 \end{aligned}$$

for all $a \in S_1(\mathfrak{A}, \phi)$; that is, $(E_\gamma)_{\gamma\in\Gamma}$ satisfies $(\mathcal{E}_\ell)$. The condition $(\mathcal{E}_c)$ is also satisfied because $E_\gamma(\xi) = T_{\pi(\mathbf{m}_\gamma)}(\xi) \in C_\xi(T, \phi)$ for all $\xi \in \mathfrak{X}$ and $\gamma \in \Gamma$.

(b)$\Rightarrow$(c). Clear.

(c)$\Rightarrow$(a). Let Λ be as in (c). Then

$$\Lambda_a \phi = \phi \cdot a = \phi$$

for all $a \in S_1(\mathfrak{A}, \phi)$. That is $\phi \in K(\Lambda, \phi)$, and hence $\phi \notin R(\Lambda, \phi)$ by Theorem 2.1(b). Using the Hahn-Banach Theorem, we may find a nonzero element F of $\mathfrak{A}^{**}$ such that $F(\phi) = 1$ and $F|_{R(\Lambda,\phi)} = 0$. Thus

$$F(f \cdot a) = F(f)$$

for all $a \in S_1(\mathfrak{A}, \phi)$ and $f \in \mathfrak{A}^*$ and hence $\mathfrak{A}$ is ϕ-amenable. ■

Recall that the Banach algebra $\mathfrak{A}$ is called two-sided ϕ-amenable if it has a two-sided ϕ-mean; i.e., an element $F \in \mathfrak{A}^{**}$ with $F(\phi) = 1$ and

$$F(f \cdot a) = F(a \cdot f) = F(f)$$

for all $a \in S_1(\mathfrak{A}, \phi)$ and $f \in \mathfrak{A}^*$. Suppose that $\mathfrak{A}$ is two-sided ϕ-amenable. Then there is a bounded net $(a_\gamma)_{\gamma\in\Gamma}$ in $S_1(\mathfrak{A}, \phi)$ such that

$$\|aa_\gamma - a_\gamma\| + \|a_\gamma a - a_\gamma\| \to 0$$

for all $a \in S_1(\mathfrak{A}, \phi)$; see [5], Proposition 3.3. Thus

$$\mathbf{m}_{\mathbf{fl}} := a_\gamma \otimes a_\gamma$$

is a two-sided ϕ-approximate diagonal in $\mathfrak{A}\widehat{\otimes}\mathfrak{A}$; i.e., $\phi(\pi(\mathbf{m}_{\mathbf{fl}})) = 1$ and

$$\|a \cdot \mathbf{m}_\gamma - \mathbf{m}_\gamma\| + \|\mathbf{m}_\gamma \cdot a - \mathbf{m}_\gamma\| \to 0$$

for all $a \in S_1(\mathfrak{A}, \phi)$. So, if we put $E_\gamma = T_{\pi(\mathbf{m}_\gamma)}$ for all $\gamma \in \Gamma$, then as in the above proof $(E_\gamma)_{\gamma\in\Gamma}$ satisfies the conditions $(\mathcal{E}_\ell)$, $(\mathcal{E}_c)$, and $(\mathcal{E}_r)$.

Notice that if $\mathfrak{A}$ is commutative and ϕ-amenable, then $\mathfrak{A}$ is automatically two-sided ϕ-amenable. Therefore, two-sided ϕ-amenable Banach algebras form a large class of Banach algebras such that for any antirepresentation T of such algebras on a Banach space $\mathfrak{X}$, there is a net $(E_\gamma)_{\gamma\in\Gamma}$ satisfying $(\mathcal{E}_\ell)$, $(\mathcal{E}_c)$, and $(\mathcal{E}_r)$.

Let $\mathfrak{A}$ be a Banach algebra and let $\phi \in \sigma(\mathfrak{A})$. For an antirepresentation T of $\mathfrak{A}$ on a Banach space $\mathfrak{X}$, let $N(T, \phi)$ denote the set of all $\xi \in \mathfrak{X}$ for which there exists a net $(a_\gamma)_{\gamma\in\Gamma}$ in $S_1(\mathfrak{A}, \phi)$ such that

$$\|T_{a_\gamma}(\xi)\| \to 0.$$

Theorem 3.2. *Let $\mathfrak{A}$ be a Banach algebra and let $\phi \in \sigma(\mathfrak{A})$. Then the following assertions are equivalent.*

(a) *$\mathfrak{A}$ is ϕ-amenable.*

(b) *$N(T, \phi) = R(T, \phi)$ for all antirepresentations T of $\mathfrak{A}$.*

(c) *$N(\Lambda, \phi) = R(\Lambda, \phi)$.*

Proof. (a)⇒(b). Since $\mathfrak{A}$ is ϕ-amenable, it follows that $\mathfrak{A}$ has a bounded ϕ-approximate diagonal $(\mathbf{m}_\gamma)_{\gamma\in\Gamma}$. Thus $\pi(\mathbf{m}_\gamma) \in S_1(\mathfrak{A},\phi)$ and $\|a\pi(\mathbf{m}_\gamma) - \pi(\mathbf{m}_\gamma)\| \to 0$ for all $a \in S_1(\mathfrak{A},\phi)$. Therefore

$$\begin{aligned}\|T_{\pi(\mathbf{m}_\gamma)}(T_a - I)\| &= \|T_{a\pi(\mathbf{m}_\gamma)-\pi(\mathbf{m}_\gamma)}\| \\ &\leq \|T\|\|a\pi(\mathbf{m}_\gamma) - \pi(\mathbf{m}_\gamma)\|.\end{aligned}$$

Thus $(T_a - I)(\xi) \in N(T,\phi)$ for all $\xi \in \mathfrak{X}$ and $a \in S_1(\mathfrak{A},\phi)$, and consequently

$$R(T,\phi) \subseteq N(T,\phi).$$

Now, let $\xi \in N(T,\phi)$. Then there exists a net $(a_\gamma)_{\gamma\in\Gamma}$ in $S_1(\mathfrak{A},\phi)$ such that

$$\|T_{a_\gamma}(\xi)\| \to 0.$$

But $(T_{a_\gamma} - I)(\xi) \in R(T,\phi)$ for all $\gamma \in \Gamma$ and

$$\lim_\gamma (T_{a_\gamma} - I)(\xi) = -\xi.$$

Since $R(T,\phi)$ is a closed subspace of $\mathfrak{X}$, it follows that $\xi \in R(T,\phi)$

(b)⇒(c). Clear.

(c)⇒(a). It is clear that $\phi \notin N(\Lambda,\phi)$. Thus the result follows by the same argument of proof of Theorem 3.1(c) with the fact that $R(\Lambda,\phi)$ is a closed subspace of $\mathfrak{X}$. ∎

4 Ergodic characterization of ϕ-contractibility

A Banach algebra $\mathfrak{A}$ is called *ϕ-contractible* if there exists a ϕ-diagonal; i.e., an element $\mathbf{m}$ in the projective tensor product $\mathfrak{A}\widehat{\otimes}\mathfrak{A}$ such that $\phi(\pi(\mathbf{m})) = 1$ and $a \cdot \mathbf{m} = \phi(a)\mathbf{m}$ for all $a \in \mathfrak{A}$. Before, we give a characterization of ϕ-contractibility, we need the following Lemma.

Lemma 4.1. *Let $\mathfrak{A}$ be a Banach algebra with $\phi \in \sigma(\mathfrak{A})$ and let $a \in S_1(\mathfrak{A},\phi)$. Consider the following conditions.*

(a) *a is a ϕ-mean in $\mathfrak{A}$.*

(b) $R(T,\phi) = \{\xi \in \mathfrak{X} : \ T_a(\xi) = 0\}$ *for all antirepresentations T of $\mathfrak{A}$.*

(c) $R(\Lambda,\phi) = \{f \in \mathfrak{A}^* : \ f \cdot a = 0\}$.

Then (a)⇒(b)⇒(c) *and if $\mathfrak{A}$ has a right approximate identity, then* (c)⇒(a).

Proof. Suppose that (a) holds. Then $\phi(a) = 1$ and $ba = a$ for all $b \in S_1(\mathfrak{A},\phi)$ and hence

$$T_a(T_b - I)(\xi) = (T_{ba-a})(\xi) = 0$$

for all $\xi \in \mathfrak{X}$. Thus $T_a(\eta) = 0$ for all $\eta \in R(T,\phi)$; that is,

$$R(T,\phi) \subseteq \{\xi \in \mathfrak{X} : \ T_a(\xi) = 0\}.$$

The reverse inclusion is clear.

That (b) implies (c) is trivial. Now, suppose that (c) holds and that there is a right approximate identity $(e_\gamma)_{\gamma\in\Gamma}$ for $\mathfrak{A}$, and let $b \in S_1(\mathfrak{A}, \phi)$. Then

$$f \cdot b - f \in R(T, \phi)$$

and so

$$f \cdot (ba - a) = (f \cdot b - f) \cdot a = 0$$

for all $f \in \mathfrak{A}^*$. Thus

$$\begin{aligned} f(ba - a) &= \lim_\gamma f((ba - a)e_\gamma) \\ &= \lim_\gamma [f \cdot (ba - a)](e_\gamma) = 0. \end{aligned}$$

Since $\mathfrak{A}^*$ separates the elements of $\mathfrak{A}$, it follows that $ba - a = 0$ for all $b \in S_1(\mathfrak{A}, \phi)$; hence $ba = \phi(b)a$ for all $b \in \mathfrak{A}$. Thus a is a ϕ-mean. □

Proposition 4.2. *Let $\mathfrak{A}$ be a Banach algebra with a right approximate identity and let $\phi \in \sigma(\mathfrak{A})$. Then the following assertions are equivalent.*

(a) *$\mathfrak{A}$ is ϕ-contractible.*

(b) *There exists $a \in S_1(\mathfrak{A}, \phi)$ such that $R(T, \phi) = \{\xi \in \mathfrak{X} : \ T_a(\xi) = 0\}$ for all antirepresentations T of $\mathfrak{A}$.*

(c) *There exists $a \in S_1(\mathfrak{A}, \phi)$ such that $R(\Lambda, \phi) = \{f \in \mathfrak{A}^* : \ f \cdot a = 0\}$.*

Proof. Suppose that $\mathfrak{A}$ is ϕ-contractible. Then there is a ϕ-diagonal $\mathbf{m} \in \mathfrak{A}\widehat{\otimes}\mathfrak{A}$ for $\mathfrak{A}$. It easy to check that $\pi(\mathbf{m})$ is a ϕ-mean in $\mathfrak{A}$. Conversely, if a is a ϕ-mean in $\mathfrak{A}$, then it is clear that $\mathbf{m} = a \otimes a$ is a ϕ-diagonal for $\mathfrak{A}$. Thus the result follows immediately from Lemma 4.1. ■

Before we give our last result, note that if a is a ϕ-mean in $\mathfrak{A}$, then $\langle a \rangle$, the Banach space generated by $\{a\}$ is a Banach algebra.

Corollary 4.3. *Let $\mathfrak{A}$ be a Banach algebra and let $\phi \in \sigma(\mathfrak{A})$. If a is a two-sided ϕ-mean in $\mathfrak{A}$, then the following assertions hold.*

(a) *$R(\Lambda, \phi) = \{f \in \mathfrak{A}^* : \ f(a) = 0\}$.*

(b) *$\mathfrak{A}^* = \Sigma(\Lambda, \phi)$ if and only if $K(\Lambda, \phi) = \langle a \rangle^*$.*

Proof. (a). By Lemma 4.1, it suffices to show that $f \cdot a = 0$ if and only if $f(a) = 0$. Since a is a two-sided ϕ-mean, it follows that for each $b \in S_1(\mathfrak{A}, \phi)$,

$$f(a) = f(ba) = f(ab) = f \cdot a(b)$$

and this completes the proof.

(b). Suppose that $\mathfrak{A}^* = \Sigma(\Lambda, \phi) = K(\Lambda, \phi) \oplus R(\Lambda, \phi)$. It is clear that

$$R(\Lambda, \phi) = \langle a \rangle^\perp,$$

where

$$\langle a \rangle^\perp = \{f \in \mathfrak{A}^* : f(b) = 0 \ \text{ for all } b \in \langle a \rangle\}.$$

Thus $\mathfrak{A}^*/\langle a \rangle^\perp \cong K(\Lambda, \phi)$. On the other hand we have

$$\mathfrak{A}^*/\langle a \rangle^\perp \cong \langle a \rangle^*$$

and so $K(\Lambda,\phi) \cong \langle a\rangle^*$.

Conversely, if $K(\Lambda,\phi) = \langle a\rangle^*$, then since

$$\mathfrak{A}^*/\langle a\rangle^{\perp} \cong \langle a\rangle^*,$$

it follows that $\mathfrak{A}^* = K(\Lambda,\phi) \oplus R(\Lambda,\phi) = \Sigma(\Lambda,\phi)$. ■

Example 4.4. Let G be a compact group with normalized Haar measure dx and consider the convolution algebra $L^p(G)$ as in [2], where $1 \leq p < \infty$. Let $\widehat{G}$ denote the set of all continuous homomorphisms from G into the circle group $\mathbb{T}$, equipped with the topology of uniform convergence. For $\rho \in \widehat{G}$, define $\phi_\rho \in \sigma(L^p(G))$ to be the character induced by ρ on $L^p(G)$; that is,

$$\phi_\rho(g) = \int_G \overline{\rho(x)}g(x)dx \quad (g \in L^p(G)).$$

Fix $\rho \in \widehat{G}$, it is clear that $\rho \in L^p(G)$ and for each $g \in L^p(G)$, we have

$$g * \rho = \rho * g = \phi_\rho(g)\rho.$$

Thus ρ is a two-sided ϕ_ρ-mean in $L^p(G)$. Consider the antirepresentation Λ of $L^p(G)$ on $L^p(G)^* = L^q(G)$ defined by

$$\Lambda_g(f) = f \cdot g$$

for all $g \in L^p(G)$ and $f \in L^q(G)$, and note that Λ is ϕ_ρ-ergodic by Theorem 3.1, where $q = p/(p-1)$. Now, we show that

$$K(\Lambda,\phi_\rho) = \{f \in L^q(G) : \ \widehat{f}(\overline{\eta}) = 0 \ \text{ for all } \ \eta \neq \rho\},$$

where

$$\widehat{f}(\eta) = \int_G \overline{\eta(x)}f(x)dx$$

for all $\eta \in \widehat{G}$. Suppose that $f \in K(\Lambda,\phi_\rho)$. Then $f \cdot g - f = 0$ for all $g \in S_1(L^p(G),\phi_\rho)$. In particular $f \cdot \rho - f = 0$; thus

$$\widehat{f \cdot \rho} = \widehat{f}.$$

Now, let $\eta \in \widehat{G}$. Then we have

$$\begin{aligned}
(\widehat{f \cdot \rho})(\overline{\eta}) &= (\widehat{f * \overline{\rho}})(\overline{\eta}) \\
&= \int_G \eta(x)\Big(\int_G f(xy)\rho(y)dy\Big)dx \\
&= \int_G\int_G f(x)\eta(x)\rho(y)\overline{\eta(y)}dxdy \\
&= \widehat{f}(\overline{\eta})\int_G \rho(y)\overline{\eta(y)}dy.
\end{aligned}$$

The orthogonality relations now imply that

$$(\widehat{f \cdot \rho})(\overline{\eta}) = 0$$

whenever $\eta \neq \rho$. Thus $\widehat{f}(\overline{\eta}) = 0$ for all $\eta \neq \rho$, and this shows that

$$K(\Lambda, \phi_\rho) \subseteq \{f \in L^q(G) : \ \widehat{f}(\overline{\eta}) = 0 \ \text{ for all } \ \eta \neq \rho\}.$$

For the reverse inclusion, let $f \in L^q(G)$ with $\widehat{f}(\overline{\eta}) = 0$ for all $\eta \neq \rho$. Then for each $\eta \in \widehat{G}$ and $g \in S_1(L^p(G), \phi_\rho)$, we have

$$\begin{aligned}
(\widehat{f \cdot g - f})(\overline{\eta}) &= (\widehat{f \cdot g})(\overline{\eta}) - \widehat{f}(\overline{\eta}) \\
&= (\widehat{f * \check{g}})(\overline{\eta}) - \widehat{f}(\overline{\eta}) \\
&= \int_G \int_G f(x) g(y) \overline{\eta(y)} \eta(x) dy dx - \widehat{f}(\overline{\eta}) \\
&= \widehat{f}(\overline{\eta}) \phi_\eta(g) - \widehat{f}(\overline{\eta}),
\end{aligned}$$

where $\check{g}(x) = g(x^{-1})$ for all $x \in G$. Thus

$$(\widehat{f \cdot g - f})(\overline{\eta}) = 0$$

whenever $\widehat{f}(\overline{\eta}) = 0$ or $\phi_\eta(g) = 1$. It follows that

$$\widehat{f \cdot g - f} = 0$$

for all $g \in S_1(L^p(G), \phi_\rho)$. Thus $f \cdot g - f = 0$ and hence

$$\{f \in L^q(G) : \ \widehat{f}(\overline{\eta}) = 0 \ \text{ for all } \ \eta \neq \rho\} \subseteq K(\Lambda, \phi_\rho)$$

as required. By Corollary 4.3(a), it is clear that

$$R(\Lambda, \phi_\rho) = \{f \in L^q(G) : \ \widehat{f}(\overline{\rho}) = 0\}.$$

Consequently $\Sigma(\Lambda, \phi_\rho) = L^q(G)$.

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References

[1] M. Alaghmandan, R. Nasr-Isfahani and M. Nemati, Character amenability and contractibility of abstract Segal algebras, *Bull. Austral. Math. Soc.* **82** (2010), 274-281.

[2] E. Hewitt and K. A. Ross , Abstract harmonic analysis I, Springer-Verlag, Berlin, 1970.

[3] Z. Hu, M. S. Monfared and T. Traynor, On character amenable Banach algebras, *Studia Math.* **193** (2009) 53-78.

[4] E. Kaniuth, A.T. Lau and J. Pym, On ϕ-amenability of Banach algebras, *Math. Proc. Camb. Phil. Soc.* **144** (2008), 85-96.

[5] E. KANIUTH, A.T. LAU AND J. PYM, On character amenability of Banach algebras, *J. Math. Anal. Appl.* **344** (2008), 942-955.

[6] E. KANIUTH, A.T. LAU AND A. ÜLGER, Multipliers of commutative Banach algebras, power boundedness and Fourier-Stieltjes algebras, *J. London Math. Soc.* **81** (2010), 255-275.

[7] E. KANIUTH, A.T. LAU AND A. ÜLGER, Power boundedness in Fourier and Fourier-Stieltjes algebras and other commutative Banach algebras, *J. Funct. Anal.*, to appear.

[8] A. T. LAU, Analysis on a class of Banach algebras with application to harmonic analysis on locally compact groups and semigroups, *Fund. Math.* **118** (1983), 161-175.

[9] A. T. LAU, Uniformly continuous functionals on Banach algebras, *Colloq. Math.* **51** (1987), 195-205.

[10] A. T. LAU AND J. C. S. WONG, Invariant subspaces for algebras of linear operators and amenable locally compact groups, *Proc. Amer. Math. Soc.* **102** (1988), 581-586.

[11] B. MOHAMMADZADEH AND R. NASR-ISFAHANI, Positive elements of left amenable Lau algebras, *Bull. Belg. Math. Soc.* **13** (2006), 319-324.

[12] M. S. MONFARED, Character amenability of Banach algebras, *Math. Proc. Camb. Phil. Soc.* **144** (2008), 697-706.

[13] R. NASR-ISFAHANI, Factorization in some ideals of Lau algebras with applications to semigroup algebras, *Bull. Belg. Math. Soc.* **7** (2000), 429-433.

[14] R. NASR-ISFAHANI, Ergodic theoretic characterization of left amenable Lau algebras, *Bull. Iranian Math. Soc.* **28** (2002), 29-35.

[15] R. NASR-ISFAHANI AND S. SOLTANI, Character contractibility of Banach algebras and homological properties of Banach modules, *Studia Math.*, to appear.

[16] J. P. PIER, Amenable Banach algebras, *Pitman research notes in mathematics series*, Vol. 172, Longman scientific and technical, Harlow, 1988.

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