Core Theorems for Subsequences of Double Complex Sequences*

Harry I. Miller Leila Miller-Van Wieren

Abstract

In this article we present core theorems for double sequences whose entries are complex numbers. These results extend work of Miller and Patterson [9] dealing with double sequences of real numbers. The proofs in this paper are much more involved then the proofs in the article just mentioned as the convex sets in the plane are, in general, much more involved then the trivial convex sets in the line. We give an answer to the following question. If w is a bounded double sequence with complex entries and A is a 4-dimensional matrix summability method, under what conditions on A does there exist z, a subsequence (rearrangement), of w such that each complex number t, in the core of w, is a limit point of Az?

1 Introduction

In [9], Miller and Patterson proved a theorem, that answers the question mentioned in the abstract, for double sequences whose entries are real numbers. In the introduction of that paper results for single sequences of reals are presented as motivation (see [9])). In this paper we answer the question asked for double complex sequences mentioned in our abstract. All double sequences, in the remainder of this paper, will be assumed to have complex entries.

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2 Definitions and Preliminaries

Definition 2.1. *A double sequence $w = (w_{u,v})$ has Pringsheim limit L (denoted by $P - lim\ \ w = L$) provided that given any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|w_{u,v} - L| < \epsilon$ whenever $u, v > N$. We shall describe such a w more briefly as "P-convergent".*

Definition 2.2. *A double sequence w is called definitely divergent, if for every $G > 0$ there exist two natural numbers n_1 and n_2 such that $|w_{u,v}| > G$ for $u \geq n_1$, $v \geq n_2$.*

Definition 2.3. *The double sequence z is called a double subsequence of the sequence w provided that there exist two strictly increasing index sequences (n_j) and (k_j) such that $z_j = w_{n_j,k_j}$, and z is formed by*

$$z = \begin{pmatrix} z_1 & z_2 & z_5 & z_{10} & \cdot \\ z_4 & z_3 & z_6 & \cdot & \cdot \\ z_9 & z_8 & z_7 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

That is: $z_{1,1} = z_1$, $z_{1,2} = z_2$, $z_{1,3} = z_5$, $z_{2,1} = z_4$, $z_{2,2} = z_3$, etc.

Definition 2.4. *A number β is called a Pringsheim limit point of the double sequence $w = (w_{u,v})$ provided that there exists a subsequence $z = (z_{u,v})$ of $w = (w_{u,v})$ that has Pringsheim limit β, i.e. $P - lim\ \ z = \beta$.*

The double sequence w is bounded if there exists a positive number M such that $|w_{u,v}| < M$ for all u and v. A two dimensional matrix transformation is said to be regular if it maps every convergent sequence onto a convergent sequence with the same limit. The Silverman-Toeplitz theorem [15], [16], characterizes the regularity of two dimensional matrix transformations.

Let $A = (a_{m,n,u,v})$ denote a four dimensional real matrix. We obtain a summability method that maps the double complex sequence w into the double complex sequence Aw where the mn^{th} term of Aw is as follows:

$$z = (Aw)_{m,n} = \sum_{u,v=1,1}^{\infty,\infty} a_{m,n,u,v} w_{u,v},$$

provided that

$$\lim_{p,q} \sum_{u=1}^{p} \sum_{v=1}^{q} a_{m,n,u,v} w_{u,v} = (Aw)_{m,n}$$

in the sense of Pringsheim convergence, for all $m, n = 1, 2, 3 \ldots$. In this case we call the terms $(Aw)_{m,n}$ the A-means of w. For a reference on the above and what follows see Moritz and Rhoades [13] . Moreover, we say that a sequence w is A-summable to the limit t if the A-means exist for all $m, n = 1, 2, 3 \ldots$, and

$$\lim_{m,n} (Aw)_{m,n} = t$$

in the sense of Pringsheim convergence.

Definition 2.5. *The four dimensional real matrix A is said to be bounded regular if every bounded P-convergent double complex sequence w with Pringsheim limit t, is also A-summable to t and the A-means of w are bounded.*

Theorem 2.6. *([6], [13], [14]) Necessary and sufficient conditions for A to be bounded regular are:*

(1) : $\lim_{m,n} a_{m,n,u,v} = 0$ *for each u and v;*

(2) : $\lim_{m,n} \sum_{u,v=1,1}^{\infty,\infty} a_{m,n,u,v} = 1$;

(3) : $\lim_{m,n} \sum_{u=1}^{\infty} |a_{m,n,u,v}| = 0$ *for each v;*

(4) : $\lim_{m,n} \sum_{v=1}^{\infty} |a_{m,n,u,v}| = 0$ *for each u;*

(5) : $\sum_{u,v=1,1}^{\infty,\infty} |a_{m,n,u,v}|$ *is P-convergent; and*

(6): *there exist positive integers A and B such that* $\sum_{u,v>B} |a_{m,n,u,v}| < A$ *for each m, n.*

We now define the concept of a λ-rearrangement for double sequences (from [9]).

Definition 2.7. *A mapping $\phi : \mathbb{N}x\mathbb{N} \longrightarrow \mathbb{N}x\mathbb{N}$ is called a λ-rearrangement, $\lambda > 1$, of $\mathbb{N}x\mathbb{N}$ if it is a one to one, onto function such that $\phi(u, v) = (u, v)$ for $(u, v) \in \mathbb{N}x\mathbb{N} \setminus (\lambda - wedge)$ where the $(\lambda - wedge) = \{(u, v) : \frac{1}{\lambda} \leq \frac{u}{v} \leq \lambda\}$. A λ-rearrangement of w , a double sequence is a double sequence of the form $(w_{\phi(u,v)})_{u,v}$ where ϕ is a λ-rearrangement of $\mathbb{N}x\mathbb{N}$.*

Definition 2.8. *The double sequence z is called a rearrangement of the double sequence w provided that there is a one-to-one, onto $\phi : \mathbb{N}x\mathbb{N} \longrightarrow \mathbb{N}x\mathbb{N}$ such that for each (u, v), $z_{(u,v)} = w_{\phi(u,v)}$.*

In [8] the following is proved.

Theorem 2.9. *If w is a bounded sequence and $A = (a_{m,n})$ is a regular matrix summability method satisfying $\lim_m(sup_n|a_{m,n}|) = 0$, then there exists a subsequence z of w such that each t in the core of w is a limit point of (Az). Here the core of w equals $[\liminf w, \limsup w]$.*

Definition 2.10. *$A = (a_{m,n,u,v})$ is said to satisfy condition (S) if the double sequence $(\sup_{u,v} |a_{m,n,u,v}|)_{m,n}$ is Pringsheim convergent to zero.*

Definition 2.11. *If w is a double sequence, then we use the following notation:*

(a) : *$C(w)$ denotes the smallest convex set containing all limit points of w. $C(w)$ is called the core of w;*

(b) : *$L(w)$ denotes the set of all limit points of w;*

(c) : *$D(w)$ denotes the set of all complex numbers of the form $\sum_{i=1}^{n} \alpha_i t_i$; $\alpha_i \geq 0$ for all i, $\sum_{i=1}^{n} \alpha_i = 1$, and $t_i \in L(w)$ for all i.*

Remark 2.12. *It is easy to show that $C(w) = D(w)$ and it is a closed set.*

3 Results

We will prove analogues of Theorems 3.1 and 3.2 from [9], for double sequences with complex entries.

Theorem 3.1. *If $A = (a_{m,n,u,v})$ is a four dimensional bounded regular real summability method satisfying condition (S) and $w = (w_{u,v})$ is a bounded double complex sequence, then there exists a double subsequence z of w such that each $t \in C(w)$ is a Pringsheim limit point of Az.*

Proof: If w is Pringsheim convergent then the result is immediate by the regularity of A. If $C(w)$ is a line segment in the complex plane then a minor modification of the proof when $w = (w_{u,v})$ is a bounded real double sequence, found in [9], is applicable. Therefore we will only consider the case when $C(w)$ contains three non-linear points in the complex plane.

We note again that it is an easy exercise to show that $C(w) = D(w)$. Now, if $C(w)$ contains three non-linear points, it is easy to see that there exists a complex sequence (s_n) that is dense in $C(w)$ and such that each s_n is an interior point of $C(w)$. Let (t_n) denote the sequence $s_1, s_1, s_2, s_1, s_2, s_3, s_1, s_2, s_3,$
$s_4, \ldots$. Let $M > 1$ be an upper bound of the double sequence $(|w_{u,v}|)$. Further, let (ϵ_n) be a strictly monotonic null sequence.

Since $C(w) = D(w)$, t_1 can be written in the form $t_1 = q_{11}l_{11} + q_{12}l_{12} + \ldots + q_{1n(1)}l_{1n(1)}$ where $q_{1i} > 0$ for all i, $\sum_{i=1}^{n(1)} q_{1i} = 1$, and $l_{11}, l_{12}, \ldots, l_{1n(1)} \in L(w)$. By (1) thru (5) and (S) there exist positive integers m_1 and r_1, both greater than 1 such that

$$(\beta)\ \sup_{u,v} |a_{m_1,m_1,u,v}| < \min\{\tfrac{\epsilon_1}{M}, \tfrac{q_{11}}{10^1 n(1)M}, \ldots, \tfrac{q_{1n(1)}}{10^1 n(1)M}\}$$

$$(\gamma)\ |\textstyle\sum_{u,v \le r_1 - 1} a_{m_1,m_1,u,v} - 1| < \min\{\tfrac{\epsilon_1}{M}, \tfrac{q_{1n(1)}}{10^1 n(1)M}\}$$

$$(\delta)\ \textstyle\sum_{(u,v):(u \ge r_1) \bigvee (v \ge r_1)} |a_{m_1,m_1,u,v}| < \tfrac{\epsilon_1}{M}.$$

Next, there exist $q_{21}, q_{22}, \ldots q_{2n(2)}$ where $q_{2i} > 0$ for all i, $\sum_{i=1}^{n(2)} q_{2i} = 1$, and $l_{21}, l_{22}, \ldots l_{2n(2)}$ in $L(w)$ such that $t_2 = \sum_{i=1}^{n(2)} q_{2i} l_{2i}$. By (1) through (5) of Theorem 2.6 and (S) there exist positive integers m_2, r_2 with $m_2 > m_1$ and $r_2 > r_1$ such that

$$\textstyle\sum_{u,v \le r_1 - 1} |a_{m_2,m_2,u,v}| < \tfrac{\epsilon_2}{M},$$

$$\sup_{u,v} |a_{m_2,m_2,u,v}| < \min\{\tfrac{\epsilon_2}{M}, \tfrac{q_{21}}{10^2 n(2)M}, \ldots, \tfrac{q_{2n(2)}}{10^2 n(2)M}\}$$

$$|\textstyle\sum_{(u,v) \in L_{r_1,r_2-1}} a_{m_2,m_2,u,v} - 1| < \min\{\tfrac{\epsilon_2}{M}, \tfrac{q_{2n(2)}}{10^2 n(2)M}\}$$

$$\textstyle\sum_{(u,v):(u \ge r_2) \bigvee (v \ge r_2)} |a_{m_2,m_2,u,v}| < \tfrac{\epsilon_2}{M}.$$

In the above and in what follows, if t and s are positive integers, with $t > s$ let

$$L_{st} = (\mathbb{N}x\mathbb{N}) \setminus (\{(u,v) : u > t \text{ or } v > t\} \bigcup \{(u,v) : u, v < s\}).$$

Continuing, as in the first two steps, we obtain two strictly increasing sequences (m_k) and (r_k) of positive integers satisfying:

(A) $\sum_{u,v\leq r_{k-1}-1} |a_{m_k,m_k,u,v}| < \frac{\epsilon_k}{M}$,

(B) $\sup_{u,v} |a_{m_k,m_k,u,v}| < \min\{\frac{\epsilon_k}{M}, \frac{q_{k1}}{10^k n(k)M}, \ldots, \frac{q_{kn(k)}}{10^k n(k)M}\}$

(C) $|\sum_{(u,v)\in L_{r_{k-1},r_k-1}} a_{m_k,m_k,u,v} - 1| < \min\{\frac{\epsilon_k}{M}, \frac{q_{kn(k)}}{10^k n(k)M}\}$

(D) $\sum_{(u,v):(u\geq r_k)\vee(v\geq r_k)} |a_{m_k,m_k,u,v}| < \frac{\epsilon_k}{M}$,

where $t_k = \sum_{i=1}^{n(k)} q_{ki}l_{ki}$, $q_{ki} > 0$ for all i, $\sum_{i=1}^{n(k)} q_{ki} = 1$, and $l_{k1}, l_{k2}, \ldots l_{kn(k)}$ in $L(w)$.

We now construct the subsequence z of w eluded to in the statement of the theorem. To achieve this consider the sequence of sets: $L_1, L_{r_1,r_2-1}, L_{r_2,r_3-1}, L_{r_3,r_4-1}, \ldots$ where $L_1 = \{(u,v) : u,v \leq r_1 - 1\}$. Notice that these sets are pairwise disjoint and their union is $\mathbb{N}x\mathbb{N}$. We now proceed to partition these sets into $n(1), n(2), n(3), \ldots$ sets respectively using the linear ordering in Definition 2.3, namely we consider $z_1 < z_2 < z_3 \ldots$.

Start with L_1. We partition it into $n(1)$ pairwise disjoint subsets having union L_1. Denote by $<$ the linear ordering of $\mathbb{N}x\mathbb{N}$ from Definition 2.3. Let x_{11} be the first positive integer satisfying:

a) $\sum[a_{m_1,m_1,u,v} : (u,v)$ is at most the x_{11} -st element in $L_1] \leq q_{11}$
$\sum[a_{m_1,m_1,u,v} : (u,v)$ is at most the $x_{11}+1$ -st element in $L_1] \geq q_{11}$

Let x_{12}, $x_{12} > x_{11}$ be the first positive integer satisfying:

b) $\sum[a_{m_1,m_1,u,v} : (u,v) \in L_1, x_{11} < (u,v) \leq x_{12}] \leq q_{12}$
$\sum[a_{m_1,m_1 u,v} : (u,v) \in L_1, x_{11} < (u,v) \leq x_{12}+1] \geq q_{12}$.

Continue, in order, defining the strictly increasing sequence $x_{11}, x_{12}, \ldots,$ $x_{1n(1)}$ satisfying

c) $\sum[a_{m_1,m_1,u,v} : (u,v) \in L_1, x_{1i} < (u,v) \leq x_{1,i+1}] \leq q_{1,i+1}$
$\sum[a_{m_1,m_1,u,v} : (u,v) \in L_1, x_{1i} < (u,v) \leq x_{1,i+1}+1] \geq q_{1,i+1}$.

for $i = 1, 2, \ldots, n(1) - 2$ and

d) $\sum[a_{m_1,m_1,u,v} : (u,v) \in L_1, x_{1,n(1)-1} < (u,v) \leq \omega_1] \in$ $(0.8q_{1n(1)}, 1.2q_{1n(1)})$

where ω_1 is the last element in L_1 with the ordering again being the one from Definition 2.3. For each k, $k = 2, 3, \ldots,$ in a similar way, we partition L_{r_{k-1},r_k-1}. Namely, there exist $x_{k1}, x_{k2}, \ldots, x_{kn(k)}$ in L_{r_{k-1},r_k-1} that are strictly increasing, such that

e) $\sum[a_{m_k,m_k,u,v} : (u,v) \in L_{r_{k-1},r_k-1}, (u,v) \leq x_{k1}] \leq q_{k1}$
$\sum[a_{m_k,m_k,u,v} : (u,v) \in L_{r_{k-1},r_k-1}, (u,v) \leq x_{k1}+1] \geq q_{k1}$.

f) $\sum[a_{m_k,m_k,u,v} : (u,v) \in L_{r_{k-1},r_k-1}, x_{ki} < (u,v) \leq x_{k,i+1}] \leq q_{k,i+1}$
$\sum[a_{m_k,m_k,u,v} : (u,v) \in L_{r_{k-1},r_k-1}, x_{ki} < (u,v) \leq x_{k,i+1}+1] \geq q_{k,i+1}$.

for $i = 1, 2, \ldots, n(k) - 2$ and

g) $\sum[a_{m_k,m_k,u,v} : (u,v) \in L_{r_{k-1},r_k-1}, x_{1,n(k)-1} < (u,v) \leq \omega_k] \in ((1-\frac{2}{10^k})q_{kn(k)}, (1+\frac{2}{10^k})q_{kn(k)})$

where ω_k is the last element in L_{r_{k-1},r_k-1}. Now a) through g) are easy consequences of (β), (γ), (B) and (C). And now, finally, after having achieved the partitions of L_1, L_{r_1,r_2-1}, L_{r_2,r_3-1}, …, we construct the required subsequence z of w. We define in pieces, z_{uv} for (u,v) in L_1, L_{r_1,r_2-1}, L_{r_2,r_3-1}, … respectively. Namely we can construct a subsequence z of w (see the definition of a subsequence) such that:

$$|z_{uv} - l_{11}| < \frac{\epsilon_1}{C_1 2^j}$$

where (u,v) is the j-th element in L_1 such that $(u,v) \leq x_{11}$,

$$|z_{uv} - l_{12}| < \frac{\epsilon_1}{C_1 2^j}$$

where (u,v) is the j-th element in L_1 such that $x_{11} < (u,v) \leq x_{12}, \ldots$

$$|z_{uv} - l_{1,n(1)-1}| < \frac{\epsilon_1}{C_1 2^j}$$

where (u,v) is the j-th element in L_1 such that $x_{1,n(1)-2} < (u,v) \leq x_{1,n(1)-1}$,

$$|z_{uv} - l_{1,n(1)}| < \frac{\epsilon_1}{C_1 2^j}$$

where (u,v) is the j-th element in L_1 such that $x_{1,n(1)-1} < (u,v) \leq \omega_1$ where $C_1 = \sum_{u,v} |a_{m_1,m_1,u,v}| < \infty$ (by (5) in Theorem 2.6).

Now also, for each k, $k = 2, 3, \ldots$ z_{uv} is defined on L_{r_{k-1},r_k-1} so that:

$$|z_{uv} - l_{k1}| < \frac{\epsilon_k}{C_k 2^j}$$

where (u,v) is the j-th element in L_{r_{k-1},r_k-1} such that $(u,v) \leq x_{k1}, \ldots$

$$|z_{uv} - l_{ki}| < \frac{\epsilon_k}{C_k 2^j}$$

where (u,v) is the j-th element in L_{r_{k-1},r_k-1} such that $x_{k,i-1} < (u,v) \leq x_{k,1}$, $(i = 2, 3, 4 \ldots, n(i) - 1)$

…

$$|z_{uv} - l_{k,n(i)}| < \frac{\epsilon_k}{C_k 2^j}$$

where (u,v) is the j-th element in L_{r_{k-1},r_k-1} such that $x_{k,n(i)-1} < (u,v) \leq \omega_k$,

where $C_k = \sum_{u,v} |a_{m_k m_k uv}| < \infty$ (as before). Let us simplify notation by setting $L_k = L_{r_{k-1},r_k-1}$.

By the definition of a subsequence and the fact that each l_{ki} is a limit point of w, a subsequence satisfying all of the above inequalities exists.

Now we show that such a subsequence z satisfies the required conditions, namely each $t \in C(w)$ is a limit point of Az. To see this, let k be a positive integer, then we have

$$\begin{aligned}(xx)_k &= |\sum_{(u,v)} a_{m_k,m_k,u,v}z_{uv} - t_k| \leq |\sum_{(u,v)\in L_k} a_{m_k,m_k,u,v}z_{uv} - t_k| \\ &+ M \sum_{(u,v)<r_{k-1}} |a_{m_k,m_k,u,v}| \\ &+ |\sum_{\{(u,v):u>r_k-1 \bigvee v>r_k-1\}} a_{m_k,m_k,u,v}z_{uv}|.\end{aligned}$$

Note that the second term on the right hand side of $(xx)_k$ is taken to be zero in case when $k = 1$. Let us examine the first term on the right hand side of $(xx)_k$, denote it $(xxx)_k$. Then we have

$$\begin{aligned}(xxx)_k &= |(l_{k1} \sum_{(u,v)\in B_{k1}} a_{m_k,m_k,u,v} + l_{k2} \sum_{(u,v)\in B_{k2}} a_{m_k,m_k,u,v} + \dots \\ &+ l_{kn(k)} \sum_{(u,v)\in B_{kn(k)}} a_{m_k,m_k,u,v} - t_k) + (\sum_{(u,v)\in B_{k1}} a_{m_k,m_k,u,v}\gamma_{uv} \\ &+ \sum_{(u,v)\in B_{k2}} a_{m_k,m_k,u,v}\gamma_{uv} + \dots + \sum_{(u,v)\in B_{kn(k)}} a_{m_k,m_k,u,v}\gamma_{uv})|\end{aligned}$$

where here $z_{uv} = l_{ki} + \gamma_{uv}$ if $(u,v) \in B_{ki}, i = 1,2,\dots,n(k)$ and $B_{ki}, i = 1,2,\dots,n(k)$ are the $n(k)$ sets into which L_k has been partitioned. The first set of parentheses, inside the absolute value signs, on the right side of $(xxx)_k$ has absolute value less than $\frac{1}{10^k}$. The second set of parentheses, inside the absolute value signs, on the right side of $(xxx)_k$ has absolute value less than ϵ_k.

Finally, the second and third terms on the right hand side of $(xx)_k$ satisfy $M\sum_{(u,v)<r_{k-1}} |a_{m_k,m_k,u,v}| < \epsilon_k$ and $|\sum_{\{(u,v):u>r_k-1 \bigvee v>r_k-1\}} a_{m_k,m_k,u,v}z_{uv}| < \epsilon_k$, so that $(xx)_k < \frac{1}{10^k} + 3\epsilon_k$ for each k, which shows that each t_k, and consequently each s_k is a limit point of Az. Since $\{s_n : n \in N\}$ is dense in $C(w)$, each $t \in C(w)$ is a limit point of Az.

We finish, by stating a theorem that is the exact analogue of Theorem 3.2 in [9], and whose proof follows that of the mentioned theorem .

Theorem 3.2. *If $A = (a_{m,n,u,v})$ is a four dimensional bounded regular summability matrix satisfying (S) and w is a bounded double sequence, then there exists a rearrangement z of w such that each t in the Pringsheim core of w is a P-limit point of Az.*

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Faculty of Engineering and Natural Sciences
International University of Sarajevo
Sarajevo, 71000
Bosnia-Herzegovina
E–mail: himiller@hotmail.com, , lejla.miller@yahoo.com