Existence and multiplicity of solutions to fourth order elliptic equations with critical exponent on compact manifolds

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Abstract

This paper deals with some perturbation of the so called prescribed scalar Q-curvature type equations on compact Riemannian manifolds; these equations are fourth order elliptic and of critical Sobolev growth. Sufficient conditions are given to have at least two distinct solutions first without using the concentration-compactness technic but with a suitable range of the parameters and secondly by using the concentration-compactness methods.

1 Introduction

Let (M, g) be a Riemannian compact smooth $n-$manifold, $n \geq 5$, with metric g, we let $H_2^2(M)$ be the standard Sobolev space which is the completion of the space

$$C_2^2(M) = \left\{ u \in C^{\infty}(M) : \|u\|_{2,2} < +\infty \right\}$$

with respect to the norm $\|u\|_{2,2} = \sum_{l=0}^{2} \left\| \nabla^l u \right\|_2$.

We denotes by H_2, the space H_2^2 endowed with the equivalent norm

$$\|u\|_{H_2} = \left(\|\Delta u\|_2^2 + \|\nabla u\|_2^2 + \|u\|_2^2 \right)^{\frac{1}{2}}.$$

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We investigate multiple solutions of the equation

$$\Delta^2 u + \nabla^i(a(x)\nabla_i u) + h(x)u = f(x)\,|u|^{N-2}\,u + \lambda\,|u|^{q-2}\,u + \epsilon g(x) \tag{1.1}$$

where a, h, f and g are smooth functions on M , $N = \frac{2n}{n-4}$ is the critical exponent, $2 < q < N$ a real number, $\lambda > 0$ a real parameter and $\epsilon > 0$ any small real number. Since the embedding $H_2 \hookrightarrow H_N^k$, ($k = 0, 1$) fails to be compact, as known, one encounters serious difficulties in solving equations like (1.1).

In 1983, Paneitz [8] introduced a conformal fourth order operator defined on 4-dimensional Riemannian manifolds which was generalized by Branson [3] to higher dimensions.

$$PB_g(u) = \Delta^2 u + div(-\frac{(n-2)^2+4}{2(n-1)(n-2)}R.g + \frac{4}{n-2}Ric)du + \frac{n-4}{2}Q^n u$$

where $\Delta u = -div\,(\nabla u)$, R is the scalar curvature, Ric is the Ricci curvature of g and where

$$Q^n = \frac{1}{2(n-1)}\Delta R + \frac{n^3-4n^2+16n-16}{8(n-1)^2(n-2)^2}R^2 - \frac{2}{(n-2)^2}\,|Ric|^2$$

is associated to the notion of Q -curvature.

We refer to a Paneitz-Branson type operator as an operator of the form

$$P_g u = \Delta^2 u + \nabla^i(a(x)\nabla_i u) + h(x)u.$$

Equation (1.1) is a perturbation of the equation

$$\Delta^2 u + \nabla^i(a(x)\nabla_i u) + h(x)u = f(x)\,|u|^{N-2}\,u. \tag{1.2}$$

Since 1990 many results have been established for the equation (1.2) and for precise functions a, h and f. D.E. Edmunds, D. Fortunato, E. Jannelli [7] proved for $n \geq 8$ that if $\lambda \in (0, \lambda_1)$, with λ_1 the first eigenvalue of Δ^2 on the euclidean open ball B, the problem

$$\begin{cases} \Delta^2 u - \lambda u = u\,|u|^{\frac{8}{n-4}} \text{ in } B \\ u = \frac{\partial u}{\partial n} = 0 \text{ on } \partial B \end{cases}$$

has a non trivial solution.

In 1995, R. Vander Vorst [9] obtained the same results as D.E. Edmunds, D. Fortunato, E. Jannelli. when he considered the problem

$$\begin{cases} \Delta^2 u - \lambda u = u\,|u|^{\frac{8}{n-4}} \text{ in } \Omega \\ u = \Delta u = 0 \text{ on } \partial\Omega \end{cases}$$

where Ω is an open bounded set of R^n and moreover he showed that the solution is positive.

In [5] D.Caraffa studied the equation (1.2) in the case $f(x)$ =constant; and in the particular case where the functions $a(x)$ and $h(x)$ are precise constants she obtained the existence of positive regular solutions. In [6], P. Esposito and

F. Robert studied the existence of solutions to fourth order equations involving Paneitz-Branson type operators and critical Sobolev exponent.

In this paper we show that, under conditions on the operator $Lu = \Delta^2 u + \nabla^i(a(x)\nabla_i u) + h(x)u$ and on the function f, the existence of at least two solutions of equation (1.1) first without using the concentration compactness methods but with a suitable range of the parameter λ and secondly by mean of the concentration compactness technique we prove the existence of at least two solutions. Merely speaking, we prove the following results

Theorem 1. *Let (M, g) be a compact Riemannian $n-$manifold, $n \geq 5$, a, h, f, g be smooth real functions on M with*

(i) $f(x) > 0$ everywhere on M

(ii) the operator $Lu = \Delta^2 u + \nabla^i(a(x)\nabla_i u) + h(x)u$ is coercive.

Then there exist $\lambda_o > 0$ and $\varepsilon_o > 0$ such that the equation 1.1 admits at least two distinct solutions in $H_2(M)$ for any $\lambda \geq \lambda_o$ and $0 < \varepsilon \leq \varepsilon_o$.

Remark 1. *The above result was already obtained by the author in [2], but with an incomplete proof, so I deliberately reconsidered this theorem with a complete proof.*

Theorem 2. *Let (M, g) be a compact Riemannian $n-$manifold, $n \geq 6$, a, h, f, g be smooth real functions on M with*

(i) $f(x) > 0$ and $g(x) > 0$ everywhere on M

(ii) the operator $Lu = \Delta^2 u + \nabla^i(a(x)\nabla_i u) + h(x)u$ is coercive

(iii) if $n > 6$, we suppose $\frac{\Delta f(x_o)}{2f(x_o)} + C_1(n)R(x_o) - C_2(n)a(x_o) > 0$ and if $n = 6$, we suppose that $\frac{4}{3n}R(x_o) - \frac{1}{(n-4)}a(x_o) > 0$

Then equation (1.1) has at least two distinct solutions in H_2.

2 Palais-Smale conditions

We quote after D. Caraffa the following Sobolev's inequality

Lemma 1. *[5]Let (M, g) be a compact $n-$Riemannian manifold ($n \geq 4$) and q a real $1 \leq q < \frac{n}{2}$. The best constant K_2 in the Sobolev inequality corresponding to the embedding $H_2^q \subset L_p$ with $\frac{1}{p} = \frac{1}{q} - \frac{2}{n}$ depends only on n and q and for any $\epsilon > 0$ there is a constant $A(\epsilon)$ such that for any $\varphi \in H_2^q$*

$$\|\varphi\|_p \leq K_2(1+\epsilon)\|\varphi\|_{H_2^q} + A(\epsilon)\|\varphi\|_q$$

Consider the functional $I_{\epsilon,\lambda}$ defined on H_2 by

$$I_{\epsilon,\lambda}(u) = \|\Delta u\|_2^2 - \int_M a(x)|\nabla u|^2 dv_g + \int_M h(x)u^2 dv_g - \frac{2}{N}\int_M f(x)|u|^N dv_g$$
$$-\frac{2}{q}\lambda\int_M |u|^q dv_g - 2\epsilon\int_M g(x)u dv_g. \tag{2.1}$$

Lemma 2. *The the functional $I_{\epsilon,\lambda}(u)$ is of class C^1 on H_2.*

Proof. It suffices to show that the functional $F(u) = \int_M f(x) |u|^N dv_g$ is of class C^1 on M. Let $u, v \in H_2$, we have

$$\left| F(u+v) - F(u) - N \int_M |u|^{N-2} u.vdv_g \right|$$

$$= \left| \int_M f(x) \left(|u+v|^N - |u|^N - Nf(x) |u|^{N-2} u.v \right) dv_g \right|$$

and using the Taylor expansion

$$|u+v|^N = |u|^N + N \int_0^1 |u+tv|^{N-2} (u+tv)dt$$

we obtain

$$|u+v|^N - |u|^N - N |u|^{N-2} u.v =$$

$$= N \left[\int_0^1 \left(|u+tv|^{N-2} (u+tv)v - |u|^{N-2} u \right) vdt \right].$$

Since $N > 2$,(with $t \in [0,1]$) we write

$$\left(|u+tv|^{N-2} (u+tv)v - |u|^{N-2} u \right) v = \left(|u+tv|^{N-2} - |u|^{N-2} \right) uv$$

$$+ |u+tv|^{N-2} tv^2$$

so if $2 < N \leq 3$, we get

$$\left| \left(|u+tv|^{N-2} (u+tv)v - |u|^{N-2} u \right) v \right| \leq |v|^{N-1} |u| + |u+v|^{N-2} v^2$$

and by Hölder inequality, we obtain

$$\left| F(u+v) - F(u) - N \int_M f(x) |u|^{N-2} u.vdv_g \right| \leq$$

$$N \max_{x \in M} f(x) \int_M \left(|v|^{N-1} |u| + |u+v|^{N-2} v^2 \right) dv_g \leq$$

$$N \max_{x \in M} f(x) \left(\|u\|_N + \|u+v\|_N^{N-2} \|v\|_N^{3-N} \right) \|v\|_N^{N-1}.$$

The case $N > 3$, we have

$$\left| \left(|u+tv|^{N-2} (u+tv)v - |u|^{N-2} u \right) v \right| \leq \left(|u+v|^{N-2} - |u|^{N-2} \right) |u| |v|$$

$$+ (|u| + |v|)^{N-2} v^2$$

and using the following formula, which can be derived from the the Taylor expansion, for any $x > 1$ and any real $p > 1$

$$(1+x)^p < x^p + px^{p-1} + \frac{1}{2} p(p-1)x^{p-2} + ...$$

$$+\frac{1}{E(p)}p(p-1)...(p-E(p)+1)x^{p-E(p)}$$

where $E(p)$ is the entire part of the integer p, we obtain

$$\left(|u+v|^{N-2}-|u|^{N-2}\right)|u|\,|v| \le \Big[(N-2)\,|u|^{N-1}+...+$$

$$\frac{1}{E(N-2)}(N-2)....(N-1-E(N-2))\,|u|^{N-1-E(N-2)}\,|v|^{E(N-2)-1}\Big]\,|v|^2$$

and using again the Hölder inequality, we get

$$\left|F(u+v)-F(u)-N\int_M f(x)\,|u|^{N-2}\,u.vdv_g\right|$$

$$\le N\sup_{x\in M}f(x)\,\Big[(N-2)\,\|u\|_N^{N-1}+...+$$

$$\frac{1}{E(N-2)}(N-2)....(N-1-E(N-2))\,\|u\|_N^{N-1-E(N-2)}\Big]\,\|v\|_N^2$$

and finally by the Sobolev inequality given in Lemma 1, we deduce that in the two cases we have

$$\left|F(u+v)-F(u)-N\int_M f(x)\,|u|^{N-2}\,u.vdv_g\right| = o(\|v\|_{H_2})$$

which shows that the functional $F(u)$ is differentiable with derivative at the point u given by $F'(u)v = N\int_M f(x)\,|u|^{N-2}\,uvdv_g$. ■

3 Existence of solution with negative energy

In this section, we aim to prove the existence of a positive solution to equation (1.1) with negative energy. To do so, we establish the following results.

Lemma 3. *There exits $\rho > 0$, such that for any $\lambda > 0$ and $\epsilon > 0$ the functional $I_{\epsilon,\lambda}$ is weakly lower semi-continuous on the closed ball $\left\{u \in H_1^p(M) : \|u\|_{H_2} \le \rho\right\}$.*

Proof. Let $(u_k)_k$ be a sequence in $H_2(M)$ such that $u_k \to u$ weakly in $H_2(M)$ and $\|u_k\|_{H_2} \le \rho$. Up to a subsequence, we obtain

$$\nabla u_k \to \nabla u \text{ weakly in } H_2(M)$$

$$u_k \to u \text{ strongly in } L^r(M) \text{ with } r < p^*$$

$$u_k \to u \text{ strongly in } H_2^1(M)$$

and

$$u_k \to u \text{ a.e. in } M.$$

We have to show that

$$\liminf_k I_{\epsilon,\lambda}(u_k) \ge I_{\epsilon,\lambda}(u).$$

By the Brezis-Lieb Lemma [4], we have

$$\|\Delta u_k\|_2^2 - \|\Delta u\|_2^2 = \|\Delta (u_k - u)\|_2^2 + o(1)$$

and

$$\int_M f(x)\left(|u_k|^N - |u|^N\right) dv(g) = \int_M f(x)\, |u_k - u|^N\, dv(g) + o(1).$$

On the other hand the Sobolev inequality given by Lemma 1.1 allows us to write

$$\int_M f(x)\, |u_k - u|^N\, dv(g) \le \sup_{x\in M} f(x) \left[\max\left(K_2^2 + \epsilon_1, A(\epsilon_1)\right)\right]^{\frac{N}{2}} \|u_k - u\|_{H_2}^N$$

where ϵ_1 is any positive number, K_2 and $A(\epsilon_1)$ are the constants appearing in the Sobolev embedding. So

$$I_{\epsilon,\lambda}(u_k) - I_{\epsilon,\lambda}(u) \ge \|u_k - u\|_{H_2}^2$$

$$\times\left(1 - \sup_{x\in M} f(x)\left[\max\left(K_2^2 + \epsilon_1, A(\epsilon_1)\right)\right]^{\frac{N}{2}} 2^{N-2}\max(\|u_k\|_{H_2}^{N-2}, \|u\|_{H_2}^{N-2})\right) + o(1).$$

We choose the radius of the ball $\left\{u \in H_2(M) : \|u\|_{H_2} \le \rho\right\}$ small enough so that it satisfies our claim. ■

Lemma 4. *For each fixed $\lambda > 0$, there exist $\epsilon_o > 0$ sufficiently small, $\rho > 0$ and $\eta > 0$ such that for any $u \in H_2$ with $\|u\|_{H_2} = \rho$ it holds $I_{\epsilon,\lambda}(u) > \eta$ for any $0 < \epsilon < \epsilon_o$.*

Proof. Consider the functional $I_{\epsilon,\lambda}(u)$ defined by (2.1). By the coerciveness of the operator $L(u) = \Delta^2 u + \nabla^i (a(x)\nabla_i u) + h(x)u$ and the Sobolev inequality given by Lemma 1, we get

$$I_{\epsilon,\lambda}(u) \ge \Lambda \|u\|_{H_2}^2 - \frac{2}{N}\max_{x\in M} f(x)\, \|u\|_N^N - \frac{2}{q}\lambda vol(M)^{1-\frac{q}{N}} \|u\|_N^q$$

$$-2\epsilon \max_{x\in M} |g(x)|\, vol(M)^{1-\frac{1}{N}} \|u\|_N$$

$$\ge \Bigg[\Bigg(\Lambda - \frac{2}{N}\max_{x\in M} f(x)\max\left((1+\epsilon_1)K_2, A(\epsilon_1)\right)^N \|u\|_{H_2}^{N-2}$$

$$-\lambda\frac{2}{q}\max\left((1+\epsilon_1)K_2, A(\epsilon_1)\right)^q \|u\|_{H_2}^{q-2}\Bigg) \|u\|_{H_2}$$

$$-2\epsilon \max_{x\in M} |g(x)|\, vol(M)^{1-\frac{1}{N}} \max\left((1+\epsilon_1)K_2, A(\epsilon_1)\right)\Bigg] \|u\|_{H_2}$$

where Λ is the constant of the coercivity and $\epsilon_1 > 0$ is the one appearing in the Sobolev inequality.

Then there are $\rho > 0$, $\epsilon_o > 0$ and $\eta > 0$ such that for any $u \in H_2$ with $\|u\|_{H_2} = \rho$ and any $0 < \epsilon < \epsilon_o$, $I_{\epsilon,\lambda}(u) > \eta$. ■

Now we are able to prove the existence of solution to equation (1.1) with negative energy

Theorem 3. *Let (M, g) be a compact Riemannian $n-$manifold, $n \geq 5$, a, h, f, g be smooth real functions on M with*

(i) $f(x) > 0$ everywhere on M

(ii) the operator $Lu = \Delta^2 u + \nabla^i(a(x)\nabla_i u) + h(x)u$ is coercive.

Then there exists $\varepsilon_o > 0$ small enough such that for any $0 < \varepsilon \leq \varepsilon_o$ the equation (1.1) admits a weak solution with negative energy.

Proof. Let $v \in H_2(M)$ such that $\int_M g(x)vdv_g > 0$. For any $t > 0$,

$$I_{\epsilon,\lambda}(tv) = t^2 \left(\|\Delta v\|_2^2 - \int_M a(x) |\nabla v|^2 dv_g + \int_M h(x)v^2 dv_g \right) - \frac{2}{N} t^N \int_M f(x) |v|^N dv_g$$
$$-\lambda \frac{2}{q} t^q \int_M |v|^q dv_g - 2\epsilon t \int_M g(x)vdv_g$$

so we deduce that there is a $t_1(\lambda, \epsilon) > 0$ such that for any $t \in]0, t_1(\lambda, \epsilon)[$, $I_{\epsilon,\lambda}(tv) < 0$ and for $\rho > 0$

$$\inf_{\|u\|_{H_2} \leq \rho} I_{\epsilon,\lambda}(u) < 0.$$

Now, by Lemma 3 there exist $\rho > 0$ and $w \in H_2(M)$ with $\|w\|_{H_2} \leq \rho$ such that

$$I_{\epsilon,\lambda}(w) = \inf_{\|u\|_{H_2} \leq \rho} I(u) < 0.$$

On the other hand for sufficiently small $\epsilon > 0$ and sufficiently small $\rho > 0$, w is such that $\|w\|_{H_2} < \rho$, otherwise by Lemma 4 $I_{\epsilon,\lambda}(w) \geq 0$. Hence w is a weak solution of equation (1.1) with negative energy. ∎

4 Palais-Smale condition

Lemma 5. *Suppose $n \geq 5$, a, h, f, g be smooth real functions on M with*

(i) $f(x) > 0$ everywhere on M.

(ii) the operator $Lu = \Delta^2 u + \nabla^i(a(x)\nabla_i u) + h(x)u$ is coercive.

Then there exists $\varepsilon_o > 0$ sufficiently small such for any $0 < \varepsilon \leq \varepsilon_o$, each $(PS)_c$-sequence is bounded in H_2.

Proof. Take $(u_n) \subset H_2$ such that $I_{\epsilon,\lambda}(u_k) \to c$ and $I'_{\epsilon,\lambda}(u_k) \to 0$ strongly in $H'_2(M)$ the dual space of $H_2(M)$; then

$$I_{\epsilon,\lambda}(u_k) - \frac{1}{q} I'_{\epsilon,\lambda}(u_k)(u_k) \geq \left(1 - \frac{2}{q}\right) \left(\|\Delta u_k\|_2^2 - \int_M a(x) |\nabla u_k(x)|^2 dv_g \right.$$
$$\left. + \int_M h(x) u_k(x)^2 dv_g \right) + 2\epsilon(-1 + \frac{1}{q}) \max_{x \in M} |g(x)| \, vol(M)^{1-\frac{1}{N}} \|u_k\|_N$$

and from the coerciveness of the operator L, and the Sobolev inequality formulated in Lemma 1 one gets for any $\eta > 0$, there is an integer $k_o > 0$ such that for any $k \geq k_o$

$$c + \eta \geq \left[(1 - \frac{2}{q}) \Lambda \|u_k\|_{H_2} + 2\epsilon(-1 + \frac{1}{q}) \max_{x \in M} |g(x)| \, vol(M)^{1-\frac{1}{N}} \right.$$

$$\times \max(K_2(1+\epsilon_1), A(\epsilon_1))] \|u_k\|_{H_2}$$

where Λ denotes the coefficient of the coerciveness, and letting ϵ sufficiently small, the boundedness of the $(PS)_c$- sequence follows. ∎

Now, we are going to show that the Palais-Smale condition is satisfied.

Lemma 6. *Let (u_k) be a $(PS)_{c_{\epsilon,\lambda}}$- sequence. Suppose that the conditions of Lemma 2 are satisfied and*

$$c_{\epsilon,\lambda} < \frac{4}{n} \max_{x\in M} f(x)^{1-\frac{n}{4}} K_2^{-\frac{n}{2}}$$

then there exists a strongly convergent subsequence of (u_k).

Proof. Let (u_k) be a $(\mathrm{PS})_c$- sequence , then by Lemma 5 (u_k) is bounded in H_2. From the reflexivity of H_2 and the compactness of the embedding $H_2 \subset H_q^k$, ($k = 0, 1$; $q < N$) we have a subsequence of (u_k) still denoted (u_k) such that
$u_k \to u$ weakly in H_2
$u_k \to u$ and $\nabla u_k \to \nabla u$ strongly in $L_q(M), q < N$.

Now by standard variational method we obtain that u is a weak solution of the equation (1.1) that is to say: for any $v \in H_2$, we have

$$\int_M \Delta u \Delta v dv_g - \int_M a(x) \langle \nabla u, \nabla v \rangle dv_g + \int_M h(x) uv dv_g =$$

$$= \int_M f |u|^{N-2} uv dv_g + \lambda \int_M |u|^{q-2} uv dv_g + \epsilon \int_M g(x) v dv_g$$

where $\langle .,. \rangle = g(.,.)$ denotes the Riemannian metric. Letting $v = u$, we get the expression of $I_{\epsilon,\lambda}(u)$

$$I_{\epsilon,\lambda}(u) = \left(1 - \frac{2}{N}\right) \int_M f |u|^N dv_g + \left(1 - \frac{2}{q}\right) \lambda \int_M |u|^q dv_g - \epsilon \int_M g(x) u dv_g$$

$$\geq \left(1 - \frac{2}{N}\right) \int_M f |u|^N dv_g + \left[\left(1 - \frac{2}{q}\right) \lambda \|u\|_q^{q-1} - \epsilon \max_{x\in M} |g(x)| \, vol(M)\right] \|u\|_q .$$

Letting $w_k = u_k - u$, thanks to the Brezis-Lieb lemma [4], we have

$$\|\nabla w_k\|_2^2 = \|\nabla u_k\|_2^2 - \|\nabla u\|_2^2 + o(1)$$

and

$$\|\Delta w_k\|_2^2 = \|\Delta u_k\|_2^2 - \|\Delta u\|_2^2 + o(1). \tag{4.1}$$

By standard integration theory we can write

$$\int_M f |u_k - u|^N dv_g = \int_M f |u_k|^N dv_g - \int_M f |u|^N dv_g + o(1) \tag{4.2}$$

Since $\int_M a(x) |\nabla u_k|^2 dv_g \to \int_M a(x) |\nabla u|^2 dv_g$, $\int_M h(x) u_k^2 dv_g \to \int_M h(x) u^2 dv_g$, and $\int_M g(x) u_k dv_g \to \int_M g(x) u dv_g$, and taking into account of (4.1) and (4.2), we obtain

$$I_{\epsilon,\lambda}(u_k) - I_{\epsilon,\lambda}(u) = \int_M (\Delta u_k)^2 dv_g - \int_M (\Delta u)^2 dv_g$$

$$-\frac{2}{N}\int_M f(x)(|u_k|^N - |u|^N)dv_g + o(1)$$

$$= \int_M (\Delta(u_k - u))^2\, dv_g - \frac{2}{N}\int_M f(x)\,|u_k - u|^N\, dv_g + o(1). \tag{4.3}$$

Now, testing the function $DI(u_k)$ in the weak convergence $u_k \to u$ in H_2, we get

$$o(1) = DI(u_k)(u_k - u)$$

$$\int_M (\Delta(u_k - u))^2\, dv_g - \int_M f(x)\,|u_k - u|^N\, dv_g + o(1) \tag{4.4}$$

and combining (4.3) and (4.4), we obtain

$$\int_M (\Delta(u_k - u))^2\, dv_g = \int_M f(x)\,|u_k - u|^N\, dv_g + o(1)$$

$$= \int_M f(x)\,|u_k|^{N-2}\,(u_k - u)^2\, dv_g + o(1). \tag{4.5}$$

Hence

$$I_{\epsilon,\lambda}(u_k) - I_{\epsilon,\lambda}(u) = \left(1 - \frac{2}{N}\right)\int_M (\Delta(u_k - u))^2\, dv_g + o(1). \tag{4.6}$$

On the other hand using (4.1), and (4.2) we write

$$I(u_k) - I(u) = \int_M (\Delta(u_k - u))^2 - \frac{2}{N}\int_M f(x)\,|u_k|^{N-2}\,(u_k - u)^2 + o(1)$$

and from the Hölder's inequality, one gets

$$I(u_k) - I(u) \geq \|\Delta(u_k - u)\|_2^2 - \frac{2}{N}\max_{x\in M} f(x)\,\|u_k\|_N^{N-2}\,\|u_k - u\|_N^2 + o(1)$$

and by the Sobolev inequality given by Lemma 1 one writes

$$I(u_k) - I(u) \geq \|\Delta(u_k - u)\|_2^2 - \frac{2}{N}\max_{x\in M} f(x)\,\|u_k\|_N^{N-2}$$

$$\times\left[(K_2^2 + \epsilon_1)\,\|\Delta(u_k - u)\|_2^2 + A(\epsilon_1)\,\|u_k - u\|_2^2\right] + o(1)$$

so

$$I(u_k) - I(u) \geq \left(1 - \frac{2}{N}\left(K_2^2 + \epsilon_1\right)\max_{x\in M} f(x)\,\|u_k\|_N^{N-2}\right) \tag{4.7}$$

$$\times\,\|\Delta(u_k - u)\|_2^2 + o(1)$$

and taking account of the equality (4.6), we get

$$\left(1 - \frac{2}{N}\right)\|\Delta(u_k - u)\|_2^2\, dv_g \geq$$

$$\left(1 - \frac{2}{N}\left(K_2^2 + \epsilon_1\right)\max_{x\in M} f(x)\,\|u_k\|_N^{N-2}\right)\|\Delta(u_k - u)\|_2^2 + o(1)$$

so

$$\left(1-\left(K_2^2+\epsilon_1\right)\max_{x\in M} f(x)\,\|u_k\|_N^{N-2}\right)\|\Delta(u_k-u)\|_2^2 \le o(1).$$

Consequently if

$$\limsup_k \|u_k\|_N < \left(\left(K_2^2+\epsilon_1\right)\max_{x\in M} f(x)\right)^{-\frac{1}{N-2}} \tag{4.8}$$

we get that

$$\|\Delta(u_k-u)\|_2 = o(1)$$

that is to say the strong convergence of the sequence u_k to u in $H_2(M)$.

Now, from $I_{\epsilon,\lambda}(u_k)\to c_{\epsilon,\lambda}$, we deduce that

$$\int_M (\Delta u_k)^2\,dv_g+\int_M a(x)\,|\nabla u_k|^2\,dv_g-\frac{2}{N}\int_M f(x)\,|u_k|^N\,dv_g = \tag{4.9}$$

$$-\int_M h(x)u_k^2dv_g+\lambda\frac{2}{q}\int_M |u_k|^q\,dv_g+2\epsilon\int_M g(x)u_kdv_g+c_{\epsilon,\lambda}+o(1)$$

and from $I'_{\epsilon,\lambda}(u_k)(u_k)\to 0$, we obtain

$$\int_M (\Delta u_k)^2\,dv_g+\int_M a(x)\,|\nabla u_k|^2\,dv_g-\int_M f(x)\,|u_k|^N\,dv_g = \tag{4.10}$$

$$=-\int_M h(x)u_k^2dv_g+\lambda\int_M |u_k|^q\,dv_g+\epsilon\int_M k(x)\,|u_k|^p\,dv_g+o(1).$$

By combining (4.9) and (4.10), we get

$$\left(1-\frac{2}{N}\right)\int_M f(x)\,|u_k|^N\,dv_g+\lambda(1-\frac{2}{q})\int_M |u_k|^q\,dv_g-\epsilon\int_M g(x)u_kdv_g = c_{\epsilon,\lambda}+o(1).$$

Now since $\lambda>0$, the sequence (u_k) is bounded and $\epsilon>0$ small enough, to have (4.8) satisfied, we must assume

$$c_{\epsilon,\lambda} < \frac{4}{n}\max_{x\in M} f(x)^{1-\frac{n}{4}}\left(K_2^2+\epsilon_1\right)^{-\frac{n}{4}} \tag{4.11}$$

and a fortiori

$$c_{\epsilon,\lambda} < \frac{4}{n}\max_{x\in M} f(x)^{1-\frac{n}{4}}K_2^{-\frac{n}{2}}.$$

∎

5 Generic existence theorem of a second solution

Using the Mountain Pass theorem, we get a second weak solution with positive energy.

Lemma 7. *Suppose that*

(i) $f(x) > 0$ everywhere on M.

(ii) the operator $Lu = \Delta^2 u + \nabla^i(a(x)\nabla_i u) + h(x)u$ is coercive

(iii) $0 < c_{\epsilon,\lambda} < \left(\frac{N}{2}\right)^{\frac{1}{N-2}} \left(K_2^2 \max_{x\in M} f(x)\right)^{-\frac{1}{N-2}}$

1) there exists a positive constants r and ρ such that $I(u) > r > 0$ for any u with $\|u\|_{H_2} = \rho$.

2) there exists $v \in H_2(M)$ with $I(v) < 0$ and $\|v\|_{H_2} > \rho$.

Proof. The condition(i) is obtained similarly as in the proof of Lemma 4. The second condition follows, since $I_{\epsilon,\lambda}(tu)$ goes to $-\infty$ as $t \to +\infty$. Let $v \in H_2$ with $I_{\epsilon,\lambda}(v) < 0$,

$$\Gamma = \{\gamma \in C\left([0,1]\right),\ H_2)\ ;\ \gamma(0) = 0, \gamma(1) = v\}$$

and $c_{\epsilon,\lambda} = \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} I(\gamma(t))$. By the Mountain Pass Theorem there exists a $(\mathrm{PS})_{c_{\epsilon,\lambda}}$-sequence $(u_k) \subset H_2$ and by the condition(iii) the $(\mathrm{PS})_{c_{\epsilon,\lambda}}$ condition holds and therefore $c_{\epsilon,\lambda}$ is a critical level for the functional $I_{\epsilon,\lambda}$. ∎

6 Proof of the main results

First, we prove the following results which is crucial to the proof of the existence of multiple solutions without the use of the concentration-compactness method.

Lemma 8. *There exist $\lambda_o > 0$ and $\epsilon_o > 0$ such that for any $\lambda \geq \lambda_o$ and $0 < \epsilon \leq \epsilon_o$, we have*

$$0 < c_{\epsilon,\lambda} < \frac{4}{n} \max_{x\in M} f(x)^{1-\frac{n}{4}} K_2^{-\frac{n}{2}}. \tag{6.1}$$

Proof. Let $\phi \in H_2(M)$ be such that $\int_M g(x)\phi dv_g > 0$ and $\int_M f(x) |\phi|^N dv_g = 1$, then we have

$$\lim_{t\to+\infty} I_{\lambda,\epsilon}(t\phi) = -\infty$$

so there exists $t_{\epsilon,\lambda} > 0$ such that

$$I_{\lambda,\epsilon}(t_{\epsilon,\lambda}\phi) = \sup_{t\geq 0} I_{\lambda,\epsilon}(t\phi) > 0. \tag{6.2}$$

Hence

$$t_{\epsilon,\lambda}^{q-1}\left\{t_{\lambda,\epsilon}^{2-q}\left(\|\Delta\phi\|_2^2 - \int_M a(x)|\nabla\phi|^2 dv_g\right) - \frac{2}{N} t_{\epsilon,\lambda}^{N-q} - \frac{2}{q}\lambda\|\phi\|_q^q\right\} = 2\epsilon\int_M g(x)\phi dv_g. \tag{6.3}$$

Noting that

$$\lim_{\lambda\to+\infty}\left(\frac{2}{N}t_{\epsilon,\lambda}^{N-q} + \frac{2}{q}\lambda\|\phi\|_q^q\right) = +\infty$$

it follows by (6.3) that

$$\lim_{\lambda\to\infty} t_{\lambda,\epsilon} = 0$$

and taking into account of (6.2), we obtain

$$\lim_{\lambda\to+\infty} \sup_{t\geq 0} I_{\epsilon,\lambda}(t_{\lambda,\epsilon}\phi) = 0.$$

So there exists λ_o with

$$0 < \sup_{t\geq 0} I_{\epsilon,\lambda}(t\phi) < \frac{4}{n} \max_{x\in M} f(x)^{1-\frac{n}{4}} K_2^{-\frac{n}{2}} \tag{6.4}$$

for any $\lambda \geq \lambda_o$.

Let $\psi = t\phi$ with t large enough so that $I_{\epsilon,\lambda}(\psi) < 0$ and let

$$\Gamma = \{\gamma \in C([0,1], H_2(M) : \gamma(0) = 0, \gamma(1) = \psi)\}$$

and

$$c_{\epsilon,\lambda} = \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} I_{\epsilon,\lambda}(\gamma(t)).$$

Taking into account of Lemma 6, there exist $\epsilon_o > 0$ and a sequence (u_k) in $H_2(M)$ such that for any $0 < \epsilon \leq \epsilon_o$

$I_{\epsilon,\lambda}(u_k) \to c_{\epsilon,\lambda}$ and $I'_{\epsilon,\lambda}(u_k) \to 0$ with

$$0 < c_{\epsilon,\lambda} = \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} I_{\epsilon,\lambda}(\gamma(t)) \leq \sup_{t\geq 0} I_{\epsilon,\lambda}(t\phi).$$

$I_{\epsilon,\lambda}$ satisfies the $(PS)_{c_{\epsilon,\lambda}}$ condition.

Hence by (6.4), we obtain

$$0 < c_{\epsilon,\lambda} < \frac{4}{n} \max_{x\in M} f(x)^{1-\frac{n}{4}} K_2^{-\frac{n}{2}}$$

for any $0 < \epsilon \leq \epsilon_o$ and any $\lambda \geq \lambda_o$. ■

7 Proof of the main theorems

Proof. (of Theorem 1) Theorem 1 is a corollary of Lemma 8 . ■

Proof. (of Theorem 2)

Let $x_o \in M$ where the function f is maximum, $\delta > 0$ sufficiently small so that $\delta < \frac{1}{2} i_g(M)$, where $i_g(M)$ denotes the injectivity radius of M and $\eta \in C^\infty(M)$ a cutting function

$$\eta(r) = \begin{cases} 1 & \text{if } x \in B_{x_o}(\delta) \\ 0 & \text{if } x \notin B_{x_o}(2\delta) \end{cases}$$

and consider the function

$$\varphi_k(r) = \left(\frac{n(n-2)(n+2)(n-4)}{2} f(x_o)^{-1} k^4 \right)^{\frac{n-4}{8}} \frac{\eta(r)}{(k^2+r^2)^{\frac{n-4}{2}}}.$$

Theorem 1 will be proven if the condition (6.1) holds that is

$$0 < c_{\epsilon,\lambda} < \frac{4}{n} f(x_o)^{1-\frac{n}{4}} K_2^{-\frac{n}{2}}.$$

Now since the function g is positive everywhere on M, we have

$$I_{\epsilon,\lambda}(\varphi_k) \leq I(\varphi_k) = \left[\|\Delta\varphi_k\|_2^2 + \int_M a(x) \left|\nabla\varphi_k\right|^2 dv_g + \int_M h(x)\varphi_k^2 dv_g \right] - \frac{2}{N} \int_M f(x)\varphi_k^N dv_g.$$

So by Lemma 5 to prove Theorem 1, it suffices to show that

$$I(\varphi_k) < \frac{4}{n} f(x_o)^{1-\frac{n}{4}} K_2^{-\frac{n}{2}} .$$

Let ω_{n-1} be the volume of the Euclidean unit sphere and R be the scalar curvature and let

$$I_p^q = \int_0^\infty (1+t)^{-p} t^q dt$$

for any real numbers p, q with $p > q+1$.

We have

$$I_{p+1}^q = \frac{p-q-1}{p}$$

and

$$I_{p+1}^{q+1} = \frac{q+1}{p-q-1}.$$

If $\delta \in R^+$,

$$\lim_{k \to 0^+} \left\{ \int_0^\delta (r+k)^{-p} t^p dt - k^{p-q-1} I_p^q \right\}$$

is finite if $p-q-1 > 0$.

Similarly

$$\lim_{k \to 0^+} \left\{ \int_0^\delta (r+k)^{-p} t^p dt - \log\frac{1}{k} \right\}$$

if $p-q-1 = 0$.

Now, the computations given in [6] lead to, for $n > 6$ and $k \to 0$

$$A = \int_M (\Delta\varphi_k)^2 dv_g = \frac{n^{\frac{n}{4}} \left[(n-2)(n+2)\,(n-4)\right]^{\frac{n}{4}}}{2^{\frac{n}{4}}} f(x_o)^{1-\frac{n}{4}} \omega_{n-1} I_n^{\frac{n}{2}-1} \left\{ 1 - k^2 R(x_o) \left[\frac{(n^2+4)\,(n-4)}{6n(n-2)(n+2)(n-6)} + \frac{n-1}{2n(n+2)} \right] + O(k^3) \right\}.$$

Also

$$B=\int_M a(x)\left|\nabla\varphi_k\right|^2dv_g=\frac{n^{\frac{n}{4}}(n-1)\left[(n-4)\,(n-2)(n+2)\right]^{\frac{n}{4}-1}(n-4)^3}{2^{\frac{n}{4}-2}(n-6)}$$
$$f(x_o)^{1-\frac{n}{4}}\omega_{n-1}I_n^{\frac{n}{2}-1}k^2\left\{a(x_o)+O(k^3)\right\}$$

and

$$C=\int_M h(x)\varphi_k^2dv_g=\frac{\left[n\,(n-2)\,(n+2)\,(n-4)\right]^{\frac{n}{4}-1}}{2^{\frac{n}{4}-1}}f(x_o)^{1-\frac{n}{4}}O(k^4).$$

Finally

$$D=\frac{2}{N}\int_M f(x)\varphi_k^N dv_g=\frac{n^{\frac{n}{4}}\left[(n-4)\,(n-2)(n+2)\right]^{\frac{n}{4}}}{2^{\frac{n}{4}+1}}f(x_o)^{1-\frac{n}{4}}\omega_{n-1}I_n^{\frac{n}{2}-1}$$
$$\left\{1-\frac{k^2}{n-2}\left(\frac{\Delta f(x_o)}{2f(x_o)}+\frac{R(x_o)}{6}+O(k^3)\right)\right\}.$$

Consequently

$$I(\varphi_k)=A+B+C-D=\frac{n^{\frac{n}{4}}\left[(n-4)\,(n-2)(n+2)\right]^{\frac{n}{4}}}{2^{\frac{n}{4}+1}}f(x_o)^{1-\frac{n}{4}}\omega_{n-1}I_n^{\frac{n}{2}-1}$$
$$\times\left\{1-\frac{k^2}{n-2}\left(\frac{\Delta f(x_o)}{2f(x_o)}+\frac{5n^2(n-7)+52(n-1)}{6n(n+2)(n-6)}R(x_o)\right.\right.$$
$$\left.\left.-\frac{8(n-1)}{(n+2)(n-6)}a(x_o)\right)+O(k^3)\right\}.$$

On the other, the best constant K_2 in the Sobolev embedding $H_2^2(R^n)\hookrightarrow L^N(R^n)$ is

$$K_2^{-2}=n(n+2)(n-2)(n-4)\left(\frac{\omega_{n-1}I_n^{\frac{n}{2}-1}}{2}\right)^{\frac{n}{4}} \tag{7.1}$$

so letting

$$C_1(n)=\frac{5n^2(n-7)+52(n-1)}{6n(n+2)(5n-6)}$$

and

$$C_2(n)=\frac{8(n-1)}{(n+2)(n-6)}$$

we get

$$I(\varphi_k)\le\frac{1}{2^{\frac{n}{4}}}K_2^{-\frac{n}{2}}f(x_o)^{1-\frac{n}{4}}$$
$$\left\{1-\frac{k^2}{n-2}\left(\frac{\Delta f(x_o)}{2f(x_o)}+C_1(n)R(x_o)-C_2(n)a(x_o)\right)+O(k^3)\right\}.$$

So if

$$\frac{\Delta f(x_o)}{2f(x_o)} + C_1(n)R(x_o) - C_2(n)a(x_o) > 0$$

then

$$I(\varphi_k) < \frac{4}{n} f(x_o)^{1-\frac{n}{4}} K_2^{-\frac{n}{2}}.$$

For $n = 6$ and $k \to 0$, the expression of $D = \int_M f(x)\varphi_k^4 dv_g$ remains unchanged, however

$$A = \frac{n^{\frac{n}{2}-1}\,(n-4)\,[(n-2)(n+2)]^{\frac{n}{4}}\,\omega_{n-1}}{2^{\frac{n}{4}}} f(x_o)^{1-\frac{n}{4}}$$

$$\times \left\{ I_n^{\frac{n}{2}-1} - \frac{4(n-4)}{3n(n-2)(n+2)} R(x_o) k^2 \log(\frac{1}{k^2}) + O(k^2) \right\}$$

$$B = \frac{n^{\frac{n}{2}-1}(n-4)\,[(n-2)(n+2)]^{\frac{n}{4}}\,\omega_{n-1}}{2^{\frac{n}{4}}} f(x_o)^{1-\frac{n}{4}}$$

$$\times \left\{ \frac{1}{(n-4)(n-2)(n+2)} a(x_o) k^2 \log(\frac{1}{k^2}) + O(k^2) \right\}$$

and

$$C = \frac{1}{k^{n-4}}.O(k^4).$$

Consequently

$$A + B + C = \frac{n^{\frac{n}{4}}\,[(n-4)\,(n-2)(n+2)]^{\frac{n}{4}}\,\omega_{n-1}}{2^{\frac{n}{4}}} f(x_o)^{1-\frac{n}{4}}$$

$$\times \left\{ I_n^{\frac{n}{2}-1} - \frac{1}{(n-2)(n+2)} \left(\frac{4(n-4)}{3n} R(x_o) - \frac{1}{(n-4)} a(x_o) \right) k^2 \log(\frac{1}{k^2}) + O(k^2) \right\}$$

hence

$$A + B + C - D = \frac{[n(n-4)(n-2)(n+2)]^{\frac{n}{4}}\,\omega_{n-1}}{2^{\frac{n}{4}+1}} f(x_o)^{1-\frac{n}{4}}$$

$$\times \left\{ I_n^{\frac{n}{2}-1} - \frac{2}{(n-2)(n+2)} \left(\frac{4}{3n} R(x_o) - \frac{1}{(n-4)} a(x_o) \right) k^2 \log(\frac{1}{k^2}) + O(k^2) \right\}.$$

So if

$$\frac{4}{3n} R(x_o) - \frac{1}{(n-4)} a(x_o) > 0$$

and taking account of the value of K_2 given by (7.1) we get

$$I(\varphi_k) < \frac{1}{2^{\frac{n}{4}}} f(x_o)^{1-\frac{n}{4}} K_2^{-\frac{n}{2}} < \frac{4}{n} f(x_o)^{1-\frac{n}{4}} K_2^{-\frac{n}{2}}.$$

■

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