On analytic continuation in Hardy spaces

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Abstract

Let D be the open unit disk in $\mathbb{C}$. In this article, we construct dense subspaces of $H^p(D)$, $1 \leq p \leq \infty$, with certain barrelledness properties, such that their nonzero elements cannot be extended holomorphically outside D.

1 Introduction and notation

Throughout this paper all linear spaces are assumed to be defined over the field $\mathbb{C}$ of complex numbers. We write $\mathbb{N}$ for the set of positive integers. Given a complex number z_0 and $\rho > 0$, we put

$$D(z_0; \rho) := \{z \in \mathbb{C} : | z - z_0 | < \rho\}$$

and write

$$D := D(0; 1).$$

For $1 \leq p \leq \infty$, $H^p(D)$ stands for the Hardy space, that is, $H^\infty(D)$ is the linear space formed by the bounded holomorphic functions in D with the norm $\| \cdot \|_\infty$ such that

$$\| f \|_\infty = sup\{| f(z) | : \ z \in D\}, \quad f \in H^\infty(D),$$

and, for $1 \leq p < \infty$, $H^p(D)$ is the linear space of the holomorphic functions in D such that

$$\| f \|_p := sup_{0 \leq r < 1} \left(\int_{-\pi}^{\pi} | f(re^{i\theta}) |^p \ d\theta \right)^{\frac{1}{p}} < \infty,$$

provided with the norm $\| \cdot \|_p$.

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Let us now fix $1 \leq p \leq \infty$. We take a countable dense subset $\{z_n : n \in \mathbb{N}\}$ of the unit circle. Given $m, n, s \in \mathbb{N}$, we put

$$A_{m,n,s} := \{f \in H^p(D) : | f'(z) | \leq m, \; z \in D \cap D(z_n; 1/s)\}.$$

This subset of $H^p(D)$ is closed and absolutely convex. It is not hard to find a function g which is continuous in the closure $\overline{D}$ of D, holomorphic in D and whose derivative $g'(z)$ is not bounded in $D \cap D(z_n; 1/s)$. Since $A_{m,n,s}$ does not absorb g, which is obviously in $H^p(D)$, we have that $A_{m,n,s}$ is not a neighborhood of zero in $H^p(D)$, thus it has no interior points. Denoting by M_p the subset of $H^p(D)$ formed by those elements that cannot be extended holomorphically outside D, we have that

$$\bigcup\{A_{m,n,s} : \; m, n, s \in \mathbb{N}\} \supset H^p(D) \setminus M_p$$

from where we deduce that M_p is a set of the second category in the Banach space $H^p(D)$.

In [2], the authors construct a non-separable closed linear subspace Y of $H^\infty(D)$ such that every nonzero element of Y does not extend holomorphically outside D. In this paper we are interested in constructing dense subspaces of $H^p(D)$, $1 \leq p \leq \infty$, which, except for the zero function, are contained in M_p, at the same time possessing good barrelledness properties.

Let P be a subset of $\mathbb{N}$. Given j in $\mathbb{N}$, we write $P(j)$ to denote the set of elements of P which are not greater than j. P is said to have zero density whenever

$$\lim_{j \to \infty} \frac{P(j)}{j} = 0.$$

We say that a sequence (a_j) of complex numbers has zero density whenever the set

$$\{j \in \mathbb{N} : \; a_j \neq 0\}$$

has zero density. For $1 \leq p < \infty$, we write $\ell^p_{(0)}$ to represent the subspace of ℓ^p whose elements have zero density.

ℓ^∞_0 will stand for the subspace of ℓ^∞ formed by those sequences taking only a finite number of values, or, equivalently, ℓ^∞_0 is the linear span in ℓ^∞ of the sequences which take only the values 0 and 1.

2 The space $H^\infty(D)$

The interpolation theorem in $H^\infty(D)$ refers to the existence of sequences (z_n) in D such that, given an arbitrary bounded sequence of complex numbers (a_n), there is an element f in $H^\infty(D)$ such that

$$f(z_n) = a_n, \quad n \in \mathbb{N}.$$

Whenever a sequence (z_n) has such a property, we say that it is an interpolating sequence.

Working independently, L. Carleson [3], W. Hayman [8] and D. J. Newman [11] dealt with this kind of problem. Carleson showed that a necessary and sufficient condition for (z_n) to be an interpolating sequence is that there exist $\delta > 0$ such that

$$\prod_{n \neq k} \mid \frac{z_n - z_k}{1 - \overline{z_n}\, z_k} \mid \ \geq \ \delta, \quad k \in \mathbb{N}. \tag{1}$$

Newman showed that if a sequence (z_n) of D satisfies that, for each $f \in H^1(D)$,

$$\sum_{n=1}^{\infty} \mid f(z_n) \mid \cdot (1 - \mid z_n \mid) \ < \ \infty \tag{2}$$

and besides condition (1) is also satisfied, then (z_n) is an interpolating sequence.

Carleson's result clearly yields that condition (1) implies condition (2). Hayman proved that condition (1) is a necessary condition for a sequence to be interpolating and also provided a condition stronger than (1) to achieve sufficiency. This stronger condition enabled him to obtain an explicit interpolation formula for the function f that takes the previously fixed values (a_n) at (z_n). Hayman also showed that if one can interpolate sequences of one's and zero's at the points of (z_n), then condition (1) is satisfied. By applying Carleson's result, (z_n) is then an interpolating sequence. We shall obtain that (z_n) is an interpolating sequence without using Carleson's theorem. To do so, we shall make use of the following result, which is a particular case of [8, p. 296]: *a) If T is a continuous linear map from a Banach space E onto a barrelled normed space F, then F is a Banach space.*

In [6, p. 145], A. Grothendieck shows that ℓ_0^∞ is a barrelled space. In [14], motivated by a problem of localization in an *LF* space of the values of a bounded additive measure, we obtained the following result: *b) If (E_n) is an increasing sequence of subspaces of ℓ_0^∞ such that its union is ℓ_0^∞, then there is a subspace E_{n_0} which is barrelled and dense in ℓ_0^∞.* After studying this localization problem replacing the *LF* space by a webbed space of type $\mathcal{C}$, [4], we conjecture that a stronger property than that of $b)$ will still hold. We shall study this property in the next section and will later use it in the problem that we are interested in.

Theorem 1. *If (z_n) is a sequence in D such that, for every sequence (a_n) with a_n being either zero or one, $n \in \mathbb{N}$, there is f in $H^\infty(D)$ such that $f(z_n) = a_n$, $n \in \mathbb{N}$, then (z_n) is an interpolating sequence.*

Proof. Let T be the map from $H^\infty(D)$ into ℓ^∞ given by

$$T f \ := \ (f(z_n)), \quad f \in H^\infty(D).$$

We have that $F := T(H^\infty(D))$ contains ℓ_0^∞. Since ℓ_0^∞ is barrelled and dense in ℓ^∞, it follows that F is barrelled and dense in ℓ^∞. Thus, $T : H^\infty(D) \to F$ is continuous, linear and onto. We apply result $a)$ and so we have that F is a Banach space. Consequently, $T(H^\infty(D)) = \ell^\infty$ and the conclusion follows. ∎

3 The space ℓ_0^∞

We consider the following tree of infinitely many ramification points:

$$T_\infty := \bigcup\{\mathbb{N}^k : k \in \mathbb{N}\}.$$

An increasing web in a set E is a family

$$\mathcal{W} = \{E_t : t \in T_\infty\}$$

of subsets of E such that

$$E_1 \subset E_2 \subset ... \subset E_n \subset ..., \ \cup_{n=1}^\infty E_n = E,$$

and such that, for each t of T_∞,

$$E_{t,1} \subset E_{t,2} \subset ... \subset E_{t,n} \subset ..., \ \cup_{n=1}^\infty E_{t,n} = E_t.$$

If E is a linear space and E_t is a linear subspace of E, $t \in T_\infty$, we say that $\mathcal{W}$ is an increasing linear web.

A locally convex space E is said to be baireled whenever, for any increasing linear web in E,

$$\mathcal{W} = \{E_t : t \in T_\infty\},$$

there is an infinite branch

$$\gamma = \{(n_1), (n_1, n_2), ..., (n_1, n_2, ..., n_j), ...\}$$

such that each E_t, $t \in \gamma$, is dense in E and barrelled. It is shown in [10] that ℓ_0^∞ is baireled and this property, in a more general way, is used to obtain some new results on bounded additive measures, both scalar and vector-valued.

For the proof of the next proposition we shall make use of the following result, [12]: *c) Let F be a closed subspace of a locally convex space E and let T be the canonical mapping from E onto E/F. Let A be a closed absolutely convex subset of E. If there is an absolutely convex zero-neighborhood U of E such that $U \cap F \subset A$ and $\overline{T(A \cap U)}$ is a zero-neighborhood in E/F, then A is a zero-neighborhood in E.*

Proposition 1. *Let F be a closed subspace of the locally convex space E. If F and E/F are both baireled, then E is also baireled.*

Proof. Let

$$\mathcal{W} = \{E_t : t \in T_\infty\}$$

be an increasing linear web in E. It follows that

$$\mathcal{W}' = \{E_t \cap F : t \in T_\infty\}$$

is an increasing linear web in F and so, since this space is baireled, T_∞ has an infinite branch

$$\gamma = \{(n_1), (n_1, n_2), ..., (n_1, n_2, ..., n_j), ...\}$$

such that $E_t \cap F$ is dense in F and barrelled, for $t \in \gamma$. If in $\mathcal{W}$, for every $k \in \mathbb{N}$, we only consider the subindexes t of the form $(j_1, j_2, ..., j_k)$, with $j_1 \geq n_1$, $j_2 \geq n_2$,..., $j_k \geq n_k$, we obtain a subset $\mathcal{W}_1$ of $\mathcal{W}$ such that, if we conveniently change the subindexes of its elements, we have an increasing linear web such that the intersection of each of its elements with F is dense in F and barrelled. Hence, we may assume that $\mathcal{W}$ has the property that, for every $t \in T_\infty$, $E_t \cap F$ is dense in F and barrelled. On the other hand, we have that

$$\mathcal{W}'' = \{T(E_t) : t \in T_\infty\}$$

is an increasing linear web in E/F and, since this space is baireled, we can proceed as before and assume that $\mathcal{W}$ has the property that, for every $t \in T_\infty$, $T(E_t)$ is dense in E/F and barrelled.

Let us fix $t \in T_\infty$. Let B be a closed absolutely convex absorbing subset of E_t. Let A be the closure of B in E. We have that $B \cap F$ is a zero-neighborhood in $E_t \cap F$ and thus $A \cap F$ is a zero-neighborhood in F. We find an absolutely convex zero-neighborhood U in E such that $U \cap F \subset A$. It follows that $B \cap U$ is an absolutely convex absorbing subset of E_t and so $T(B \cap U)$ is an absolutely convex absorbing subset of $T(E_t)$. Hence, if $\overline{T(B \cap U)}$ denotes the closure of $T(B \cap U)$ in E/F, $\overline{T(B \cap U)}$ is a zero-neighborhood in this space. Since $\overline{T(A \cap U)}$ contains $\overline{T(B \cap U)}$, we have that $\overline{T(A \cap U)}$ is a zero-neighborhood in E/F. By applying result c), we obtain that A is a zero-neighborhood in E, from which we deduce that E_t is barrelled and dense in E. ■

In the coming section, besides using the three-space property before stated, we shall need the following result, [5]: *d) Let F a subspace of countable codimension of the locally convex space E. If E is baireled, then so is F.*

For a given integer $k \geq 2$, let M denote the subset of $\mathbb{N}^k$ such that $(j_1, j_2, ..., j_k) \in M$ if and only if $j_1 < j_2 < ... < j_k$. We write

$$H_{j_1,j_2,...,j_k} := \{(a_j) \in \ell_0^\infty : a_{j_1} = a_{j_2} = ... = a_{j_k}\}.$$

Then $H_{j_1,j_2,...,j_k}$ is a closed subspace of ℓ_0^∞ with codimension $k-1$. We also have that

$$\bigcup\{H_{j_1,j_2,...,j_k} : (j_1, j_2, ..., j_k) \in M\} = \ell_0^\infty,$$

from where it follows that ℓ_0^∞ is not a Baire space.

In [15], a locally convex space E is said to be totally barrelled whenever, for an arbitrary countable cover of E by subspaces $\{E_n : n \in \mathbb{N}\}$, there is an integer n_0 such that E_{n_0} is barrelled and its closure has finite codimension. Noticing that, for any $m \in \mathbb{N}$, ℓ_0^∞ may be covered by a countable collection of closed subspaces with codimension m, one may wonder whether ℓ_0^∞ is totally barrelled. The answer to this is found in [1], where the following is shown: *e) There is a sequence (F_n) of closed subspaces of ℓ_0^∞, with infinite codimension, which covers ℓ_0^∞.*

In the proof of the next proposition we shall need the following result, to be found in [15]: *f) Let $\{E_n : n \in \mathbb{N}\}$ be a sequence of subspaces of a locally convex space E which covers E. If E is totally barrelled, then there is a positive integer n_0 such that E_{n_0} is totally barrelled and its closure has finite codimension.*

Proposition 2. *Let E be a locally convex space. If E is totally barrelled, then it is baireled.*

Proof. Let

$$\mathcal{W} := \{E_t : t \in T_\infty\}$$

be a linear increasing web in E. The sequence (E_n) covers E, hence there is a positive integer n_1 such that E_{n_1} is totally barrelled and its closure has finite codimension. Since (E_n) is increasing, we may take E_{n_1} being dense in E. Now, the sequence $(E_{n_1,n})$ is increasing and covers E_{n_1}. Thus, we may find $n_2 \in \mathbb{N}$ such that E_{n_1,n_2} is dense in E_{n_1}, therefore dense in E, and totally barrelled. Proceeding in this way, we obtain a branch in T_∞

$$\gamma = \{(n_1), (n_1, n_2), ..., (n_1, n_2, ..., n_j), ...\}$$

in such a way that E_t is barrelled and dense in E for every $t \in \gamma$. ∎

Since ℓ_0^∞ is baireled, result $e)$ and the former proposition tell us that being totally barrelled is a property which is strictly stronger than that of being baireled.

4 On certain dense subspaces of $H^\infty(D)$

Theorem 2. *There exists in $H^\infty(D)$ a dense subspace G which is baireled and such that every non-zero element f of G does not extend holomorphically outside D.*

Proof. We choose in D an interpolating sequence (z_n) such that its closure coincides with the unit circle. Let T be the map from $H^\infty(D)$ into ℓ^∞ such that

$$Tf := (f(z_n)), \quad f \in H^\infty(D).$$

Let F denote the subspace of $H^\infty(D)$ given by the kernel of T. We put $E := T^{-1}(\ell_0^\infty)$. Since ℓ_0^∞ is dense in ℓ^∞, we have that E is dense in $H^\infty(D)$ and so ℓ_0^∞ identifies canonically with E/F. Since F is the kernel of a continuous operator, it is automatically a closed, hence Banach, subspace. Thus F is baireled. E/F is also baireled. Proposition 1 applies yielding that E is baireled.

We take an element g of E such that it extends holomorphically outside D. We find u_0 in the unit circle and $\rho > 0$ for which there exists a function h holomorphic in $D(u_0; \rho)$ and coinciding with g in $D \cap D(u_0; \rho)$. Since $(g(z_n))$ takes only a finite number of distinct values, we have that h is constant in $D(u_0; \rho)$ and so g is also constant. If k is the element of $H^\infty(D)$ such that $k(z) = 1$, $z \in D$, it follows that k belongs to E. Let G be a hyperplane dense in E with $k \notin G$. Then after result $d)$, G is baireled. Besides, G is dense in $H^\infty(D)$ and each non-zero element of G does not extend holomorphically outside D. ∎

Let us now consider a simply connected domain Ω of $\mathbb{C}$, distinct from $\mathbb{C}$. Let $H^\infty(\Omega)$ be the linear space formed by the bounded holomorphic functions in Ω. If f is in $H^\infty(\Omega)$, we put

$$\| f \|_\infty := sup\{| f(z) | : z \in \Omega\}.$$

We consider $H^\infty(\Omega)$ provided with the norm $\| \cdot \|_\infty$.

Theorem 3. *There is in $H^\infty(\Omega)$ a dense subspace G which is baireled and such that each non-zero element f of G does not extend holomorphically outside Ω.*

Proof. We apply Riemann's theorem and obtain a function φ holomorphic in Ω, which defines a homeomorphism onto D. We find a sequence (z_n) in Ω such that its closure coincides with the boundary $\partial\Omega$ of Ω and so that $(\varphi(z_n))$ is an interpolating sequence in $H^\infty(D)$. Let T be the map from $H^\infty(\Omega)$ into ℓ^∞ such that

$$Tf \;:=\; (f(z_n)), \quad f \in H^\infty(\Omega).$$

We have that T is linear and bounded. We show that it is also onto. If $(a_n) \in \ell^\infty$, we find an element g of $H^\infty(D)$ such that

$$g(\varphi(z_n)) = a_n, \quad n \in \mathbb{N}.$$

It follows that $g \circ \varphi \in H^\infty(\Omega)$ and

$$(g \circ \varphi)(z_n) = \; g(\varphi(z_n)) \;=\; a_n, \;\; n \in \mathbb{N}.$$

Proceeding as in the proof of the previous theorem, the conclusion follows. ∎

5 The spaces $\ell^p_{(0)}$, $1 \leq p < \infty$.

In [9, p. 369], the space $\ell^1_{(0)}$ is shown to be barrelled and, in [15], it is proved to be totally barrelled. We shall see in this section that the spaces $\ell^p_{(0)}$, $1 < p < \infty$ also possess these properties. Moreover, we prove that $\ell^p_{(0)}$, $1 \leq p < \infty$ enjoys a strictly stronger property than that of being totally barrelled.

We put $B(\ell^p)$ for the closed unit ball of ℓ^p and $e_n := (a_j)$, with $a_j = 0$, $j \neq n$, $a_n = 1$. Given a positive integer r, we write $B^r(\ell^p)$ to denote the set of elements (a_j) of $B(\ell^p)$ such that $a_j = 0$, $j = 1, 2, ..., r$. If s is an integer such that $0 \leq s < r$, by $B^r_s(\ell^p)$ we represent the set of the elements (a_j) of $B^r(\ell^p)$ for which

$$a_j \;=\; 0, \quad j \notin \{rm + s : \; m \in \mathbb{N}\}.$$

Following [17], we say that a subset A of a locally convex space E is sum-absorbing whenever there exists $\lambda > 0$ such that $\lambda(A + A)$ is contained in A.

Lemma 1. *Let (A_n) be a sequence of closed balanced sum-absorbing subsets of ℓ^p such that they cover $\ell^p_{(0)}$. Then there are positive integers n_0 and r_0 such that the linear span of A_{n_0} contains the closed linear span of $\{e_j \;:\; j = r_0, r_0 + 1, ...\}$.*

Proof. Without loss of generality, we may assume that the homethetics of A_n, with ratio a positive integer, that is, all sets of the form rA_n, $r \in \mathbb{N}$, are contained in (A_n). We proceed by contradiction assuming that the property does not hold. It is clear that the linear span of A_n coincides with $\cup_{j=1}^\infty \, j \; A_n$. Hence, for every pair of positive integers n, r, A_n does not absorb $B^r(\ell^p)$.

We put $r_1 = 2$. Proceeding inductively, let us assume that, for a positive integer i, we have obtained an integer $r_i > 1$. Since A_i is balanced and sum-absorbing and

$$B^{r_i^2}(\ell^p) \subset \sum_{s=0}^{r_i^2-1} B_s^{r_i^2}(\ell^p),$$

there exists s_i, $0 \leq s_i < r_i^2$, such that A_i does not absorb $B_{s_i}^{r_i^2}(\ell^p)$. We find $(b_j^{(i)})$ in $B_{s_i}^{r_i^2}(\ell^p)$ such that

$$(b_j^{(i)}) \notin A_i.$$

Since A_i is closed in ℓ^p, there is an integer $r_{i+1} > r_i^2$ with the form $\dot{\overline{r_i^2}} + s_i$, i.e., r_{i+1} congruent with s_i modulo r_i^2, such that, if

$$b_{j,i} := \begin{cases} b_j^{(i)}, & j = 1, 2, ..., r_{i+1}, \\ 0, & j = r_{i+1}+1, r_{i+1}+2, ..., \end{cases}$$

then

$$(b_{j,i}) \notin A_i. \tag{3}$$

This concludes the complete induction procedure. We write

$$P := \{ j \in \mathbb{N} : b_{j,i} \neq 0, \textit{for some } i \in \mathbb{N}\}.$$

We take an element h in P. Let $i \in \mathbb{N}$ be such that $b_{h,i} \neq 0$. This integer i is clearly unique and we have that

$$r_i^2 < h \leq r_{i+1}.$$

There is a positive integer q such that $h = qr_i^2 + s_i$. Now, the elements $j \in \mathbb{N}$ such that $j \leq h$ and $b_{j,i} \neq 0$ belong to one of the following sets

$$\{lr_i^2 : l = 2, 3, ..., q\}, \ \textit{if } s_i = 0, \ \{lr_i^2 + s_i : l = 1, 2, ..., q\}, \ \textit{if } s_i \neq 0.$$

If $i = 1$, then

$$P(h) \leq q.$$

If $i > 1$, and i', k are in $\mathbb{N}$ such that $i' \neq i$, $k < h$ and $b_{k,i'} \neq 0$, then $i' < i$ and so

$$k \leq r_{i'+1} \leq r_i,$$

from where we have

$$P(h) \leq r_i + q.$$

In any case we obtain that

$$P(h) \leq r_i + q$$

and thus

$$\frac{P(h)}{h} \leq \frac{r_i + q}{qr_i^2 + s_i} \leq \frac{1}{qr_i} + \frac{1}{r_i^2},$$

so

$$\lim_{h \to \infty} \frac{P(h)}{h} = 0,$$

that is, P has zero density. We now put $\ell^p(P)$ to denote the closed linear span of $\{e_j : j \in P\}$ in ℓ^p. Then $\ell^p(P)$ is a Banach space which is contained in $\ell^p_{(0)}$. Since (A_n) covers $\ell^p(P)$, there is a positive integer n_0 such that $A_{n_0} \cap \ell^p(P)$ has non-empty interior in $\ell^p(P)$, and, since A_{n_0} is balanced and sum-absorbing, it follows that $A_{n_0} \cap \ell^p(P)$ is a zero-neighborhood in $\ell^p(P)$. Consequently, there is a positive integer s such that

$$B(\ell^p) \cap \ell^p(P) \subset s\, A_{n_0}.$$

Since the sequence (A_n) contains all the homothetics of A_{n_0} with ratio a positive integer, there is $m \in \mathbb{N}$ such that $sA_{n_0} \subset A_m$. Hence

$$(b_{j,m}) \in B(\ell^p) \cap \ell^p(P) \subset A_m,$$

which is in contradiction with (3). ■

In [16], the following definition is given: *A locally convex space E is semi-Baire whenever, for every sequence (A_j) of closed balanced sum-absorbing subsets of E covering E, there is $j_0 \in \mathbb{N}$ such that A_{j_0} is a zero-neighborhood in its linear hull $L(A_{j_0})$, and this space has finite codimension in E.*

Note. Notice that, if in $\ell^p_{(0)}$ we put H_j to denote the closed linear span of

$$\{e_n : n \in \mathbb{N} \setminus \{j\}\},$$

then (H_j) is a sequence of closed hyperplanes of $\ell^p_{(0)}$ which covers $\ell^p_{(0)}$.

Theorem 4. *The space $\ell^p_{(0)}$ is semi-Baire.*

Proof. In $\ell^p_{(0)}$, let (B_n) be a sequence of closed balanced and sum-absorbing subsets which covers $\ell^p_{(0)}$. Let A_n be the closure of B_n in ℓ^p. It is quite clear that A_n is balanced and sum-absorbing. By applying the former lemma, we obtain two positive integers n_0, r_0 such that $L(A_{n_0})$ contains the closed linear span in ℓ^p of $\{e_j : j = r_0, r_0 + 1, ...\}$. Hence, $L(A_{n_0})$ is closed and has finite codimension in ℓ^p, thus yielding that A_{n_0} is a zero-neighborhood in $L(A_{n_0})$. Since $B_{n_0} = A_{n_0} \cap \ell^p_{(0)}$ and $L(B_{n_0}) = L(A_{n_0}) \cap \ell^p_{(0)}$, it follows that B_{n_0} is a zero-neighborhood in $L(B_{n_0})$ and this space has finite codimension in $\ell^p_{(0)}$. ■

Corollary 1. *$\ell^p_{(0)}$ is barrelled.*

Corollary 2. *$\ell^p_{(0)}$ is totally barrelled.*

Corollary 1 may be found in [9, p. 369] for $p = 1$ and Corollary 2 is proved in [14] for $p = 1$.

The example given in [16, pp. 154-155] shows that there exist locally convex spaces which are totally barrelled but not semi-Baire.

Proposition 3. *Let E be a semi-Baire locally convex space. If (E_n) is a sequence of subspaces of E which covers E, then there is a positive integer n_0 such that E_{n_0} is semi-Baire and whose closure has finite codimension.*

Proof. Assuming the property is not true, let (E_n) be a sequence of subspaces covering E and not satisfying the statement. We have that $(\overline{E_n})$ is a sequence of closed balanced sum-absorbing subsets of E covering E. Hence, one of them has finite codimension. Let M be the subset of $\mathbb{N}$ consisting of all integers n such that $\overline{E_n}$ has finite codimension. Thus, since none of the subspaces E_n, $n \in \mathbb{N} \setminus M$, has finite codimension, it follows that $\{E_n : n \in \mathbb{N} \setminus M\}$ does not cover E. We take

$$x \in E \setminus \cup\{E_n : n \in \mathbb{N} \setminus M\}.$$

Let us assume that $\{E_n : n \in M\}$ does not cover E. Then, we may take

$$y \in E \setminus \cup\{E_n : n \in M\}.$$

We have that x and y are distinct from zero and $x \neq y$. Let

$$L := \{ x + \lambda y : \lambda \in \mathbb{R} \}.$$

The sequence (E_n) covers L and each E_n meets L in at most one point, which is a contradiction. Therefore, $(E_n)_{n \in M}$ covers E. We may thus assume to finish with the proof that all $\overline{E_n}$, $n \in \mathbb{N}$, have finite codimension.

Let us now assume that each E_n is covered by a sequence of subspaces $(E_{n,i})_{i \in \mathbb{N}}$ such that $\overline{E_{n,i}}$ does not have finite codimension in E, $n, i \in \mathbb{N}$. Recalling that

$$E = \cup\{ E_{n,i} : n, i \in \mathbb{N}\}$$

we obtain a contradiction. Arguing as before, we may assume that, for each $n \in \mathbb{N}$, for an arbitrary sequence $(E_{n,i})_{i \in \mathbb{N}}$ of subspaces of E_n which covers E_n, there is $i_0 \in \mathbb{N}$ such that $\overline{E_{n,i_0}}$ has finite codimension in E.

Since E_n is not semi-Baire, we may take in E_n a sequence $(A_{n,i})_{i \in \mathbb{N}}$ of closed balanced sum-absorbing subsets covering E_n and such that, if for $i \in \mathbb{N}$ the set $A_{n,i}$ is a zero-neighborhood in $L(A_{n,i})$, then $\overline{L(A_{n,i})}$ does not have finite codimension in E. Proceeding as before, we may assume that, for every $i \in \mathbb{N}$, $\overline{L(A_{n,i})}$ has finite codimension in E, $n \in \mathbb{N}$. It follows now that the sets $\overline{A_{n,i}}$, $n, i \in \mathbb{N}$, are closed balanced sum-absorbing subsets of E which cover E. Consequently, there are positive integers m, s such that $\overline{A_{m,s}}$ is a zero-neighborhood in $L(\overline{A_{m,s}})$. Given that

$$A_{m,s} = \overline{A_{m,s}} \cap L(A_{m,s}),$$

we have that $A_{m,s}$ is a zero-neighborhood in $L(A_{m,s})$, which is a contradiction. ■

Proposition 4. *If F is a countable codimensional subspace of a semi-Baire space E, then F is semi-Baire.*

Proof. Let us assume first that F is a hyperplane. Let (A_n) be a sequence of closed balanced and sum-absorbing subsets of F. We also assume that the homothetics of each A_n, with ratio a positive integer, are contained in (A_n). We put $B_n := \overline{A_n}$, $n \in \mathbb{N}$. We then have that B_n is closed balanced and sum-absorbing in E. If $\cup_{n=1}^{\infty} B_n = E$, there is a positive integer n_0 such that B_{n_0} is a zero-neighborhood in $L(B_{n_0})$ and this space has finite codimension in E. Hence,

$A_{n_0} = B_{n_0} \cap L(A_{n_0})$ and so A_{n_0} is a zero-neighborhood in $L(A_{n_0})$. Also this space has finite codimension in F. Assume now that

$$\cup\{ B_n \; : \; n \in \mathbb{N} \} \neq E.$$

Then, there is x in E such that

$$\{ \lambda x \; : \; \lambda \in \mathbb{C}, \lambda \neq 0\}$$

does not intersect B_n, $n \in \mathbb{N}$. Let

$$B := \{ \lambda x \; : \; \lambda \in \mathbb{C}, \mid \lambda \mid \leq 1\}.$$

We set

$$B_{n,m} := B_n + m\, B.$$

In E, $B_{n,m}$ is closed balanced and sum-absorbing. We take $y \in E$. Then

$$y = z + \mu\, x, \;\; z \in F, \;\; \mu \in \mathbb{C}.$$

We find r in $\mathbb{N}$ such that $z \in A_r$ and choose $s \in \mathbb{N}$ such that $\mid \mu \mid \leq s$. Thus

$$y \in B_r + s\, B = Br, s.$$

Therefore $\{B_{n,m} : n, m \in \mathbb{N}\}$ covers E and so there are $n_0, m_0 \in \mathbb{N}$ such that B_{n_0,m_0} is a zero-neighborhood in $L(B_{n_0,m_0})$ and this space has finite codimension in E. It follows that

$$B_{n_0,m_0} \cap L(B_{n_0}) = B_{n_0},$$

from where we get that B_{n_0} is a zero-neighborhood in $L(B_{n_0})$ and this space has finite codimension in E. Thus, since

$$A_{n_0} = B_{n_0} \cap L(A_{n_0}),$$

it follows that A_{n_0} is a zero-neighborhood in $L(A_{n_0})$ and this space has finite codimension in F.

From what was said before, if F has finite codimension in E, we have that F is semi-Baire.

Let us assume now that F has countably infinite codimension in E. Let $\{x_j : j \in \mathbb{N}\}$ be a cobasis of F in E. We denote by E_n the linear span of $F \cup \{x_1, x_2, ..., x_n\}$, $n \in \mathbb{N}$. Hence, $E = \cup_{n=1}^{\infty} E_n$ and, from Proposition 3, there is $n_0 \in \mathbb{N}$ such that E_{n_0} is a semi-Baire space. Now, since F has finite codimension in E_{n_0}, we obtain that F is also semi-Baire. ∎

For the proof of our next proposition, we shall need the following result which is found in [17]: *g) Let F be a closed subspace of a locally convex space E and let T be the canonical mapping from E onto E/F. Let A be a closed balanced sum-absorbing subset of E. If there is an absolutely convex zero-neighborhood U in E such that $U \cap F \subset A$ and $\overline{T(A \cap U)}$ is a zero-neighborhood in E/F, then A is a zero-neighborhood in E.*

Proposition 5. *Let F be a closed subspace of a locally convex space E. If F and E/F are semi-Baire spaces, then E is also a semi-Baire space.*

Proof. Let (A_n) be a sequence of closed balanced sum-absorbing subsets of E which covers E. We also assume the homothetics, with positive ratio, of each A_n, $n \in \mathbb{N}$, are also contained in the sequence. It follows that the sequence $(A_n \cap F)$ is formed by closed balanced sum-absorbing subsets of F which cover F.

Let us define the set $M \subset \mathbb{N}$, such that $n \in M$ if and only if $A_n \cap F$ is a zero-neighborhood in $L(A_n \cap F)$ and this space has finite codimension in F. We show that the family $L(A_n)$, $n \in \mathbb{N} \setminus M$, does not cover F. Otherwise, since

$$\cup\{j\, A_n : j \in \mathbb{N}\} \;=\; L(A_n),$$

we would have that the family $A_n \cap F$, $n \in \mathbb{N} \setminus M$, would cover F. So there would be a positive integer n_0 in $\mathbb{N} \setminus M$ such that $A_{n_0} \cap F$ is a zero-neighborhood in $L(A_{n_0} \cap F)$ and this space has finite codimension in F. Hence $n_0 \in M$, which is a contradiction.

Proceeding similarly as in the proof of the former proposition, we obtain that

$$\cup\{L(A_n) : n \in M\}$$

covers E and thus $(A_n)_{n \in M}$ also covers E. We may thus assume that, for each $n \in \mathbb{N}$, $A_n \cap F$ is a zero-neighborhood in $L(A_n \cap F)$ and that this space has finite codimension in F.

Let us assume now that G_n is a topological complement of $L(A_n \cap F)$ in F and let K_n be a compact balanced absolutely convex subset of G_n which is a zero-neighborhood in G_n. We put $B_n := A_n + K_n$. Then $B_n \cap F$ is a zero-neighborhood in F. We have that $B_n \cap L(A_n) = A_n$ and $L(A_n)$ has finite codimension in $L(B_n)$. Therefore it suffices to find a positive integer s such that B_s is a zero-neighborhood in $L(B_s)$ and this space has finite codimension in E. So, in order to prove this proposition, we may assume that $A_n \cap F$ is a zero-neighborhood in F. Then $L(A_n) \supset F$.

Let T denote the canonical mapping from E onto E/F. Since E/F is semi-Baire and the sequence $(T(L(A_n)))$ covers E/F, after Proposition 3, we have that there is a positive integer n_0 for which $T(L(A_{n_0}))$ is a semi-Baire subspace of E/F and its closure $\overline{T(L(A_{n_0}))}$ has finite codimension in E/F. Let T_1 be the mapping from $L(A_{n_0})$ onto $T(L(A_{n_0}))$ such that

$$T_1 x \;=\; T\,x, \quad x \in L(A_{n_0}).$$

We have that F is the kernel of T_1. We find an absolutely convex zero-neighborhood U in $L(A_{n_0})$ such that $U \cap F \subset A_{n_0}$. It follows that $T_1(U \cap A_{n_0})$ is a balanced absorbing and sum-absorbing subset of $T(L(A_{n_0}))$. If M_{n_0} stands for the closure of $T_1(U \cap A_{n_0})$ in $T(L(A_{n_0}))$, we have that the sequence $(j\, M_{n_0})_{j=1}^{\infty}$ covers $T(L(A_{n_0}))$ and, since these sets are balanced and sum-absorbing, we obtain that M_{n_0} is a zero-neighborhood in $T(L(A_{n_0}))$. Applying result *g*) we have that A_{n_0} is a zero-neighborhood in $L(A_{n_0})$ and, since A_{n_0} is closed in E, it follows that $L(A_{n_0})$ is closed in E. Then

$$L(A_{n_0}) \;=\; T^{-1}(\overline{T(L(A_{n_0}))}),$$

from where we deduce that $L(A_{n_0})$ has finite codimension in E. ∎

6 On certain dense subspaces of $H^p(D)$, $1 \leq p < \infty$

The weighted interpolation problem in $H^p(D)$, $1 \leq p < \infty$, refers to the existence of sequences (z_n) in D such that, given an arbitrary sequence (a_n) in ℓ^p, there is an element $f \in H^p(D)$ satisfying that

$$f(z_n)\,(\,1 - \mid z_n \mid)^{1/p} \;=\; a_n, \quad n \in \mathbb{N},$$

and also that, for each $g \in H^p(D)$,

$$(g(z_n)(1 - \mid z_n \mid)^{1/p}) \;\in\; \ell^p.$$

Whenever (z_n) satisfies these conditions, we shall say that it is a weight interpolating sequence for $H^p(D)$.

In [13], it is shown that a sequence (z_n) in D is a weight interpolating sequence for every $H^p(D)$ if and only if condition (1) is satisfied.

Theorem 5. *In $H^p(D)$, there is a dense subspace E which is semi-Baire and such that every non-zero element of E cannot be extended holomorphically outside D.*

Proof. We take a weight interpolating sequence (v_n) in D such that the set of all its cluster points coincides with the unit circle Γ. We choose a sequence (u_n) in Γ such that each u_r, $r \in \mathbb{N}$, appears infinitely many times in the sequence (u_n) and the elements of this sequence form a dense subset of Γ. We consider an element t_{n_1} in (v_n) such that $\mid t_{n_1} - u_1 \mid < 1/2$.

Proceeding inductively, let us assume that, for a positive integer r, we have found a positive integer n_r and a term t_{n_r} of (v_n). We choose a finite subsequence $t_{n_r+1}, t_{n_r+2}, ..., t_{n_{r+1}}$ of (v_n) such that $n_{r+1} > 2n_r$, the term t_{n_r+1} is posterior to t_{n_r} in the sequence (v_n) and

$$\mid t_j - u_{r+1} \mid < \frac{1}{2^{j+1}}, \quad j = n_r + 1, n_r + 2, ..., n_{r+1}. \tag{4}$$

We write the sequence

$$t_{n_1}, t_{n_1+1}, ..., t_{n_2}, ..., t_{n_r}, ..., t_{n_r+1}, t_{n_r+2}, ..., t_{n_{r+1}}, ...$$

in the form (z_j). Clearly, (z_j) is a weight interpolating sequence for $H^p(D)$. Let T be the map from $H^p(D)$ into ℓ^p such that

$$T\,f \;:=\; (f(z_n)(1 - \mid z_n \mid)^{1/p}), \quad f \in H^p(D).$$

Then, T is an onto bounded linear map. Setting $E := T^{-1}(\ell^p_{(0)})$, we apply Theorem 4 and Proposition 5 to obtain that E is a semi-Baire space. Let us now assume there is a non-zero element f of E admitting continuation outside D. We find positive integers m, s such that there is a holomorphic function h in $D(u_s; 1/m)$ which coincides with f in $D \cap D(u_s; 1/m)$ and so that $h(z) \neq 0$, $z \in D(u_s; 1/m)$. For an arbitrary positive integer q, we find $r > q$ such that

$$u_{r+1} \;=\; u_s, \quad \frac{1}{2^{n_r}} \;<\; \frac{1}{m}.$$

It follows from (4) that $f(t_j) \neq 0$ for those values of j. Consequently, if j_0 is the positive integer for which $z_{j_0} = t_{n_r+1}$, we have that

$$\frac{P(j_0)}{j_0} \geq \frac{n_{r+1} - n_r}{n_{r+1}} = 1 - \frac{n_r}{n_{r+1}} > \frac{1}{2},$$

from where we obtain that the sequence $(f(z_j))$ does not have zero density, which is a contradiction. ■

References

[1] ARIAS DE REYNA, J.: $\ell_0^\infty(\Sigma)$ no es totalmente tonelado, *Rev. Real Acad. Cienc. Exact. Fis. Natur.*, **Madrid 79**, 77-78 (1980).

[2] ARON, R., GARCIA, D., MAESTRE, M.: Linearity in non-linear problems, *Rev. R. Acad. Cienc.*, Serie A. Mat., **95**(1), 7-12 (2001).

[3] CARLESON, L.: An interpolation problem for bounded analytic functions, *Amer. J. of Math.* **80**, 921-930 (1958).

[4] De WILDE, M.: Closed Graph Theorem and Webbed Spaces, *Pitman*, London, 1978.

[5] FERRANDO, J. C., SANCHEZ RUIZ, L. M.: A maximal class of spaces with strong barrelledness conditions, *Proc. Roy. Irish Acad.*, Sect. A **92**, 69-75 (1992).

[6] GROTHENDIECK, A.: Espaces vectoriels topologiques, *Departamento de Matematica da Universidade di Sao Paulo*, Brasil, 1954.

[7] HAYMAN, W.: Interpolation by bounded functions, *Ann. Inst. Fourier*, Grenoble, **13**, 277-290 (1958).

[8] HORVATH, J.: Topological Vector spaces and Distributions I, *Addison-Wesley*, Reading, Massachussets, 1966.

[9] KÖTHE, G.: Topological Vector Spaces I, *Springer-Verlag*, Berlin-Heidelberg-New York, 1984.

[10] LOPEZ PELLICER, M.: Webs and bounded additive measures, *J. of Math. Anal. and App.*, Vol. **210**, 257-267 (1997).

[11] NEWMAN, D. J.: Interpolation in H^∞, *Trans. Amer. Math. Soc.*, **92**, 501-507 (1959).

[12] ROELCKE, W., DIEROLF, S.: On the three-space problem for topological vector spaces, *Collect. Math.* , **Vol. XXXII**, 87-106 (1972).

[13] SHAPIRO, H.S., SHIELDS, A.L.: On some interpolation problems for analytic functions, *Amer. J. Math.*, **83**, 513-532 (1961).

[14] VALDIVIA, M.: On certain barrelled normed spaces, *Ann. Inst. Fourier*, **29**, 39-56 (1979).

[15] VALDIVIA, M., PEREZ CARRERAS, P.: On totally barrelled spaces, *Math. Z.*, **178**, 263-269 (1981).

[16] VALDIVIA, M.: On certain spaces of holomorphic functions, *Proceedings of the Second International School. Advanced Courses of Mathematical Analysis II*, World Scientific Publishing Co. Pte. Ltd., 151-173 (2007).

[17] VALDIVIA, M.: The space $\mathcal{H}(\Omega, (z_j))$ of holomorphic functions, *J. of Math. Anal. and App.*, Vol. **337/2**, 821-839 (2008).

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