Multiple periodic solutions of some Liénard equations with p-Laplacian

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Abstract

The existence, non-existence and multiplicity of solutions to periodic boundary value problems of Liénard type

$$(|u'|^{p-2}u')' + f(u)u' + g(u) = e(t) + s, \quad u(0) - u(T) = 0 = u'(0) - u'(T),$$

is discussed, where $p > 1$, f is arbitrary and g is assumed to be bounded, positive and $g(\pm\infty) = 0$. The function e is continuous on $[0, T]$ with mean value 0 and s is a parameter.

1 Introduction and the main result

Consider periodic boundary value problems of the form

$$(\phi(u'))' + g(u) = e(t) + s, \quad u(0) - u(T) = 0 = u'(0) - u'(T), \tag{1}$$

where $\phi : (-a, a) \to \mathbb{R}$ is an increasing homeomorphism such that $\phi(0) = 0$ and $0 < a \leq +\infty$, $g : \mathbb{R} \to \mathbb{R}$, $e : [0, T] \to \mathbb{R}$ are continuous functions and $s \in \mathbb{R}$ is a parameter. Assume that the following assumptions are satisfied.

(H1) $\int_0^T e(t)dt = 0$.

(H2) $g(u) > 0$ for all $u \in \mathbb{R}$.

(H3) $g(\pm\infty) = \lim_{u\to\pm\infty} g(u) = 0$.

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By solution of (1) we mean a function $u \in C^1([0,T])$ such that $\phi \circ u' \in C^1([0,T])$ and which verifies (1). The main result in [1] is the following one.

Theorem 1. *If $\phi : (-a,a) \to \mathbb{R}$ with $0 < a \leq +\infty$ and conditions (H1)-(H3) hold, there exists $s^*(e) \in (0, \sup_R g]$ such that problem (1) has zero, at least one or at least two solutions according to $s \notin (0, s^*(e)]$, $s = s^*(e)$ or $s \in (0, s^*(e))$.*

This type of result has been initiated by Ward [6] without multiplicity conclusion and $\phi(v) = v$. In the case $\phi(v) = |v|^{p-2}v$ for some $p > 1$, we generalize the result above as follows.

Consider periodic boundary value problems of Liénard type

$$(|u'|^{p-2}u')' + f(u)u' + g(u) = e(t) + s, \quad u(0) - u(T) = 0 = u'(0) - u'(T), \tag{2}$$

where $p > 1$, $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and g, e and s are as above. The main result of this paper is the following one.

Theorem 2. *If conditions (H1)-(H3) hold, there exists $s^*(e) \in (0, \sup_R g]$ such that problem (2) has zero, at least one or at least two solutions according to $s \notin (0, s^*(e)]$, $s = s^*(e)$ or $s \in (0, s^*(e))$.*

To prove our main result, we use an approach similar to that in [1], but with technical differences due to the presence of $f(u)u'$. In what follows $\phi : \mathbb{R} \to \mathbb{R}$ denotes the increasing homeomorphism defined by

$$\phi(v) = |v|^{p-2}v.$$

If $\Omega \subset X$ is an open set of a normed space X and if $S : \overline{\Omega} \to X$ is completely continuous and such that $0 \notin (I - S)(\partial\Omega)$, then $d_{LS}[I - S, \Omega, 0]$ denotes the Leray-Schauder degree with respect to Ω and 0. For the definition and properties of the Leray-Schauder degree see [3].

2 Notation and auxiliary results

Let C denote the Banach space of continuous functions on $[0,T]$ endowed with the uniform norm $\|\cdot\|_\infty$, C^1 denotes the Banach space of continuously differentiable functions on $[0,T]$, equipped with the norm $\|u\| = \|u\|_\infty + \|u'\|_\infty$. We consider its closed subspace

$$C^1_\# = \{u \in C^1 : u(0) = u(T), u'(0) = u'(T)\},$$

and denote corresponding open balls of center 0 and radius r by B_r. We denote by $P, Q : C \to C$ the continuous projectors defined by

$$P, Q : C \to C, \quad Pu(t) = u(0), \quad Qu(t) = \frac{1}{T}\int_0^T u(\tau)\, d\tau \quad (t \in [0,T]),$$

and define the continuous linear operator $H : C \to C^1$ by

$$Hu(t) = \int_0^t u(\tau)\, d\tau \quad (t \in [0,T]).$$

A technical result from [4] is needed for the construction of the equivalent fixed point problems.

Proposition 1. *For each $h \in C$, there exists a unique $\alpha := Q_\phi(h) \in$ Range h such that*

$$\int_0^T \phi^{-1}(h(t) - \alpha)\, dt = 0.$$

Moreover, the function $Q_\phi : C \to \mathbb{R}$ is continuous.

The following fixed point reformulation of periodic boundary value problems like (2) is taken from [4].

Proposition 2. *Assume that $F : C^1 \to C$ is continuous and takes bounded sets into bounded sets. Then u is a solution of the abstract periodic problem*

$$(\phi(u'))' = F(u), \quad u(0) - u(T) = 0 = u'(0) - u'(T)$$

if and only if $u \in C^1_\#$ is a fixed point of the operator $M^F_\#$ defined on $C^1_\#$ by

$$M^F_\#(u) = Pu + QF(u) + H \circ \phi^{-1} \circ (I - Q_\phi) \circ [H(I - Q)F](u).$$

Furthermore, $M^F_\#$ is completely continuous on $C^1_\#$.

The following result is a continuation theorem due to Manásevich and Mawhin [4].

Proposition 3. *Let $h : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ be a continuous function and assume that there exists $R > 0$ such that the following conditions hold.*

(i) For each $\lambda \in (0, 1]$ the problem

$$(\phi(u'))' = \lambda h(t, u, u'), \quad u(0) - u(T) = 0 = u'(0) - u'(T),$$

has no solution on ∂B_R.

(ii) The continuous function $\eta : \mathbb{R} \to \mathbb{R}$

$$\eta(d) := \frac{1}{T}\int_0^T h(t, d, 0)dt = 0,$$

is such that $\eta(-R)\eta(R) < 0$.

Then problem

$$(\phi(u'))' = h(t, u, u'), \quad u(0) - u(T) = 0 = u'(0) - u'(T), \tag{3}$$

has a least one solution in B_R, and

$$|d_{LS}[I - M^h_\#, B_R, 0]| = 1,$$

where $M^h_\#$ denotes the fixed point operator associated to (3).

Let us decompose any $u \in C^1_\#$ in the form

$$u = \overline{u} + \tilde{u} \quad (\overline{u} = u(0), \quad \tilde{u}(0) = 0),$$

and let

$$\widetilde{C^1_\#} = \{u \in C^1_\# : u(0) = 0\}.$$

The following inequality will be very useful in the sequel:

$$||\tilde{u}||_\infty \le T^{1/q}||u'||_p \quad \forall u \in C^1_\#, \qquad (Sobolev),$$

where $1/p + 1/q = 1$ and $||u||_p = (\int_0^T u(t)dt)^{1/p}$ for all $u \in C$.

3 Proof of the main result

For $s \in \mathbb{R}$, we define the continuous nonlinear operator $N_s : C^1 \to C$ by

$$N_s(u)(t) = e(t) + s - g(u(t)) - f(u(t))u'(t) \quad (t \in [0,T]).$$

Using Proposition 2, it follows that $u \in C^1_\#$ is a solution of (2) if and only if

$$u = Pu + QN_s(u) + H \circ \phi^{-1} \circ (I - Q_\phi) \circ [H(I-Q)N_s](u) =: \mathcal{G}(s,u),$$

and the nonlinear operator $\mathcal{G}(s,\cdot) : C^1_\# \to C^1_\#$ is completely continuous.

A *strict lower solution* α (resp. *strict upper solution* β) of (2) is a function $\alpha \in C^1$ such that $\phi(\alpha') \in C^1$, $\alpha(0) = \alpha(T)$, $\alpha'(0) \geq \alpha'(T)$ (resp. $\beta \in C^1$, $\phi(\beta') \in C^1$, $\beta(0) = \beta(T)$, $\beta'(0) \leq \beta'(T)$) and

$$(\phi(\alpha'(t)))' + f(\alpha(t))\alpha'(t) + g(\alpha(t)) > e(t) + s$$

(resp.

$$(\phi(\beta'(t)))' + f(\beta(t))\beta'(t) + g(\beta(t)) < e(t) + s)$$

for all $t \in [0,T]$.

Lemma 1. *If $f, g : \mathbb{R} \to \mathbb{R}$ are continuous functions, $e \in C$ and if (2) has a strict lower solution α and a strict upper solution β such that $\alpha(t) \leq \beta(t)$ for all $t \in [0,T]$, then problem (2) has a solution u such that $\alpha(t) < u(t) < \beta(t)$ for all $t \in [0,T]$. Moreover,*

$$|d_{LS}[I - \mathcal{G}(s,\cdot), \Omega^r_{\alpha,\beta}, 0]| = 1,$$

where

$$\Omega^r_{\alpha,\beta} = \{u \in C^1_\# : \alpha(t) < u(t) < \beta(t) \quad \textit{for all} \quad t \in [0,T], \quad \|u'\|_\infty < r\},$$

and r is sufficiently large.

Proof. I. A modified problem.
Let $\gamma : [0,T] \times \mathbb{R} \to \mathbb{R}$ be the continuous function defined by

$$\gamma(t,u) = \begin{cases} \beta(t), & u > \beta(t) \\ u, & \alpha(t) \leq u \leq \beta(t) \\ \alpha(t), & u < \alpha(t). \end{cases}$$

We consider the modified problem

$$\begin{aligned}(|u'|^{p-2}u')' - [u - \gamma(t,u)] + f(\gamma(t,u))u' + g(\gamma(t,u)) = e(t) + s,\\ u(0) - u(T) = 0 = u'(0) - u'(T).\end{aligned} \tag{4}$$

It is not difficult to show that if u is a solution of (4), then $\alpha(t) < u(t) < \beta(t)$ for all $t \in [0,T]$ and hence u is a solution of (2) (see [5], [2]).
II. A priori estimations.

In order to apply Manásevich-Mawhin continuation theorem to problem (4), we consider the family of problems

$$(|u'|^{p-2}u')' - \lambda[u - \gamma(t,u)] + \lambda f(\gamma(t,u))u' + \lambda g(\gamma(t,u)) = \lambda(e(t)+s),$$
$$u(0) - u(T) = 0 = u'(0) - u'(T), \tag{5}$$

where $\lambda \in (0,1]$. Let u be a possible solution of (5). Let $\tau \in [0,T)$ be such that $u(\tau) = \max_{[0,T]} u$. This implies that $(\phi(u'(\tau)))' \leq 0$ and $u'(\tau) = 0$. Hence, using (5), it follows that

$$\lambda u(\tau) \leq \lambda[\gamma(\tau, u(\tau)) + g(\gamma(\tau, u(\tau))) - e(\tau) - s],$$

and there exists a constant $C_1 > 0$ which not depends upon λ and u such that $u(\tau) < C_1$. Analogously, we can prove that there exists a constant C_2 which not depends upon λ and u such that $\min_{[0,T]} u > C_2$. So, there exists $C_3 > 0$ such that

$$||u||_\infty < C_3. \tag{6}$$

Multiplying both members of (5) by u, integrating over $[0,T]$ and using (6), we deduce that there exists $C_4, C_5 > 0$ such that

$$||u'||_p^p < C_4 + C_5||u'||_p,$$

which implies that there exists a constant $C_6 > 0$ such that

$$||u'||_p < C_6. \tag{7}$$

Using (5), (6) and (7) it follows easily that there exists $R > 0$ such that $||u|| < R$, and because, in this case, the function η is given by

$$\eta(d) = d - \frac{1}{T}\int_0^T [\gamma(t,d) + g(\gamma(t,d))]dt + \frac{1}{T}\int_0^T e(t)dt + s,$$

we deduce that R can be chosen such that $\eta(-R)\eta(R) < 0$.
III. End of the proof.
Using II and Manásevich-Mawhin continuation theorem, we deduce that

$$|d_{LS}[I - \mathcal{H}(s,\cdot), B_R, 0]| = 1,$$

where $\mathcal{H}(s,\cdot)$ is the fixed point operator associated to (4). On the other hand, using I, II and Proposition 2, it follows that every fixed point of the nonlinear operator $\mathcal{H}(s,\cdot)$ belongs to $\Omega^r_{\alpha,\beta}$ for r sufficiently large, and by excision property of the Leray-Schauder degree, we deduce that

$$|d_{LS}[I - \mathcal{H}(s,\cdot), \Omega^r_{\alpha,\beta}, 0]| = 1.$$

Because $\mathcal{G}(s,\cdot) = \mathcal{H}(s,\cdot)$ on $\overline{\Omega^r_{\alpha,\beta}}$, it follows that

$$|d_{LS}[I - \mathcal{G}(s,\cdot), \Omega^r_{\alpha,\beta}, 0]| = 1,$$

and by existence property of the Leray-Schauder degree, $\mathcal{G}(s,\cdot)$ has a fixed point in $\Omega^r_{\alpha,\beta}$, which is a solution of (2). ∎

Let $M : C^1 \to C$ be the continuous nonlinear operator defined by

$$M(u)(t) = e(t) - g(u(t)) - f(u(t))u'(t) \quad (t \in [0,T]),$$

and $\widetilde{M} : \mathbb{R} \times \widetilde{C^1_\#} \to \widetilde{C^1_\#}$ be the completely continuous operator defined by

$$\widetilde{M}(\overline{u}, \widetilde{u}) = H \circ \phi^{-1} \circ (I - Q_\phi) \circ [H(I-Q)M](\overline{u} + \widetilde{u}).$$

If u is a solution of (2), then

$$\frac{1}{T}\int_0^T g(u(t))dt = s, \tag{8}$$

and $\widetilde{u} = \widetilde{M}(\overline{u}, \widetilde{u})$. Reciprocally, if $(\overline{u}, \widetilde{u}) \in \mathbb{R} \times \widetilde{C^1_\#}$ is such that $\widetilde{u} = \widetilde{M}(\overline{u}, \widetilde{u})$, then $u = \overline{u} + \widetilde{u}$ is a solution of (2) with $s = \frac{1}{T}\int_0^T g(u(t))dt$. In other words, $(\overline{u}, \widetilde{u}) \in \mathbb{R} \times \widetilde{C^1_\#}$ satisfies $\widetilde{u} = \widetilde{M}(\overline{u}, \widetilde{u})$ if and only if

$$(|\widetilde{u}'|^{p-2}\widetilde{u}')' + f(\overline{u} + \widetilde{u})\widetilde{u}' + g(\overline{u} + \widetilde{u}) = e(t) + \frac{1}{T}\int_0^T g(\overline{u} + \widetilde{u}(t))dt$$

Lemma 2. *If $f, g : \mathbb{R} \to \mathbb{R}$ are continuous functions such that g is bounded and if $e \in C$ satisfies (H1), then the set $\mathcal{S}$ of solutions $(\overline{u}, \widetilde{u}) \in \mathbb{R} \times \widetilde{C^1_\#}$ of problem*

$$\widetilde{u} = \widetilde{M}(\overline{u}, \widetilde{u})$$

contains a subset $\mathcal{C}$ whose projection on $\mathbb{R}$ is $\mathbb{R}$. Moreover, there exists $\rho_1 > 0$ such that

$$||\widetilde{u}||_\infty \le \rho_1 \quad \forall (\overline{u}, \widetilde{u}) \in \mathcal{S} \tag{9}$$

and for all $\epsilon > 0$, there exists $r_\epsilon > 0$ such that

$$||\widetilde{u}'||_\infty \le r_\epsilon \quad \forall (\overline{u}, \widetilde{u}) \in \mathcal{S}, |\overline{u}| \le \epsilon. \tag{10}$$

Proof. For each $\lambda \in [0,1]$ consider the problem

$$(|\widetilde{u}'|^{p-2}\widetilde{u}')' + \lambda f(\overline{u} + \widetilde{u})\widetilde{u}' + \lambda g(\overline{u} + \widetilde{u}) = \lambda e(t) + \frac{\lambda}{T}\int_0^T g(\overline{u} + \widetilde{u}(t))dt, \tag{11}$$

and assume that $(\overline{u}, \widetilde{u}) \in \mathbb{R} \times \widetilde{C^1_\#}$ is a solution of (11). Integrating (11) over $[0,T]$ after multiplication by $\widetilde{u}$, we get, after integration by parts

$$||\widetilde{u}'||_p^p = \lambda \int_0^T [g(\overline{u} + \widetilde{u}(t)) - e(t)]\widetilde{u}dt - \frac{\lambda}{T}\int_0^T g(\overline{u} + \widetilde{u}(t))dt \int_0^T \widetilde{u}(t)dt.$$

Hence, using Sobolev inequality it follows that

$$||\widetilde{u}||_\infty^p \le T^{p/q}||\widetilde{u}'||_p^p \le T^{p/q}[2T \sup_{\mathbb{R}} |g| + ||e||_1]||\widetilde{u}||_\infty,$$

and hence

$$||\widetilde{u}||_\infty \le \{T^{p/q}[2T \sup_{\mathbb{R}} |g| + ||e||_1]\}^{1/p-1} =: \rho_1 \tag{12}$$

and

$$||\widetilde{u}'||_p \leq \{[2T \sup_{\mathbb{R}} |g| + ||e||_1]\rho_1\}^{1/p}. \tag{13}$$

Let $\epsilon > 0$ be fixed and assume that $|\overline{u}| \leq \epsilon$. Using (11), (12) and (13) it follows that

$$||(|\widetilde{u}'|^{p-2}\widetilde{u}')'||_1 \leq C_\epsilon,$$

where C_ϵ depends only on $e, \sup_{\mathbb{R}} |g|$ and $\sup_{[-(\rho_1+\epsilon),\rho_1+\epsilon]} |f|$. As $\widetilde{u}'$ necessarily vanishes at one point, this gives

$$||\widetilde{u}'||_\infty \leq \phi^{-1}(C_\epsilon) =: r_\epsilon. \tag{14}$$

Taking $\lambda = 1$ in (11) and using (12) and (14) we deduce (9) and (10).

Let $\overline{u} \in \mathbb{R}$ be fixed and $\mathcal{M}_{\overline{u}} : [0,1] \times \widetilde{C^1_\#} \to \widetilde{C^1_\#}$ be the completely continuous operator defined by

$$\mathcal{M}_{\overline{u}}(\lambda, \widetilde{u}) = H \circ \phi^{-1} \circ (I - Q_\phi) \circ [\lambda H(I - Q)M](\overline{u} + \widetilde{u}).$$

For $(\lambda, \widetilde{u}) \in [0,1] \times \widetilde{C^1_\#}$, we have that $\mathcal{M}_{\overline{u}}(\lambda, \widetilde{u}) = \widetilde{u}$ if and only if $(\overline{u}, \widetilde{u})$ is a solution of (11). Hence, using (12), (14) and the homotopy invariance property of the Leray-Schauder degree, it follows that

$$\begin{aligned} d_{LS}[I - \mathcal{M}_{\overline{u}}(1, \cdot), B_r, 0] &= d_{LS}[I - \mathcal{M}_{\overline{u}}(0, \cdot), B_r, 0] \\ &= d_{LS}[I, B_r, 0] = 1, \end{aligned}$$

for some r sufficiently large. This, together with the existence property of the Leray-Schauder degree give the existence of some $\widetilde{u} \in \widetilde{C^1_\#}$ such that $\widetilde{M}(\overline{u}, \widetilde{u}) = \mathcal{M}_{\overline{u}}(1, \widetilde{u}) = \widetilde{u}$. This completes the proof. ∎

In what follows we assume that (H1)-(H3) hold.

Let us define

$$S_j = \{s \in \mathbb{R} : (2) \text{ has at least j solutions } \} \quad (j \geq 1).$$

Lemma 3. *If $s \in S_1$, then $0 < s \leq \sup_{\mathbb{R}} |g|$.*

Proof. Assumptions (H2) and (H3) imply that g is bounded and $0 < g(u) \leq \sup_{\mathbb{R}} |g|$ for all $u \in \mathbb{R}$. Hence, if u is a solution of (2) then, using (H1), it follows that (8) holds and $0 < s \leq \sup_{\mathbb{R}} |g|$. ∎

Let $\gamma : \mathbb{R} \times \widetilde{C^1_\#} \to \mathbb{R}$ be the continuous function defined by

$$\gamma(\overline{u}, \widetilde{u}) = \frac{1}{T} \int_0^T g(\overline{u} + \widetilde{u}(t))\, dt.$$

Lemma 4. $S_1 \neq \varnothing$.

Proof. Let $(\overline{u}, \widetilde{u}) \in \mathcal{C}$, where $\mathcal{C}$ is given in Lemma 2. Then $u = \overline{u} + \widetilde{u}$ is a solution of (2) with $s = \gamma(\overline{u}, \widetilde{u})$. ∎

Let us consider

$$s^*(e) = \sup S_1.$$

Lemma 5. *We have that* $0 < s^*(e) \le \sup_{\mathbb{R}} |g|$ *and* $s^*(e) \in S_1$.

Proof. The first assertion follows from Lemma 3. Let $\{s_n\}$ be a sequence belonging to S_1 which converges to $s^*(e)$. Let $u_n = \overline{u}_n + \widetilde{u}_n$ be a solution of (2) with $s = s_n = \gamma(\overline{u}_n, \widetilde{u}_n)$. It follows that $\widetilde{u}_n = \widetilde{M}(\overline{u}_n, \widetilde{u}_n)$. Hence, if up to a subsequence $\overline{u}_n \to \pm\infty$, then using (9) and (H3), it follows that $\gamma(\overline{u}_n, \widetilde{u}_n) \to 0$, which means that $s^*(e) = 0$, contradiction. We have proved that $\{\overline{u}_n\}$ is a bounded sequence in $\mathbb{R}$ and using (9) and (10) it follows that $\{(\overline{u}_n, \widetilde{u}_n)\}$ is a bounded sequence in $\mathbb{R} \times \widetilde{C^1_{\#}}$. Because $\widetilde{M}$ is completely continuous, we can assume, passing to a subsequence, that $\widetilde{M}(\overline{u}_n, \widetilde{u}_n) \to \widetilde{u}$ and $\overline{u}_n \to \overline{u}$. We deduce that $\widetilde{u} = \widetilde{M}(\overline{u}, \widetilde{u})$, $\gamma(\overline{u}, \widetilde{u}) = s^*(e)$ and u is a solution of (2) with $s = s^*(e)$. ■

Arguing as in the proof of Lemma 5 we deduce the following a priori estimate result.

Lemma 6. *Let* $0 < s_1 < s^*(e)$. *Then, there is* $\rho' > 0$ *such that any possible solution* u *of (2) with* $s \in [s_1, s^*(e)]$ *belongs to* $B_{\rho'}$.

Lemma 7. *We have* $(0, s^*(e)) \subset S_2$.

Proof. Let $s_1, s_2 \in \mathbb{R}$ such that $0 < s_1 < s^*(e) < s_2$. Using Lemma 3, Lemma 6 and the invariance property of the Leray-Schauder degree, it follows that there is $\rho' > 0$ sufficiently large such that $d_{LS}[I - \mathcal{G}(s, \cdot), B_{\rho'}, 0]$ is well defined and independent of $s \in [s_1, s_2]$. However, using Lemma 3 we deduce that $u - \mathcal{G}(s_2, u) \neq 0$ for all $u \in C^1_{\#}$. This implies that $d_{LS}[I - \mathcal{G}(s_2, \cdot), B_{\rho'}, 0] = 0$, so that $d_{LS}[I - \mathcal{G}(s_1, \cdot), B_{\rho'}, 0] = 0$ and, by excision property of the Leray-Schauder degree,

$$d_{LS}[I - \mathcal{G}(s_1, \cdot), B_{\rho''}, 0] = 0 \quad \text{if} \quad \rho'' \ge \rho'. \tag{15}$$

Let u_* be a solution of (2) with $s = s^*(e)$ given by Lemma 5. Then, u_* is a strict lower solution of (2) with $s = s_1$. Using Lemma 2 and (H3), there is $(\overline{u}^*, \widetilde{u}^*) \in \mathcal{C}$ such that $u^* = \overline{u}^* + \widetilde{u}^* > u_*$ on $[0, T]$ and $\gamma(\overline{u}^*, \widetilde{u}^*) < s_1$. It follows that u^* is a strict upper solution of (2) with $s = s_1$. So, using Lemma 1, we have that

$$|d_{LS}[I - \mathcal{G}(s_1, \cdot), \Omega^r_{u_*, u^*}, 0]| = 1, \tag{16}$$

for some $r > 0$, and (1) has a solution in $\Omega^r_{u_*, u^*}$. Taking ρ'' sufficiently large and using (15) and (16), we deduce from the additivity property of the Leray-Schauder degree that

$$\begin{aligned} |d_{LS}[I - \mathcal{G}(s_1, \cdot), B_{\rho''} \backslash \overline{\Omega^r}_{u_*, u^*}, 0]| &= |d_{LS}[I - \mathcal{G}(s_1, \cdot), B_{\rho''}, 0] \\ -d_{LS}[I - \mathcal{G}(s_1, \cdot), \Omega^r_{u_*, u^*}, 0]| &= |d_{LS}[I - \mathcal{G}(s_1, \cdot), \Omega^r_{u_*, u^*}, 0]| = 1, \end{aligned}$$

and (2) with $s = s_1$ has a second solution in $B_{\rho''} \setminus \overline{\Omega^r}_{u_*, u^*}$. ■

End of the proof of Theorem 2. The conclusion of Theorem 2 follows from Lemmas 3, 5 and 7. ■

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