On maximal $t-$orthogonal sequences in c_0

Albert Kubzdela

Abstract

Let K be a non-Archimedean, complete, densely valued field. For a given $t \in (0,1)$ we study a maximality of $t-$orthogonal sequences in c_0 over K. In particular we prove that for every $t \in (0,1)$ there exists a maximal $t-$orthogonal sequence in c_0 which is not a base.

1 Introduction

Throughout this paper K denotes a non-Archimedean valued field which is complete with respect to the metric induced by the non-trivial dense valuation $|.| : K \to [0, \infty)$ (recall that a valuation $|.|$ is *dense* if the set of its values is dense in $[0, \infty)$). Let E be a normed space over K; we assume that the norm defined on E is *non-Archimedean* (i.e. it satisfies 'the strong triangle inequality': $||x + y|| \leq \max \{||x||, ||y||\}$ for all $x, y \in E$). By E' we mean the topological dual of E which is a normed space with the norm $||f|| = \sup_{x \in E, x \neq 0} \frac{|f(x)|}{||x||}$.

For the basic notions and properties concerning normed spaces over K we refer the reader to [1]. However we recall the following. We say that for a closed linear subspace D of E and for $x \in E \backslash D$ the distance $dist(x, D) := \inf_{d \in D} ||x - d||$ is *not attained* if $||x - d|| > dist(x, D)$ for all $d \in D$. If there exists $d_0 \in D$ such that $||x - d_0|| = dist(x, D)$ we say that $dist(x, D)$ is *attained*. Two linear subspaces $D, G \subset E$ are called *orthocomplemented* if $||x + y|| = \max \{||x||, ||y||\}$ for all $x \in D$ and $y \in G$.

Let $t \in (0, 1]$ and let $M \subseteq N$. We say that a sequence (finite or infinite) $(x_i)_{i \in M}$ of nonzero elements of E is called $t-$*orthogonal* (*orthogonal* if $t = 1$) if for every finite subset $J \subset M$ and all scalars $\{\lambda_j\}_{j \in J}$ we have $\left\| \sum_{j \in J} \lambda_j x_j \right\| \geq t \cdot \max_{j \in J} \{||\lambda_j x_j||\}$. If, additionally $\overline{\left[(x_i)_{i \in M}\right]} = E$, the sequence $(x_i)_{i \in M}$ is called a *base* of E. By Theorem 3.16 of [1], every infinite-dimensional E contains an infinite $t-$*orthogonal* sequence

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if $t < 1$ and if K is spherically complete (i.e. every centered sequence of closed balls in K has a non-empty intersection), then such E contains an infinite orthogonal sequence. Clearly, every infinite $t-$orthogonal sequence is a basic sequence in E. We say that a $t-$orthogonal sequence $(x_i)_{i\in M}$ of E is *maximal* if $\{z\}\cup\{x_i : i \in M\}$ is not $t-$orthogonal for any nonzero $z \in E$. It is easy to observe that every $t-$orthogonal sequence in E can be extended to a maximal one. Obviously, every $t-$orthogonal sequence which is a base of E is maximal in E. But, it was noted (see Remark after Theorem 3.16 of [1]) that c_0 contains a maximal orthogonal sequence which is not a base. Hence, it is natural to formulate the following question.
problem Is for a given $t \in (0, 1)$ every maximal $t-$orthogonal sequence in c_0 a base of c_0?

This paper contains the answer to this question. In Theorem 1, for every $t \in (0, 1)$ we construct a maximal $t-$orthogonal sequence in c_0 which is not a base.

2 Results

We start with simple observations.

Lemma 1. *Let $D \subset E$ be a closed, proper, infinite-dimensional linear subspace of E. If there exists $a_0 \in E\backslash D$ such that $dist(a_0, D)$ is not attained, then $dist(a_0, F) > dist(a_0, D)$ for every F, a finite-dimensional linear subspace of D.*

proof: Assume that there exists $F \subset D$ with $dist(a_0, F) = dist(a_0, D)$. Then, by Theorem 5.7 and Theorem 5.13 of [1], F is orthocomplemented in $F + [a_0]$; hence, there exists $x \in F$ with $||a_0 - x|| = dist\,(a_0, F) = dist\,(a_0, D)$, a contradiction.

Recall that a linear subspace $D \subset E$ is called a hyperplane of E if $\dim(E/D) = 1$.

Lemma 2. *Let D be a closed hyperplane of E. Let $x_0 \in E\backslash D$. If $dist(x_0, D)$ is attained (not attained), then $dist(x, D)$ is attained (not attained) for all $x \in E\backslash D$.*

Proof. Taking $x \in E\backslash D$, we can write $x = \lambda x_0 + d_x$ for some $\lambda \in K$ $(\lambda \neq 0)$ and some $d_x \in D$. Suppose that $dist(x_0, D)$ is not attained and assume that there exists $d_0 \in D$ such that $dist(x, D) = ||x - d_0||$. Then

$$||x - d_0|| = |\lambda| \cdot \left|\left| x_0 + \frac{d_x - d_0}{\lambda} \right|\right|.$$

By assumption, there exists $d \in D$ such that

$$||x_0 + d|| < \left|\left| x_0 + \frac{d_x - d_0}{\lambda} \right|\right|.$$

Thus,

$$\begin{aligned} ||x + (\lambda d - d_x)|| &= ||(\lambda x_0 + d_x) + (\lambda d - d_x)|| = |\lambda| \cdot ||x_0 + d|| \\ &< |\lambda| \cdot \left|\left| x_0 + \frac{d_x - d_0}{\lambda} \right|\right| = ||\lambda x_0 + d_x - d_0|| = ||x - d_0||, \end{aligned}$$

a contradiction. Assuming that $dist(x_0, D)$ is attained, we conclude from the above that $dist(x, D)$ is attained for all $x \in E \backslash D$. ■

Proposition 1. *Let $t \in (0,1]$ and $(x_n)_{n\in N}$ be a $t-$orthogonal sequence in E. Let $D = \overline{[(x_n)_{n\in N}]}$. If there exists $a \in E\backslash D$ such that $dist(a, D)$ is attained then $(x_n)_{n\in N}$ is not maximal $t-$orthogonal sequence in E.*

Proof. Let $a \in E\backslash D$ and assume that there exists $x \in D$ such that $||a - x|| = dist(a, D)$. Denoting $a_0 = a - x$, we get $||a_0 - d|| \geq ||a_0||$ for all $d \in D$. Thus, for every $m \in N$ and for all $\mu_1, ..., \mu_m \in K$ we obtain

$$\left|\left|a_0 + \sum_{j=1}^{m} \mu_j x_j\right|\right| \geq \max\left\{\left|\left|\sum_{j=1}^{m} \mu_j x_j\right|\right|, ||a_0||\right\} \geq t \cdot \max\left\{\max_{j=1,...,m} ||\mu_j x_j||, ||a_0||\right\},$$

since, by assumption $\left|\left|\sum_{j=1}^{m} \mu_j x_j\right|\right| \geq t \cdot \max_{j=1,...,m} ||\mu_j x_j||$. Hence, $\{a_0, x_1, x_2, ...\}$ is a $t-$orthogonal sequence in E. ■

From now on in this paper we assume that $E = c_0$. By $\{e_1, e_2, ...\}$ we will denote a standard base of E.

Remark 1. *Taking $x_n := e_{n+1}$ $(n \in N), a := e_1$ we get a simple example of an orthogonal sequence in E which satisfies conditions of Proposition 1.*

Note that linear subspaces of E which do not satisfy assumptions of Proposition 1 exist. Examples can be constructed using the next proposition. Recall that by Exercise 3.Q of [1], $E' = l^\infty$ and every $f \in E'$ is given by the formula

$$f(x) = \sum_{n\in N} u_n x_n,$$

for some $u = (u_1, u_2, u_3, ...) \in l^\infty$, where $x = (x_1, x_2, x_3, ...) \in E$.

Proposition 2. *Let $u = (u_1, u_2, u_3, ...) \in l^\infty$ and $f \in E'$ be defined by $f(z) = \sum_{n\in N} u_n z_n$, where $z = (z_1, z_2, z_3, ...) \in E$. Denote by $D = \ker(f)$. Then, $dist(x, D)$ is attained for every $x \in E\backslash D$ if and only if $\max_{n\in N} |u_n|$ exists.*

Proof. Let $z = (z_1, z_2, z_3, ...) \in E$. Since

$$\frac{|f(z)|}{||z||} = \frac{|\sum_{n\in N} u_n z_n|}{||\sum_{n\in N} z_n e_n||} \leq \frac{\max_{n\in N} |u_n z_n|}{\max_{n\in N} |z_n|} \leq \sup_{n\in N} |u_n|$$

and

$$\sup_{n\in N} \frac{|f(e_n)|}{||e_n||} = \sup_{n\in N} \frac{|u_n|}{||e_n||} = \sup_{n\in N} |u_n|,$$

we note that the norm of f is reached on $\{e_1, e_2, ...\}$; *i.e.*

$$||f|| = \sup_{n\in N} \frac{|f(e_n)|}{||e_n||} = \sup_{n\in N} |u_n|.$$

Assume that $\max_{n\in N}|u_n|$ exists, then $||f||=|u_m|$ for some $m\in N$ and $f(e_m)\notin D$. More, $dist\,(e_m,D)=||e_m||$. If not, then there exists $d\in D$ with $||e_m-d||<||e_m||$. But then

$$\frac{|f(e_m-d)|}{||e_m-d||}=\frac{|f(e_m)|}{||e_m-d||}>\frac{|f(e_m)|}{||e_m||}=||u_m||=||f||,$$

a contradiction. It follows from Lemma 2 that $dist(x,D)$ is attained for every $x\in E\backslash D$.

Suppose now that $dist(x,D)$ is attained for every $x\in E\backslash D$ and assume that $\max_{n\in N}|u_n|$ does not exist (thus, $||f||>\frac{|f(z)|}{||z||}$ for all $z\in E$). Then, we can choose a strictly increasing sequence $(n_k)_k\subset N$ with $||f||=\lim_{k\to\infty}|u_{n_k}|$. Taking $p>1$, we see that $f\left(e_{n_p}\right)\neq 0$, $x_r=e_{n_p}-\frac{u_{n_p}}{u_{n_r}}e_{n_r}\in D$ for all $r>p$ and $\lim_{r\to\infty}\left|\left|e_{n_p}-x_r\right|\right|=\lim_{r\to\infty}\left|\frac{u_{n_p}}{u_{n_r}}\right|$; thus, $dist(e_{n_p},D)\le\lim_{r\to\infty}\left|\frac{u_{n_p}}{u_{n_r}}\right|$ and by assumption we can choose $d\in D$ such that $\left|\left|e_{n_p}-d\right|\right|\le\lim_{r\to\infty}\left|\frac{u_{n_p}}{u_{n_r}}\right|$. But then, we get

$$\frac{\left|f(e_{n_p}-d)\right|}{\left|\left|e_{n_p}-d\right|\right|}=\frac{\left|f(e_{n_p})\right|}{\left|\left|e_{n_p}-d\right|\right|}\ge\lim_{r\to\infty}|u_{n_r}|,$$

a contradiction. ■

Now, we prove the main theorem.

Theorem 1. *For every $t\in(0,1)$ there exists a maximal $t-$orthogonal sequence in E which is not a base.*

Proof. Let $0<t<1$. Choose a sequence $(a_n)_{n\in N}\subset K$ (recall that by assumption K is densely valued) such that

$$1=|a_1|<...<|a_n|<|a_{n+1}|<...<\frac{1}{t}.$$

Now, define elements of E as follows

$$b_{3n-2}=e_{3n-2}+a_{k_n}e_{k_n}-\frac{a_{3n-2}}{a_{3(n+1)-2}}e_{3(n+1)-2}$$

$$b_{3n-1}=e_{k_n}+\frac{a_{l_n}}{a_{k_n}}e_{l_n}$$

$$b_{3n}=e_{3n}\quad(n\in N),$$

selecting $k_n,l_n\in N$ such that $k_n=3i_n,l_n=3j_n$ for some $i_n,j_n\in N$, $k_n\ge 3n$, $l_n>k_n$,

$$|a_{3n-2}|<t\cdot\left|a_{3(n+1)-2}\right|\cdot|a_{l_n}| \tag{1}$$

and $l_n<k_{n+1}$ for all $n\in N$. Let $N_k=\{k_n:n\in N\}$, $N_l=\{l_n:n\in N\}$ (observe that $N_k\cap N_l=\emptyset$) and let $N_0=N\backslash N_k$.

Now, we prove that $X_0=\{b_k:k\in N_0\}$ is a $t-$orthogonal sequence in E. To this end take a finite subset $J\subset N_0$, $\{\lambda_i\}_{i\in J}\subset K$ and assume that $\max_{i\in J}||\lambda_ib_i||=||\lambda_{i_0}b_{i_0}||>0$ for some $i_0\in J$.

First, we note that applying properties of the sequence $(a_n)_{n\in N}$ we have

$$\left|\left|b_{l_n}-\frac{a_{k_n}}{a_{l_n}}b_{3n-1}+\frac{1}{a_{l_n}}b_{3n-2}\right|\right|$$
$$=\left|\left|e_{l_n}-(\frac{a_{k_n}}{a_{l_n}}e_{k_n}+e_{l_n})+(\frac{1}{a_{l_n}}e_{3n-2}+\frac{a_{k_n}}{a_{l_n}}e_{k_n}-\frac{a_{3n-2}}{a_{l_n}a_{3(n+1)-2}}e_{3(n+1)-2})\right|\right|$$
$$=\left|\left|\frac{1}{a_{l_n}}e_{3n-2}-\frac{a_{3n-2}}{a_{l_n}a_{3(n+1)-2}}e_{3(n+1)-2}\right|\right|=\left|\frac{1}{a_{l_n}}\right| \quad (2)$$

for every $n\in N$ and

$$||b_{l_n}-\frac{a_{k_n}}{a_{l_n}}b_{3n-1}+\frac{1}{a_{l_n}}b_{3n-2}$$
$$+\frac{1}{a_{l_n}}\frac{a_{3n-2}}{a_{3(n-1)-2}}b_{3(n-1)-2}-\frac{a_{k_{n-1}}}{a_{l_n}}\frac{a_{3n-2}}{a_{3(n-1)-2}}b_{3(n-1)-1}+\frac{a_{l_{n-1}}}{a_{l_n}}\frac{a_{3n-2}}{a_{3(n-1)-2}}b_{l_{n-1}}||$$
$$=||e_{l_n}-(\frac{a_{k_n}}{a_{l_n}}e_{k_n}+e_{l_n})+(\frac{1}{a_{l_n}}e_{3n-2}+\frac{a_{k_n}}{a_{l_n}}e_{k_n}-\frac{a_{3n-2}}{a_{l_n}a_{3(n+1)-2}}e_{3(n+1)-2})$$
$$+(\frac{1}{a_{l_n}}\frac{a_{3n-2}}{a_{3(n-1)-2}}e_{3(n-1)-2}+\frac{a_{k_{n-1}}}{a_{l_n}}\frac{a_{3n-2}}{a_{3(n-1)-2}}e_{k_{n-1}}-\frac{1}{a_{l_n}}e_{3n-2})$$
$$-(\frac{a_{k_{n-1}}}{a_{l_n}}\frac{a_{3n-2}}{a_{3(n-1)-2}}e_{k_{n-1}}+\frac{a_{l_{n-1}}}{a_{l_n}}\frac{a_{3n-2}}{a_{3(n-1)-2}}e_{l_{n-1}})+\frac{a_{l_{n-1}}}{a_{l_n}}\frac{a_{3n-2}}{a_{3(n-1)-2}}e_{l_{n-1}}||$$
$$=||-\frac{a_{3n-2}}{a_{l_n}a_{3(n+1)-2}}e_{3(n+1)-2}+\frac{1}{a_{l_n}}\frac{a_{3n-2}}{a_{3(n-1)-2}}e_{3(n-1)-2}||$$
$$=\left|\frac{1}{a_{l_n}}\frac{a_{3n-2}}{a_{3(n-1)-2}}\right|>\left|\frac{1}{a_{l_n}}\right| \quad (3)$$

for $n=2,3,\ldots$.

Now, consider the following cases:

- $i_0=3n$ for some $n\in N$. If $i_0\notin N_l$ then $||\sum_{i\in J}\lambda_i b_i||=\max_{i\in J}||\lambda_i b_i||=||\lambda_{i_0}b_{i_0}||$. Suppose that $i_0\in N_l$, then $i_0=l_n$ for some $n\in N$. We get $||\lambda_{l_n}b_{l_n}||=||\lambda_{l_n}e_{l_n}||=|\lambda_{l_n}|$ and applying (2) and (3) we obtain

$$\left|\left|\sum_{i\in J}\lambda_i b_i\right|\right|\geq\left|\left|\lambda_{l_n}b_{l_n}-\lambda_{l_n}\frac{a_{k_n}}{a_{l_n}}b_{3n-1}+\lambda_{l_n}\frac{1}{a_{l_n}}b_{3n-2}\right|\right|$$
$$=|\lambda_{l_n}|\cdot\left|\left|b_{l_n}-\frac{a_{k_n}}{a_{l_n}}b_{3n-1}+\frac{1}{a_{l_n}}b_{3n-2}\right|\right|=\left|\frac{\lambda_{l_n}}{a_{l_n}}\right|>t\cdot|\lambda_{l_n}|=t\cdot\max_{i\in J}||\lambda_i b_i||$$

  (note that $||\lambda_{l_n}b_{l_n}+\lambda_j b_j||<||\lambda_{l_n}b_{l_n}||$ only if $j=3n-1$ and $||\lambda_{l_n}b_{l_n}+\lambda_j b_j+\lambda_l b_l||<||\lambda_{l_n}b_{l_n}+\lambda_j b_j||$ only if $l=3n-2$).

- If $i_0=3n-1$ for some $n\in N$, then we obtain

$$||\lambda_{3n-1}b_{3n-1}||=\left|\left|\lambda_{3n-1}e_{k_n}+\lambda_{3n-1}\frac{a_{l_n}}{a_{k_n}}e_{l_n}\right|\right|=\left|\lambda_{3n-1}\frac{a_{l_n}}{a_{k_n}}\right|$$

and using (2) and (3) we get

$$\left\|\sum_{i\in J}\lambda_i b_i\right\| \geq \left\|\lambda_{3n-1}b_{3n-1}-\lambda_{3n-1}\frac{a_{l_n}}{a_{k_n}}b_{l_n}-\lambda_{3n-1}\frac{1}{a_{k_n}}b_{3n-2}\right\|$$
$$=\left|\lambda_{3n-1}\frac{a_{l_n}}{a_{k_n}}\right|\cdot\left\|\frac{a_{k_n}}{a_{l_n}}b_{3n-1}-b_{l_n}-\frac{1}{a_{l_n}}b_{3n-2}\right\|$$
$$=\left|\lambda_{3n-1}\frac{1}{a_{k_n}}\right| > t\cdot\left|\lambda_{3n-1}\frac{a_{l_n}}{a_{k_n}}\right| = t\cdot\max_{i\in J}\|\lambda_i b_i\|,$$

since $\|\lambda_{3n-1}b_{3n-1}+\lambda_j b_j\| < \|\lambda_{3n-1}b_{3n-1}\|$ only if $j=l_n$ and $\|\lambda_{3n-1}b_{3n-1}+\lambda_{l_n}b_{l_n}+\lambda_l b_l\| < \|\lambda_{3n-1}b_{3n-1}+\lambda_{l_n}b_{l_n}\|$ only if $l=3n-2$.

- Assuming that $i_0=3n-2$ for some $n\in N$, we have

$$\|\lambda_{3n-2}b_{3n-2}\| = |\lambda_{3n-2}a_{k_n}|,$$

observing that $\|\lambda_{3n-2}b_{3n-2}+\lambda_j b_j+\lambda_l b_l\| < \|\lambda_{3n-2}b_{3n-2}\|$ only if $j=l_n$ and $l=3n-1$ and applying (2) and (3) again, we calculate

$$\left\|\sum_{i\in J}\lambda_i b_i\right\| \geq \|\lambda_{3n-2}b_{3n-2}-\lambda_{3n-2}a_{k_n}b_{3n-1}+\lambda_{3n-2}a_{l_n}b_{l_n}\|$$
$$=|\lambda_{3n-2}a_{l_n}|\cdot\left\|\frac{1}{a_{l_n}}b_{3n-2}-\frac{a_{k_n}}{a_{l_n}}b_{3n-1}+b_{l_n}\right\|$$
$$=|\lambda_{3n-2}| > t\cdot|\lambda_{3n-2}a_{k_n}| = t\cdot\max_{i\in J}\|\lambda_i b_i\|.$$

In this way we prove that X_0 is $t-$orthogonal.

Note that, doing simple calculations, we have

$$\begin{aligned}
&\left\|e_1-\sum_{n=1}^{m}\frac{a_1}{a_{3n-2}}\left(b_{3n-2}-a_{k_n}b_{3n-1}+a_{l_n}b_{l_n}\right)\right\| \\
&\quad=\|e_1-(e_1+a_{k_1}e_{k_1}-\frac{a_1}{a_4}e_4-a_{k_1}e_{k_1}-a_{l_1}e_{l_1}+a_{l_1}e_{l_1}) \\
&\quad-\frac{a_1}{a_4}(e_4+a_{k_2}e_{k_2}-\frac{a_4}{a_7}e_7-a_{k_2}e_{k_2}-a_{l_2}e_{l_2}+a_{l_2}e_{l_2})-... \\
&...-\frac{a_1}{a_{3m-2}}(e_{3m-2}+a_{k_m}e_{k_m}-\frac{a_{3m-2}}{a_{3(m+1)-2}}e_{3(m+1)-2}-a_{k_m}e_{k_m}-a_{l_m}e_{l_m}+a_{l_m}e_{l_m})\| \\
&\quad=\left|\frac{a_1}{a_{3(m+1)-2}}\right|<1 \qquad (4)
\end{aligned}$$

and easily observe that

$$\begin{aligned}
dist\,(e_1,[X_0]) &= \lim_{m\to\infty}\left\|e_1-\sum_{n=1}^{m}\frac{a_1}{a_{3n-2}}\left(b_{3n-2}-a_{k_n}b_{3n-1}+a_{l_n}b_{l_n}\right)\right\| \\
&= \lim_{m\to\infty}\left|\frac{a_1}{a_{3(m+1)-2}}\right| = t\cdot|a_1| = t. \qquad (5)
\end{aligned}$$

Clearly, $dist\,(e_1, [X_0])$ is not attained.

Now, we prove that X_0 is a maximal $t-$orthogonal sequence in $\overline{[\{e_1\} \cup X_0]}$. Taking $w \in [X_0]$, (we can write $w = \sum_{i=1}^{m_0} \lambda_i b_i$ for some $m_0 \in N$ and $\lambda_1, ..., \lambda_{m_0} \in K$) we show that $\{e_1 + w\} \cup X_0$ is not $t-$orthogonal sequence in $\overline{[\{e_1\} \cup X_0]}$. Since $dist\,(e_1, [X_0])$ is not attained, using (4) and (5), we can select $m > m_0 + 3$ such that

$$\left\|e_1 - \sum_{n=1}^{m} \frac{a_1}{a_{3n-2}} \left(b_{3n-2} - a_{k_n} b_{3n-1} + a_{l_n} b_{l_n}\right)\right\| < \|e_1 + w\|.$$

Let

$$z = w + \sum_{n=1}^{m} \frac{a_1}{a_{3n-2}} \left(b_{3n-2} - a_{k_n} b_{3n-1} + a_{l_n} b_{l_n}\right).$$

Since $z \in [X_0]$, we can write, choosing proper scalars $\beta_1, ..., \beta_{l_m} \in K$, $z = \sum_{i=1}^{l_m} \beta_i b_i$. In particular we have

$$\beta_{l_m} = \frac{a_1}{a_{3m-2}} a_{l_m}, \beta_{3m-1} = \frac{a_1}{a_{3m-2}} a_{k_m}, \; \beta_{3m-2} = \frac{a_1}{a_{3m-2}},$$

thus, we get

$$\max_{i=1,\dots,l_m} \{\|\beta_i b_i\|\} \geq \left|\frac{a_1}{a_{3m-2}} a_{l_m}\right|.$$

On the other hand, using (4) and (1) we obtain

$$\begin{aligned} \|e_1 + w - z\| = & \left\|e_1 - \sum_{n=1}^{m} \frac{a_1}{a_{3n-2}} \left(b_{3n-2} - a_{k_n} b_{3n-1} + a_{l_n} b_{l_n}\right)\right\| \\ & = \left|\frac{a_1}{a_{3(m+1)-2}}\right| < t \cdot \left|\frac{a_1}{a_{3m-2}} a_{l_m}\right| \leq t \cdot \max_{i=1,\dots,l_m} \{\|\beta_i b_i\|\} \end{aligned}$$

and conclude that X_0 is maximal in $\overline{[\{e_1\} \cup X_0]}$.

It is easy to check that $E = \overline{[\{e_1\} \cup X_0 \cup \{e_{3n-1} : n \in N\}]}$ and that $\overline{[\{e_1\} \cup X_0]}$ is orthocomplemented to $\overline{[\{e_{3n-1} : n \in N\}]}$. Hence, taking $X_m = X_0 \cup \{e_{3n-1} : n \in N\}$ we get a maximal $t-$orthogonal sequence in E which is not a base of E and complete the proof. ∎

Remark 2. *Note, that the closed hyperplane $D = \overline{[X_m]}$ of E, where X_m is the $t-$orthogonal sequence constructed in the proof of Theorem 1, can be obtained as a* $\ker(f)$, *for $f \in E'$, induced by $(a_1, 0, 0, a_4, 0, 0, a_7, ...) \in l^\infty$ (where $a_1, a_4, a_7, ...$ are defined in the proof of Theorem 1). Observe that $f(e_{3n-2}) = a_{3n-2}$, $f(e_{3n-1}) = f(e_{3n}) = 0 \quad (n \in N)$ and $f(b_k) = 0$ for all $k \in N_0$. Since* $\sup_{n \in N} |a_{3n-2}|$ *is not attained, it follows from Lemma 2 and Proposition 2 that dist (x, D) is not attained for every $x \in E \backslash D$.*

References

[1] Rooij, A.C.M. van - Non-Archimedean Functional Analysis. Marcel Dekker, New York (1978).