Composition operators acting on $\mathcal{N}_p$-spaces

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Abstract

We introduce a new class of functions, called the $\mathcal{N}_\mathrm{p}$-spaces and study the boundedness and compactness of composition operators on $\mathcal{N}_\mathrm{p}$-spaces as well as between $\mathcal{N}_\mathrm{p}$-spaces and Bergman-type spaces. The paper is intended to give a self-contained introduction the the $\mathcal{N}_\mathrm{p}$-spaces.

1 Introduction

Let $H(\mathbb{D})$ denote the space of analytic functions on the unit disk $\mathbb{D}$. In this paper, in order to simplify some calculations, we will identify functions differing by a constant. Thus, the word function will from here on mean an equivalence class of functions modulo constants. The Q_p-spaces (with $p \in (0,\infty)$) were introduced by Aulaskari, Xiao and Zhao [1] and consist of functions in $H(\mathbb{D})$ such that

$$\|f\|_{Q_p} := \sup_{a\in\mathbb{D}} \left(\int_{\mathbb{D}} |f'(z)|^2 (1-|\sigma_a(z)|^2)^p dA(z) \right)^{\frac{1}{2}} < \infty.$$

Here $\sigma_a(z) := (a-z)/(1-\overline{a}z)$ is the automorphism of $\mathbb{D}$ that changes 0 and a, while dA denotes the Lebesgue area measure on the plane, normalized so that $A(\mathbb{D}) = 1$. The Q_p-spaces coincide with the classical Bloch space $\mathcal{B}$ for $p \in (1,\infty)$, while Q_1 is equal to BMOA, the space of analytic functions on $\mathbb{D}$ with bounded mean oscillation. For $p \in (0,1)$, the Q_p-spaces are all different and of independent interest. A good source of information about the Q_p-spaces are the Springer Lecture Notes by Xiao [12].

The Bergman-type spaces $\mathcal{A}^{\text{-q}}$ (with $q \in (0,\infty)$) consist of functions in $H(\mathbb{D})$ such that

$$\|f\|_{\mathcal{A}^{\text{-q}}} := \sup_{z\in\mathbb{D}} |f(z)|(1-|z|^2)^q < \infty.$$

*The research of the author was supported by the Academy of Finland Project No. 205644.

2000 *Mathematics Subject Classification* : Primary 47B33, 46E15; Secondary 30B10.

Key words and phrases : Composition operator, $\mathcal{N}_p$-spaces, Bergman-type spaces, $\mathcal{Q}_p$-spaces, Nevanlinna counting function, Hadamard gap series.

Bull. Belg. Math. Soc. Simon Stevin 14 (2007), 545–554

These Banach spaces have been intensively studied in many papers (see for example [4] and the related references therein).

Throughout the paper $\varphi \in H(\mathbb{D})$ will denote a non-constant function satisfying $\varphi(\mathbb{D}) \subseteq \mathbb{D}$, while $\psi \in H(\mathbb{D})$ will be any function not identically equal to zero. The composition operator C_φ and the multiplication operator M_ψ are the linear maps on $H(\mathbb{D})$ defined by $C_\varphi f := f \circ \varphi$ and $M_\psi f := \psi f$, respectively. The weighted composition operator $W_{\varphi,\psi}$ is obtained by a combination of these operators by defining $W_{\varphi,\psi} := M_\psi C_\varphi$. Classical books on composition operators are [9] and [3].

In this paper we formally introduce the $\mathcal{N}_{\mathrm{p}}$-spaces, (with $p \in (0, \infty)$), of which $\mathcal{N}_1$ was introduced in [5] (see also [7, Remark 4.4]). The $\mathcal{N}_{\mathrm{p}}$-spaces consist of functions in $H(\mathbb{D})$ such that

$$\|f\|_{\mathcal{N}_{\mathrm{p}}} := \sup_{a \in \mathbb{D}} \left(\int_{\mathbb{D}} |f(z)|^2 (1 - |\sigma_a(z)|^2)^p dA(z) \right)^{\frac{1}{2}} < \infty.$$

As we will see, the $\mathcal{N}_{\mathrm{p}}$-spaces coincide with $\mathcal{A}^{\text{-}1}$ for $p \in (1, \infty)$ and are all different and of independent interest for $p \in (0, 1]$. Since the Q_p-spaces have been studied intensively for over a decade now and due to both the similarities and the differences between $\mathcal{B}$ and $\mathcal{A}^{\text{-}1}$, it is only natural to also study the $\mathcal{N}_{\mathrm{p}}$-spaces. In fact, $\mathcal{N}_1$ was already of significant help in [5], where the authors studied the spectra of composition operators on BMOA. The $\mathcal{N}_{\mathrm{p}}$-spaces were also informally used in [7] to obtain a characterization of a special branch of weighted composition operators with closed range on $\mathcal{A}^{\text{-q}}$.

In Section 2, we give some background information needed for the rest of the paper, while we in Section 3 state some basic facts about the $\mathcal{N}_{\mathrm{p}}$-spaces and give some inclusion results. In Section 4 we study composition operators C_φ acting on $\mathcal{N}_{\mathrm{p}}$-spaces and relate operator-theoretic properties, like boundedness and compactness, to function-theoretic properties of the inducing function φ. Finally, in Section 5 we discuss some interesting open problems.

2 Preliminaries

We will always assume that $\{p, q\} \subset (0, \infty)$ unless stated otherwise. The notation $A \lesssim B$ implies that there is a positive constant c such that $A \leq c\, B$, while $A \approx B$ indicates that there are positive constants c_1 and c_2, such that $c_1 A \leq B \leq c_2 A$. In both cases, the constants don't depend on crucial properties of A and B (which will be clear from the context). The boundary of the unit disk will be denoted by $\partial\mathbb{D}$. We will frequently use the following easily verified equality (without any further reference):

$$1 - |\sigma_a(z)|^2 = \frac{(1 - |a|^2)(1 - |z|^2)}{|1 - \overline{a}z|^2}.$$

An analytic function $f(z) = \sum_{k=0}^{\infty} a_k z^{n_k}$ (with $n_k \in \mathbb{N}$ for all $k \in \mathbb{N}$) is said to belong to the *Hadamard gap class* (also known as Lacunary series) if there exists a constant $c > 1$ such that $n_{k+1}/n_k \geq c$ for all $k \in \mathbb{N}$.

The *generalized Nevanlinna counting function*, introduced by Shapiro [8], is defined by

$$\mathrm{N}_{\varphi,\gamma}(z) := \begin{cases} \displaystyle\sum_{w \in \varphi^{-1}\{z\}} \left(\log \frac{1}{|w|} \right)^{\gamma} & \text{if } z \in \varphi(\mathbb{D}) \\ 0 & \text{if } z \in \mathbb{D} \setminus \{\varphi(0)\} \end{cases},$$

where $\varphi^{-1}\{z\}$ denotes the sequence of preimages of z, counting multiplicity and where $\gamma > 0$. By choosing $\gamma = 1$ we get the classical Nevanlinna counting function. In [8], the author used the classical and the generalized Nevanlinna counting function to study composition operators on H^2 and on weighted Bergman spaces, respectively.

For later use, we will now gather some known results related to the generalized Nevanlinna counting function and to the so called "Change of variable formula" (see [8] and [10] for more details).

Proposition 2.1.

(i) [10, Proposition 2.4] *For* $\gamma \geq -1$,

$$\int_{\mathbb{D}} |(f\circ\varphi)(z)|^q (1-|z|^2)^\gamma dA(z) \approx \int_{\mathbb{D}} |f(z)|^{q-2}|f'(z)|^2\, \mathrm{N}_{\varphi,2+\gamma}\, dA(z).$$

(ii) [10, Lemma 2.3] *For* $\gamma > -1$,

$$\int_{\mathbb{D}} |f(z)|^q (1-|z|^2)^\gamma dA(z) \approx \int_{\mathbb{D}} |f(z)|^{q-2}|f'(z)|^2 (1-|z|^2)^{2+\gamma} dA(z).$$

(iii) [8, Corollary 6.7] *For* $\gamma > 1$ *and* $r \in (0, |\varphi(0)|)$,

$$\mathrm{N}_{\varphi,\gamma}(0) \leq \frac{1}{r^2}\int_{r\mathbb{D}} \mathrm{N}_{\varphi,\gamma}(z) dA(z).$$

3 Basic properties and inclusion

In this section we state some basic Banach space properties of the $\mathcal{N}_\mathrm{p}$-spaces as well as some inclusions. In particular, we show that for $p \in (0,1)$, the $\mathcal{N}_\mathrm{p}$-spaces are all different and of independent interest. We also show the intuitively evident fact that the $\mathcal{N}_\mathrm{p}$-spaces form a much bigger class of functions than the Q_p-spaces. Some of the techniques we use are highly inspired by the corresponding ones for the Q_p-spaces in [12]. We have chosen to give the proofs for the $\mathcal{N}_\mathrm{p}$-spaces directly instead of trying to use the isomorphism $f \mapsto f'$ together with the corresponding results by Xiao in order to give a more self-contained introduction to these spaces.

Proposition 3.1 (Basic facts about the $\mathcal{N}_\mathrm{p}$-spaces)**.**

(i) *For* $p \in (0,\infty)$ *we have that* $\|\cdot\|_{\mathcal{A}^{-1}} \lesssim \|\cdot\|_{\mathcal{N}_\mathrm{p}}$. *That is,* $\mathcal{N}_\mathrm{p} \subseteq \mathcal{A}^{-1}$.

(ii) *For* $p \in (1,\infty)$ *we have that* $\|\cdot\|_{\mathcal{A}^{-1}} \approx \|\cdot\|_{\mathcal{N}_\mathrm{p}}$. *That is,* $\mathcal{N}_\mathrm{p} = \mathcal{A}^{-1}$.

(iii) *The* $\mathcal{N}_\mathrm{p}$*-space, endowed with the norm* $\|\cdot\|_{\mathcal{N}_\mathrm{p}}$*, is a Banach space and the norm topology of* $\mathcal{N}_\mathrm{p}$ *is finer than the compact-open topology.*

Proof. Part (ii) is a direct consequence of [7, Lemma 4.3], while only minor modifications of the proofs of [5, Fact 1.1, Fact 1.2] give parts (i) and (iii). ∎

The following result is simple but important. For our convenience we state it formally as a lemma.

Lemma 3.2 (Test functions in $\mathcal{N}_{\mathrm{p}}$)**.** *For $w \in \mathbb{D}$ we define*

$$k_w(z) := \frac{1-|w|^2}{(1-\overline{w}z)^2}.$$

Then $k_w \in \mathcal{N}_{\mathrm{p}}$ and $\sup_{w\in\mathbb{D}} \|k_w\|_{\mathcal{N}_{\mathrm{p}}} \le 1$.

Proof. Trivially, $k_w \in H(\mathbb{D})$. It is also easy to see that

$$\|k_w\|^2_{\mathcal{N}_{\mathrm{p}}} = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |\sigma_w'(z)|^2 (1-|\sigma_a(z)|^2)^p dA(z) \le \int_{\mathbb{D}} |\sigma_w'(z)|^2 dA(z) = 1. \qquad \blacksquare$$

Theorem 3.3. *Let $f(z) = \sum_{k=0}^{\infty} b_k z^{n_k}$ be in the Hadamard gap class. Then for*

$$\text{(i)} \quad p \in (0,1]: \quad f \in \mathcal{N}_{\mathrm{p}} \ \textit{if and only if} \ \sum_{k=0}^{\infty} \frac{1}{2^{k(1+p)}} \sum_{2^k \le n_j < 2^{k+1}} |b_j|^2 < \infty,$$

$$\text{(ii)} \quad p \in (1,\infty): \quad f \in \mathcal{N}_{\mathrm{p}} \ \textit{if and only if} \ \sup_k \frac{|b_k|}{n_k} < \infty.$$

Proof. (i) Assume that $\sum_{k=0}^{\infty} \frac{1}{2^{k(1+p)}} \sum_{2^k \le n_j < 2^{k+1}} |b_j|^2 < \infty$. Then, using Hölder's inequality,

$$\begin{aligned}
\|f\|^2_{\mathcal{N}_{\mathrm{p}}} &\lesssim \sup_{a\in\mathbb{D}} (1-|a|^2)^p \int_0^1 \left(\sum_{k=0}^{\infty} |b_k| r^{n_k}\right)^2 (1-r)^p \left(\int_0^{2\pi} \frac{1}{|1-\overline{a}re^{i\theta}|^2} d\theta\right)^p dr \\
&= \sup_{a\in\mathbb{D}} (1-|a|^2)^p \int_0^1 \left(\sum_{k=0}^{\infty} |b_k| r^{n_k}\right)^2 (1-r)^p \left(\frac{2\pi}{1-|a|^2 r^2}\right)^p dr \\
&\lesssim \int_0^1 \left(\sum_{k=0}^{\infty} |b_k| r^{n_k}\right)^2 (1-r)^p dr.
\end{aligned}$$

By [6, Theorem 1] we know that if $\alpha > 0$, $\beta > 0$ and $a_k \ge 0$ for all $k \in \mathbb{N}$, then

$$\int_0^1 \left(\sum_{k=0}^{\infty} a_k r^k\right)^{\beta} (1-r)^{\alpha-1} dr \approx \sum_{k=0}^{\infty} 2^{-k\alpha} \left(\sum_{2^k \le j < 2^{k+1}} a_j\right)^{\beta},$$

where the constants only depend on α and β. This very useful tool can now be applied to the calculation above to obtain

$$\|f\|^2_{\mathcal{N}_{\mathrm{p}}} \lesssim \sum_{k=0}^{\infty} \frac{1}{2^{k(1+p)}} \left(\sum_{2^k \le n_j < 2^{k+1}} |b_j|\right)^2.$$

Since f is in the Hadamard gap class, there exists a constant $c > 1$ such that $n_{j+1} \ge c\, n_j$ for all $j \in \mathbb{N}$. Hence, the maximum number of n_j's between 2^k and 2^{k+1} for any $k \in \mathbb{N}$ is the integer part of $\log_c 2 + 1$. Using this together with the fact that, for any $n \in \mathbb{N}$, we always have $(a_1 + \ldots + a_n)^2 \le n(a_1^2 + \ldots + a_n^2)$, we get that

$$\|f\|^2_{\mathcal{N}_{\mathrm{p}}} \lesssim \sum_{k=0}^{\infty} \frac{1}{2^{k(1+p)}} \sum_{2^k \le n_j < 2^{k+1}} |b_j|^2 < \infty.$$

Conversely, assume that $f \in \mathcal{N}_{\mathrm{p}}$. Using Parseval's formula and Stirling's formula, we see that

$$\begin{aligned}
\|f\|^2_{\mathcal{N}_{\mathrm{p}}} &\gtrsim \int_0^1 \int_0^{2\pi} \left| \sum_{k=0}^{\infty} b_k r^{n_k} (e^{i\theta})^{n_k} \right|^2 d\theta (1-r)^p r dr \\
&\approx \sum_{k=0}^{\infty} |b_k|^2 \int_0^1 r^{2n_k+1}(1-r)^p dr \\
&= \sum_{k=0}^{\infty} \frac{\Gamma(2n_k+2)\Gamma(p+1)}{\Gamma(2n_k+p+3)} |b_k|^2 \\
&\gtrsim \sum_{k=0}^{\infty} n_k^{-(1+p)} |b_k|^2.
\end{aligned}$$

Thus,

$$\infty > \|f\|^2_{\mathcal{N}_{\mathrm{p}}} \gtrsim \sum_{k=0}^{\infty} \sum_{2^k \le n_j < 2^{k+1}} n_j^{-(1+p)} |b_j|^2 \gtrsim \sum_{k=0}^{\infty} \frac{1}{2^{k(1+p)}} \sum_{2^k \le n_j < 2^{k+1}} |b_j|^2.$$

(ii) Assume that $f \in \mathcal{A}^{-1}$. Using the Cauchy integral formula, we get that for any $r \in (0,1)$,

$$|b_k| \lesssim \int_{r\partial\mathbb{D}} \frac{|f(z)|}{|z|^{n_k+1}} |dz| \lesssim \frac{\|f\|_{\mathcal{A}^{-1}}}{r^{n_k}(1-r)}.$$

Without loss of generality we may assume that $n_0 \ge 2$. Choose $r = 1 - 1/n_k$. Then

$$\sup_k \frac{|b_k|}{n_k} \lesssim \sup_k \frac{\|f\|_{\mathcal{A}^{-1}}}{\left(1 - \frac{1}{n_k}\right)^{n_k}} \le \frac{\|f\|_{\mathcal{A}^{-1}}}{\left(1 - \frac{1}{2}\right)^2} \lesssim \|f\|_{\mathcal{A}^{-1}}.$$

Conversely, assume that $\sup_k |b_k|/n_k < \infty$. Then

$$|f(z)| \le \sum_{k=0}^{\infty} |b_k|\, |z|^{n_k} \lesssim \sum_{k=0}^{\infty} n_k |z|^{n_k}.$$

Thus,

$$\frac{|f(z)|}{1-|z|} \lesssim \left(\sum_{k=0}^{\infty} n_k |z|^{n_k} \right) \left(\sum_{k=0}^{\infty} |z|^k \right) = \sum_{k=0}^{\infty} \sum_{j=0}^{k} n_j |z|^{n_j+k-j} = \sum_{k=0}^{\infty} \left(\sum_{n_j \le k} n_j \right) |z|^k.$$

Again, since f is in the Hadamard gap class, there exists a constant $c > 1$ such that $n_{j+1} \ge c\, n_j$ for all $j \in \mathbb{N}$. A straightforward calculation shows that

$$\frac{1}{k} \sum_{n_j \le k} n_j \le \frac{c}{c-1}.$$

Hence,

$$\frac{|f(z)|}{1-|z|} \lesssim \sum_{k=0}^{\infty} k|z|^k \le \frac{1}{(1-|z|)^2},$$

which clearly implies that $f \in \mathcal{A}^{-1}$. ∎

Corollary 3.4. *For $0 < p_1 < p_2 \leq 1$ we have that $\mathcal{B} \subsetneq \mathcal{N}_{\mathrm{p}_1} \subsetneq \mathcal{N}_{\mathrm{p}_2} \subsetneq \mathcal{A}^{\text{-}1}$.*

Proof. For $f \in \mathcal{B}$ we have the well-known estimation

$$|f(z)| \lesssim \log \frac{2}{1-|z|} \|f\|_{\mathcal{B}}, \quad z \in \mathbb{D},$$

which easily gives that $\|\cdot\|_{\mathcal{N}_\mathrm{p}} \lesssim \|\cdot\|_{\mathcal{B}}$ for all $p \in (0, \infty)$. The other inclusions are obvious from the definition of the $\mathcal{N}_\mathrm{p}$-spaces and Proposition 3.1 (i). Therefore it remains to show that the inclusions are strict. Define

$$f_1(z) := \sum_{k=0}^{\infty} 2^k z^{2^k}, \quad f_2(z) := \sum_{k=0}^{\infty} 2^{\frac{k(1+p_1)}{2}} z^{2^k}, \quad f_3(z) := \sum_{k=0}^{\infty} \left(\frac{3}{2}\right)^{\frac{k}{2}} z^{2^k}.$$

Then by using Theorem 3.3 and [12, Theorem 1.2.1 (ii)], it is easy to see that

$$f_1 \in \mathcal{A}^{\text{-}1} \setminus \mathcal{N}_{\mathrm{p}_2}, \quad f_2 \in \mathcal{N}_{\mathrm{p}_2} \setminus \mathcal{N}_{\mathrm{p}_1}, \quad f_3 \in \mathcal{N}_{\mathrm{p}_1} \setminus \mathcal{B}. \qquad \blacksquare$$

4 Composition operators

In this section we study composition operators acting on $\mathcal{N}_\mathrm{p}$-spaces. The techniques we use are inspired by the ones in [10] and [12].

Theorem 4.1. *$C_\varphi : \mathcal{N}_\mathrm{p} \to \mathcal{A}^{\text{-q}}$ is bounded if and only if* $\sup_{z\in\mathbb{D}} \dfrac{(1-|z|^2)^q}{1-|\varphi(z)|^2} < \infty$.

Proof. Assume that $\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^q}{1-|\varphi(z)|^2} < \infty$. Then

$$\|f \circ \varphi\|_{\mathcal{A}^{\text{-q}}} \leq \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^q}{1-|\varphi(z)|^2} \underbrace{\sup_{z\in\mathbb{D}} |f(\varphi(z))|(1-|\varphi(z)|^2)}_{\leq \|f\|_{\mathcal{A}^{\text{-}1}}} \lesssim \|f\|_{\mathcal{N}_\mathrm{p}}.$$

Conversely, assume that $C_\varphi : \mathcal{N}_\mathrm{p} \to \mathcal{A}^{\text{-q}}$ is bounded. Fix $z_0 \in \mathbb{D}$ and let k_w be the test function in Lemma 3.2 with $w = \varphi(z_0)$. Then

$$1 \geq \|k_w\|_{\mathcal{N}_\mathrm{p}} \gtrsim \|k_w \circ \varphi\|_{\mathcal{A}^{\text{-q}}} \geq \frac{1-|w|^2}{|1-\overline{w}\varphi(z_0)|^2}(1-|z_0|^2)^q = \frac{(1-|z_0|^2)^q}{1-|\varphi(z_0)|^2}. \qquad \blacksquare$$

Even though Theorem 4.1 gives a complete characterization of the bounded composition operators on $\mathcal{N}_\mathrm{p}$ for $p > 1$, it doesn't give us any information about $p \in (0, 1]$.

Theorem 4.2.

(i) **(Sufficiency)** *If* $\sup_{z\in\mathbb{D}} \dfrac{\mathrm{N}_{\varphi,2+p}(z)}{(1-|z|^2)^{2+2p}} < \infty$, *then C_φ is bounded on $\mathcal{N}_\mathrm{p}$.*

(ii) **(Necessity)** *If C_φ is bounded on $\mathcal{N}_\mathrm{p}$, then* $\sup_{z\in\mathbb{D}} \dfrac{\mathrm{N}_{\varphi,2+p}(z)}{(1-|z|^2)^2} < \infty$.

Proof. (i) Using Hölder's inequality and a well-known integral formula (see for example [4, Theorem 1.7]), we get that

$$\begin{aligned}
\|f\circ\varphi\|_{\mathcal{N}_{\mathrm{p}}}^2 &= \sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|(f\circ\varphi)(z)|^2(1-|z|^2)^{\frac{2p}{2+p}}\frac{(1-|a|^2)^p(1-|z|^2)^{\frac{p^2}{2+p}}}{|1-\overline{a}z|^{2p}}dA(z)\\
&\le \left(\int_{\mathbb{D}}|(f\circ\varphi)(z)|^{2+p}(1-|z|^2)^p dA(z)\right)^{\frac{2}{2+p}}\sup_{a\in\mathbb{D}}(1-|a|^2)^p\left(\int_{\mathbb{D}}\frac{(1-|z|^2)^p}{|1-\overline{a}z|^{4+2p}}dA(z)\right)^{\frac{p}{2+p}}\\
&\lesssim \left(\int_{\mathbb{D}}|(f\circ\varphi)(z)|^{2+p}(1-|z|^2)^p dA(z)\right)^{\frac{2}{2+p}}.
\end{aligned}$$

Hence, by using Proposition 2.1 (i) and (ii) we obtain

$$\begin{aligned}
\|f\circ\varphi\|_{\mathcal{N}_{\mathrm{p}}}^2 &\lesssim \left(\int_{\mathbb{D}}|f(z)|^p|f'(z)|^2\,\mathrm{N}_{\varphi,2+p}(z)dA(z)\right)^{\frac{2}{2+p}}\\
&\le \left(\int_{\mathbb{D}}|f(z)|^p|f'(z)|^2(1-|z|^2)^{2+2p}dA(z)\right)^{\frac{2}{2+p}}\\
&\le \|f\|_{\mathcal{A}^{-1}}^{\frac{2p}{2+p}}\left(\int_{\mathbb{D}}|f'(z)|^2(1-|z|^2)^{2+p}dA(z)\right)^{\frac{2}{2+p}}\\
&\lesssim \|f\|_{\mathcal{N}_{\mathrm{p}}}^{\frac{2p}{2+p}}\Bigg(\underbrace{\int_{\mathbb{D}}|f(z)|^2(1-|z|^2)^p dA(z)}_{\le\|f\|_{\mathcal{N}_{\mathrm{p}}}^2}\Bigg)^{\frac{2}{2+p}}.
\end{aligned}$$

(ii) Fix $w\in\mathbb{D}$ and let k_w be the test function in Lemma 3.2. Then by using Proposition 2.1 (i) we get that

$$\begin{aligned}
1 \ge \|k_w\|_{\mathcal{N}_{\mathrm{p}}}^2 &\gtrsim \sup_{w\in\mathbb{D}}\|C_\varphi k_w\|_{\mathcal{N}_{\mathrm{p}}}^2\\
&\ge \int_{\mathbb{D}}|(k_w\circ\varphi)(z)|^2(1-|z|^2)^p dA(z)\\
&\approx \int_{\mathbb{D}}|k_w'(z)|^2\,\mathrm{N}_{\varphi,2+p}(z)dA(z)\\
&= \int_{\mathbb{D}}4|w|^2\frac{(1-|w|^2)^2}{|1-\overline{w}z|^6}\,\mathrm{N}_{\varphi,2+p}(z)dA(z).
\end{aligned}$$

A change of variable $z\mapsto\sigma_w(z)$, followed by an elementary estimation, then produce

$$\begin{aligned}
1 &\gtrsim \int_{\frac12\mathbb{D}}|w|^2\frac{|1-\overline{w}z|^2}{(1-|w|^2)^2}\,\mathrm{N}_{\varphi,2+p}(\sigma_w(z))dA(z)\\
&\gtrsim \frac{|w|^2}{(1-|w|^2)^2}\int_{\frac12\mathbb{D}}\mathrm{N}_{\sigma_w\circ\varphi,2+p}(z)dA(z)
\end{aligned}$$

Without loss of generality, we may assume that $|w|$ is close to 1 and hence that $|(\sigma_w\circ\varphi)(0)|>\frac12$. Thus, by using Proposition 2.1 (iii) we apprehend

$$1\gtrsim\frac{\mathrm{N}_{\sigma_w\circ\varphi,2+p}(0)}{(1-|w|^2)^2}=\frac{\mathrm{N}_{\varphi,2+p}(w)}{(1-|w|^2)^2}.$$

Clearly this gives the desired condition. ∎

Usually if a "big-oh" condition, like the one in Theorem 4.1, describes the bounded operators, the corresponding "little-oh" condition describes the compact operators. This is also the case here. Recall that for composition operators the standard definition of compactness can be reformulated on many spaces, including the $\mathcal{N}_{\mathrm{p}}$-spaces and the Bergman-type spaces $\mathcal{A}^{\text{-q}}$ (see for example [3, Proposition 3.11]). Thus, in this case, C_φ is *compact* if and only if for every norm bounded sequence $\{f_n\}$ with $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$, we have that $f_n \circ \varphi \to 0$ with respect to the norm topology.

Theorem 4.3. *$C_\varphi : \mathcal{N}_{\mathrm{p}} \to \mathcal{A}^{\text{-q}}$ is compact if and only if* $\displaystyle\lim_{r\to 1} \sup_{|\varphi(z)|>r} \frac{(1-|z|^2)^q}{1-|\varphi(z)|^2} = 0.$

Proof. Assume that $C_\varphi : \mathcal{N}_{\mathrm{p}} \to \mathcal{A}^{\text{-q}}$ is compact and suppose that there exists $\varepsilon_0 > 0$ and a sequence $\{z_n\} \subset \mathbb{D}$ such that

$$\frac{(1-|z_n|^2)^q}{1-|\varphi(z_n)|^2} \geq \varepsilon_0 \quad \text{whenever} \quad |\varphi(z_n)| > 1 - \frac{1}{n}.$$

Clearly, we can assume that $w_n := \varphi(z_n)$ tends to $w_0 \in \partial\mathbb{D}$ as $n \to \infty$. Let k_{w_n} be the test function in Lemma 3.2. Then $k_{w_n} \to k_{w_0}$ with respect to the compact-open topology. Define $f_n := k_{w_n} - k_{w_0}$. Then $\|f_n\|_{\mathcal{N}_{\mathrm{p}}} \leq 1$ (see Lemma 3.2) and $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$. Thus, $f_n \circ \varphi \to 0$ in $\mathcal{A}^{\text{-q}}$ by assumption. But, for n big enough,

$$\begin{aligned} \|C_\varphi f_n\|_{\mathcal{A}^{\text{-q}}} &\geq |k_{w_n}(\varphi(z_n)) - k_{w_0}(\varphi(z_n))|(1-|z_n|^2)^q \\ &= \underbrace{\frac{(1-|z_n|^2)^q}{1-|\varphi(z_n)|^2}}_{\geq \varepsilon_0} \underbrace{\left|1 - \frac{(1-|w_0|^2)(1-|w_n|^2)}{|1-\overline{w_0}w_n|^2}\right|}_{=1}, \end{aligned}$$

which is a contradiction. Conversely, assume that for all $\varepsilon > 0$ there exists $\delta \in (0,1)$ such that

$$\frac{(1-|z|^2)^q}{1-|\varphi(z)|^2} < \varepsilon \quad \text{whenever} \quad |\varphi(z)| > \delta.$$

Let $\{f_n\}$ be a norm bounded sequence in $\mathcal{N}_{\mathrm{p}}$ which converges to zero on compact subsets of $\mathbb{D}$. Clearly, we may assume that $|\varphi(z)| > \delta$. Then

$$\|C_\varphi f_n\|_{\mathcal{A}^{\text{-q}}} = \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^q}{1-|\varphi(z)|^2} |f_n(\varphi(z))|(1-|\varphi(z)|^2).$$

Thus, by Proposition 3.1 (i), we have that

$$\|C_\varphi f_n\|_{\mathcal{A}^{\text{-q}}} \leq \varepsilon \|f_n\|_{\mathcal{A}^{\text{-1}}} \lesssim \varepsilon \|f_n\|_{\mathcal{N}_{\mathrm{p}}} \leq \varepsilon.$$ ∎

5 Open problems

In this section we will mention some open problems related to the results in the preceding sections. We will begin by proving a simple theorem of which the corresponding result for the Q_p-spaces is still unsolved. Indeed, in [11, Conjecture 1.5] (see also [12, p. 86]), Xiao conjectured that M_ψ is bounded on Q_p if and only if

$$\psi \in H^\infty \quad \text{and} \quad \sup_{a\in\mathbb{D}} \log^2(1-|a|) \int_{\mathbb{D}} |\psi'(z)|^2 (1-|\sigma_a(z)|^2)^p dA(z) < \infty.$$

Theorem 5.1. *M_ψ is bounded on $\mathcal{N}_{\mathrm{p}}$ if and only if $\psi \in H^\infty$.*

Proof. Assume that $\varphi \in H^\infty$. Clearly

$$\|M_\psi f\|^2_{\mathcal{N}_{\mathrm{p}}} \leq \|\psi\|^2_{H^\infty} \|f\|^2_{\mathcal{N}_{\mathrm{p}}}.$$

Conversely, let k_w be the test function in Lemma 3.2. Then for all $w \in \mathbb{D}$,

$$1 \geq \|k_w\|_{\mathcal{N}_{\mathrm{p}}} \gtrsim \|M_\psi k_w\|_{\mathcal{N}_{\mathrm{p}}} \gtrsim \|M_\psi k_w\|_{\mathcal{A}^{-1}} \geq \frac{1-|w|^2}{|1-\overline{w}w|^2} |\psi(w)|(1-|w|^2) = |\psi(w)|. \qquad \blacksquare$$

In [12, p. 22], Xiao also stated as an open problem to characterize the bounded composition operators on Q_p, which is to the best of our knowledge also still unsolved. Therefore we state here as an open problem to give a full characterization of when C_φ is bounded on $\mathcal{N}_{\mathrm{p}}$. By doing so, one should be able to combine this with Theorem 5.1 and thereby to give a full description of when $W_{\varphi,\psi}$ is bounded on $\mathcal{N}_{\mathrm{p}}$. Having done that, the open problem about bounded composition operators on Q_p should be solvable. A similar interplay between weighted composition operators (on Bergman-type spaces) and composition operators (on Bloch-type spaces) has been done for example in [2] (see also [7]).

References

[1] R. Aulaskari, J. Xiao and R. Zhao, *On subspaces and subsets of* BMOA *and* UBC, Analysis **15**, 1995, 101–121.

[2] M.D. Contreras and A.G. Hernandez-Díaz, *Weighted composition operators in weighted Banach spaces of analytic functions,* J. Austral. Math. Soc. (Ser. A) **69**, 2000, 41–60.

[3] C.C. Cowen and B.D. MacCluer, *Composition Operators on Spaces of Analytic Functions,* CRC Press, 1995.

[4] H. Hedenmalm, B. Korenblum and K. Zhu, *Theory of Bergman Spaces,* Springer, New York, 2000.

[5] M. Lindström and N. Palmberg, *Spectra of composition operators on* BMOA*,* Integr. Equ. Oper. Theory **53**, 2005, 75–86.

[6] M. Mateljević and M. Pavlović, *L^p-behavior of power series with positive coefficients and Hardy spaces,* Proc. Amer. Math Soc. **87**, 1983, 309–316.

[7] N. Palmberg, *Weighted composition operators with closed range,* to appear in Bull. Austral. Math. Soc.

[8] J.H. Shapiro, *The essential norm of a composition operator,* Annals of Math. **127**, 1987, 375–404.

[9] J.H. Shapiro, *Composition Operators and Classical Function Theory.* Springer, 1993.

[10] W. Smith, *Composition operators between Bergman and Hardy spaces,* Trans. Amer. Math. Soc. **348**, 1996, 2331–2348.

[11] J. Xiao, *The Q_p corona theorem,* Pacific J. Math. **194**, 2000, 491–509.

[12] J. Xiao, *Holomorphic Q Classes,* Springer, Berlin Heidelberg, 2001.

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