Almost Kenmotsu manifolds and local symmetry

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Abstract

We consider locally symmetric almost Kenmotsu manifolds showing that such a manifold is a Kenmotsu manifold if and only if the Lie derivative of the structure, with respect to the Reeb vector field \( \xi \), vanishes. Furthermore, assuming that for a \((2n + 1)\)-dimensional locally symmetric almost Kenmotsu manifold such Lie derivative does not vanish and the curvature satisfies \( R_{XY}\xi = 0 \) for any \( X, Y \) orthogonal to \( \xi \), we prove that the manifold is locally isometric to the Riemannian product of an \((n+1)\)-dimensional manifold of constant curvature \(-4\) and a flat \(n\)-dimensional manifold. We give an example of such a manifold.

Introduction

An almost contact structure on a differentiable manifold \( M^{2n+1} \) is given by a tensor field \( \varphi \) of type \((1, 1)\), a vector field \( \xi \) and a 1-form \( \eta \) satisfying \( \varphi^{2} = -I + \eta \otimes \xi \) and \( \eta(\xi) = 1 \), which imply that \( \varphi(\xi) = 0 \) and \( \eta \circ \varphi = 0 \).

Furthermore, on the product manifold \( M^{2n+1} \times \mathbb{R} \) one can define an almost complex structure \( J \) by \( J \left( X, f \frac{d}{dt} \right) = (\varphi X - f \xi, \eta(X) \frac{d}{dt}) \), where \( X \) is a vector field tangent to \( M^{2n+1} \), \( t \) is the coordinate of \( \mathbb{R} \) and \( f \) is a \( \mathcal{C}^\infty \) function on \( M^{2n+1} \times \mathbb{R} \). If \( J \) is integrable, the almost contact structure is said to be normal and it is known that this is equivalent to the vanishing of the tensor field \( N = [\varphi, \varphi] + 2d\eta \otimes \xi \), where \( [\varphi, \varphi] \) is the Nijenhuis torsion of \( \varphi \) ([3]).

An almost contact metric structure \((\varphi, \xi, \eta, g)\) is given by an almost contact structure and a Riemannian metric \( g \) satisfying \( g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y) \) for any vector fields \( X \) and \( Y \). Then, the fundamental 2-form \( \Phi \) is defined by
\( \Phi(X, Y) = g(X, \varphi Y) \) for any vector fields \( X \) and \( Y \). For more details, we refer to Blair’s books [3], [5].

A contact metric structure \((\varphi, \xi, \eta, g)\) is an almost contact metric structure such that \( \Phi = d\eta \) and if the structure is normal, then it is a Sasakian structure. In [14], Z. Olszak proved that in dimension \( 2n + 1 \geq 5 \) any contact metric manifold of constant sectional curvature has sectional curvature equal to 1 and is a Sasakian manifold. In [4], D.E. Blair proved that if the Riemannian curvature of a contact metric manifold \( M^{2n+1} \) satisfies \( R_{XY} \xi = 0 \) for all vector fields \( X \) and \( Y \), then \( M^{2n+1} \) is locally the product of a flat \((n + 1)\)-dimensional manifold and an \( n \)-dimensional manifold of constant curvature 4. In particular, the tangent sphere bundle of a flat Riemannian manifold admits such a structure. More recently, in [6] E. Boeckx and J.T. Cho proved that a locally symmetric contact metric space is either Sasakian of constant curvature 1 or locally isometric to \( \mathbb{R}^{n+1} \times S^n(4) \).

In this paper, we consider the class of almost contact metric manifolds called almost Kenmotsu manifolds. In [15], Olszak proved that if such a manifold has constant sectional curvature \( \kappa \) and dimension \( 2n + 1 \geq 5 \), then it is a Kenmotsu manifold and \( \kappa = -1 \). We give another proof of the same result without restrictions on the dimension. We also study locally symmetric almost Kenmotsu manifolds \( M^{2n+1} \) showing that such a manifold is a Kenmotsu manifold if and only if the operator \( h = \frac{1}{2} \mathcal{L}_\xi \varphi \) vanishes, where \( \mathcal{L} \) denotes the Lie differentiation. Furthermore, assuming \( h \neq 0 \) and \( R_{XY} \xi = 0 \) for all vector fields \( X \) and \( Y \) orthogonal to \( \xi \), we prove that the spectrum of \( h \) is \( \{0, 1, -1\} \), with 0 as simple eigenvalue, and \( M^{2n+1} \) is locally the product of an \((n + 1)\)-dimensional manifold of constant curvature \(-4\) and an \( n \)-dimensional flat manifold. We provide an example of such a manifold. Comparing with the contact case, one can state the following question: is a locally symmetric almost Kenmotsu manifold either Kenmotsu of constant curvature \(-1\) or locally isometric to the product \( \mathbb{H}^{n+1}(-4) \times \mathbb{R}^n \)?

As usual, the manifolds involved are assumed to be connected. Furthermore, we denote by \( \mathcal{X}(M^{2n+1}) \) the space of the \( C^\infty \)-sections of \( TM^{2n+1} \).

As regards Kenmotsu manifolds, we recall here the basic data related to them. An almost contact metric manifold \( M^{2n+1} \), with structure \((\varphi, \xi, \eta, g)\), is said to be a Kenmotsu manifold if it is normal, the 1-form \( \eta \) is closed and \( d\Phi = 2\eta \wedge \Phi \). It is well known that Kenmotsu manifolds can be characterized by

\[
(\nabla_X \varphi)(Y) = g(\varphi X, Y)\xi - \eta(Y)\varphi(X),
\]

for any \( X, Y, Z \in \mathcal{X}(M^{2n+1}) \), which implies that \( \nabla_\xi \varphi = 0 \). We denote by \( \mathcal{D} \) the distribution orthogonal to \( \xi \), that is \( \mathcal{D} = \text{Im}(\varphi) = \text{Ker}(\eta) \). It can be seen that \( \nabla_\xi X \in \mathcal{D} \) and \( \nabla_X \xi \in \mathcal{D} \) for any vector field \( X \in \mathcal{D} \). Moreover, one has \( \nabla \xi = -\varphi^2 \) and \( \nabla \eta = g - \eta \otimes \eta \). Since \( \eta \) is closed, \( \mathcal{D} \) is an integrable distribution. It is known that its leaves are \( 2n \)-dimensional totally umbilical K{"a}hler manifolds with mean curvature vector field \( H = -\xi \). Kenmotsu manifolds appear for the first time in [9], where they have been locally classified.

**Theorem 1.** ([9]) Let \( (M^{2n+1}, \varphi, \xi, \eta, g) \) be a Kenmotsu manifold. Then, \( M^{2n+1} \) is locally a warped product \( M' \times f^2 \mathbb{S}^n \) where \( \mathbb{S}^n \) is a K{"a}hler manifold, \( M' \) is an open interval with coordinate \( t \), and \( f^2 = ce^{2t} \) for some positive constant \( c \).
As proved in [9], a Kenmotsu manifold is locally symmetric if and only if it is a space of constant sectional curvature $K = -1$.

## 1 Almost Kenmotsu manifolds

An almost contact metric manifold $M^{2n+1}$, with structure $(\varphi, \xi, \eta, g)$, is said to be an almost Kenmotsu manifold if the 1-form $\eta$ is closed and $d\Phi = 2\eta \wedge \Phi$. Obviously, a normal almost Kenmotsu manifold is a Kenmotsu manifold.

Let $M^{2n+1}$ be an almost Kenmotsu manifold with structure $(\varphi, \xi, \eta, g)$. Since the 1-form $\eta$ is closed, we have $\mathcal{L}_\xi \eta = 0$ and $[X, \xi] \in \mathcal{D}$ for any $X \in \mathcal{D}$. The Levi-Civita connection satisfies $\nabla_\xi \xi = 0$ and $\nabla_\xi \varphi = 0$ ([10]), which implies that and $\nabla_\xi X \in \mathcal{D}$ for any $X \in \mathcal{D}$.

Now, we set $A = -\nabla \xi$ and $h = \frac{1}{2} \mathcal{L}_\xi \varphi$. Obviously, $A(\xi) = 0$ and $h(\xi) = 0$. Moreover, the tensor fields $A$ and $h$ are symmetric operators and satisfy the following relations

\[ A \circ \varphi + \varphi \circ A = -2\varphi, \quad h \circ \varphi + \varphi \circ h = 0 \]
\[ \nabla_X \xi = -\varphi^2 X - \varphi h X, \quad X \in \mathcal{X}(M^{2n+1}), \]
\[ \nabla \eta = g - \eta \otimes \eta + g \circ (\varphi \times h), \quad \delta \eta = -2n. \tag{1} \]

Hence, $M^{2n+1}$ cannot be compact. We also remark that

\[ h = 0 \iff \nabla \xi = -\varphi^2. \tag{2} \]

From Lemma 2.2 in [10] we have

\[ (\nabla_X \varphi)Y + (\nabla_\varphi X \varphi)(\varphi Y) = -\eta(Y)\varphi X - 2g(X, \varphi Y)\xi - \eta(Y)h(X) \tag{3} \]

for any $X, Y \in \mathcal{X}(M^{2n+1})$. The following result is also proved in [10].

**Proposition 1.** Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be an almost Kenmotsu manifold. The integral manifolds of $\mathcal{D}$ are almost Kähler manifolds with mean curvature vector field $H = -\xi$. They are totally umbilical submanifolds of $M^{2n+1}$ if and only if $h$ vanishes.

**Example 1.** Let $(N^{2n}, J, \bar{g})$, $n \geq 2$, be a strictly almost Kähler manifold and consider $\mathbb{R} \times N^{2n}$, with coordinate $t$ on $\mathbb{R}$. We put $\xi = \frac{\partial}{\partial t}$, $\eta = dt$ and define the tensor field $\varphi$ on $\mathbb{R} \times N^{2n}$ such that $\varphi X = JX$, if $X$ is a vector field on $N^{2n}$, and $\varphi X = 0$ if $X$ is tangent to $\mathbb{R}$. Furthermore, we consider the metric $g = g_0 + c e^{2t} \bar{g}$, where $g_0$ denotes the Euclidean metric on $\mathbb{R}$ and $c \in \mathbb{R}_+$. Then, the warped product $\mathbb{R} \times_f N^{2n}$, $f^2 = ce^{2t}$, with the structure $(\varphi, \xi, \eta, g)$, is a strictly almost Kenmotsu manifold. Namely, it is easy to verify that the 1-form $\eta$ is closed and dual of $\xi$ with respect to $g$, $\varphi^2 = -I + \eta \otimes \xi$ and $g$ is a compatible metric. Computing $\Phi$ and $d\Phi$, we get $\Phi = ce^{2t} p_2^* (\tilde{\Omega})$, where $p_2$ is the projection on $N^{2n}$ and $\tilde{\Omega}$ is the fundamental form of $N^{2n}$. Then, since $d\tilde{\Omega} = 0$, $d\Phi = 2dt \wedge \Phi = 2\eta \wedge \Phi$. Finally, since the torsion $N_f$ does not vanish, $N^{2n}$ being strictly almost Kähler, we obtain that the structure is not normal.

**Remark 1.** In [13], Oguro and Sekigawa describe a strictly almost Kähler structure on the Riemannian product $\mathbb{H}^3 \times \mathbb{R}$. Thus, we obtain a 5-dimensional strictly almost Kenmotsu manifold on the warped product $\mathbb{R} \times_f (\mathbb{H}^3 \times \mathbb{R})$, $f^2 = ce^{2t}$. 
Theorem 2. Let \((M^{2n+1}, \varphi, \xi, \eta, g)\) be an almost Kenmotsu manifold and assume that \(h = 0\). Then, \(M^{2n+1}\) is locally a warped product \(M' \times_f \mathbb{R}^{2n}\), where \(\mathbb{R}^{2n}\) is an almost Kähler manifold, \(M'\) is an open interval with coordinate \(t\), and \(f^2 = ce^{2t}\) for some positive constant \(c\).

Proof. The vector field \(\xi\) is geodesic and the orthogonal distribution \(\mathcal{D}\) is integrable with totally umbilical almost Kähler leaves. Thus, as a manifold, \(M^{2n+1}\) is locally a product \(M' \times \mathbb{R}^{2n}\) with \(TM' = [\xi]\) and \(TN^{2n} = \mathcal{D}\). We can choose a neighborhood with coordinates \((t, x^1, \ldots, x^{2n})\) such that \(\pi_*(\xi) = \frac{\partial}{\partial t}, \pi\) denoting the projection onto \(M'\). Then \(\pi : M' \times \mathbb{R}^{2n} \to M'\) is a \(C^\infty\)-submersion with vertical distribution \(\mathcal{V} = TM'\) and horizontal distribution \(\mathcal{H} = TN^{2n}\). The splitting \(\mathcal{V} \oplus \mathcal{H}\) is orthogonal with respect to \(g\) and for any \(p \in M^{2n+1}\) we have \(g_p(\xi, \xi) = 1 = g_{\pi(p)}(\pi_*\xi, \pi_*\xi)\); hence, \(\pi\) is a Riemannian submersion. Since the horizontal distribution is integrable, the O’Neill tensor \(A\) vanishes. Moreover, the vector field \(N = 2nH = -2n\xi\) is basic.

Thus, computing the free-trace part \(T^0\) of the O’Neill tensor \(T\), for any \(U, V\) vertical vector fields, we get:

\[
T^0_U V = T_U V - \frac{1}{2n}g(U, V)N = \alpha(U, V) + g(U, V)\xi = 0,
\]

\[
T^0_U \xi = T_U \xi + \frac{1}{2n}g(N, \xi)U = \nabla_U \xi - U = 0.
\]

Thus \(T^0 = 0\) and \(M^{2n+1}\) is locally a warped product of \((M', g_0)\) and \((\mathbb{R}^{2n}, \tilde{g})\) by a positive function \(f^2\) on \(M'\), where \(g_0\) is the flat metric and \(\tilde{g}\) is an almost Kähler metric. The vector field \(N = -2n\xi\) is \(\pi\)-related to \(-\frac{\partial}{\partial t} \text{grad}_{g_0} f\) ([1], 9.104). It follows that \(\text{grad}_{g_0} f = f \frac{d}{dt}\), which implies that \(f = ke^t\) and \(f^2 = ce^{2t}\), with \(c\) a positive constant. Hence, the warped metric is given by \(dt \otimes dt + ce^{2t}\tilde{g}\).

Proposition 2. Let \((M^{2n+1}, \varphi, \xi, \eta, g)\) be an almost Kenmotsu manifold such that the integral manifolds of \(\mathcal{D}\) are Kähler. Then, \(M^{2n+1}\) is a Kenmotsu manifold if and only if \(\nabla \xi = -\varphi^2\).

Proof. An easy computation shows that \(N(X, \xi) = -2h(\varphi X)\) for any vector field \(X\). Hence, assuming that the structure is normal, then \(h(Y) = 0\) for any \(Y \in \mathcal{D}\). Being \(h(\xi) = 0\), we get \(h = 0\) and (2) implies that \(\nabla \xi = -\varphi^2\). Vice versa, if \(\nabla \xi = -\varphi^2\) then \(h = 0\) by (2), and thus \(N(X, \xi) = 0\) for any vector field \(X\). Moreover, for \(X, Y \in \mathcal{D}\) we have \(N(X, Y) = N_f(X, Y) = 0\), the leaves of \(\mathcal{D}\) being Kähler manifolds.

Proposition 3. An almost Kenmotsu manifold \(M^3\) such that \(\nabla \xi = -\varphi^2\) is a Kenmotsu manifold.

Proof. In this case the integral manifolds of the distribution \(\mathcal{D}\) are almost Kähler of dimension 2 and thus they are Kähler. The result follows from the previous proposition.
2 Curvature properties and local symmetry

A simple computation gives:

**Proposition 4.** Let \((M^{2n+1}, \varphi, \xi, \eta, g)\) be an almost Kenmotsu manifold. Then, for any \(X, Y \in \mathcal{X}(M^{2n+1})\),

\[
R_{XY}\xi = \eta(X)(Y - \varphi h Y) - \eta(Y)(X - \varphi h X) + (\nabla_Y \varphi h)X - (\nabla_X \varphi h)Y.
\]

**Proposition 5.** Let \((M^{2n+1}, \varphi, \xi, \eta, g)\) be an almost Kenmotsu manifold. For any \(X \in \mathcal{X}(M^{2n+1})\) we have:

\[
R_{\xi \varphi X} = -\varphi^2 X - 2\varphi h X + h^2 X - \varphi(\nabla_{\xi} h)(X),
\]

(5)

\[
(\nabla_{\xi} h)X = -\varphi X - 2h X - \varphi^2 X - \varphi(R_{\xi X} \xi),
\]

(6)

\[
\frac{1}{2}(R_{\xi X} \xi - \varphi R_{\xi \varphi X} \xi) = -\varphi^2 X + h^2 X.
\]

(7)

**Proof.** (5) follows by direct computation, using \(\nabla_{\xi} \varphi = 0\) and (1). Applying \(\varphi\) to (5) and remarking that \(g((\nabla_{\xi} h)X, \xi) = 0\), we get (6). Finally, we write (5) for \(\varphi X\) obtaining

\[
R_{\xi \varphi X} = \varphi X + 2\varphi^2 h X + \varphi h^2 X - \varphi(\nabla_{\xi} h)(\varphi X).
\]

Then, we get

\[
R_{\xi X} \xi - \varphi R_{\xi \varphi X} \xi = -2\varphi^2 X + 2h^2 X - \varphi(\nabla_{\xi} h)(X) + \varphi^2(\nabla_{\xi} h)(\varphi X)
\]

which reduces to (7), since \((\nabla_{\xi} h) \circ \varphi = -\varphi \circ (\nabla_{\xi} h)\).

**Proposition 6.** Let \(M^{2n+1}\) be a locally symmetric almost Kenmotsu manifold. Then, \(\nabla_{\xi} h = 0\).

**Proof.** We notice that (7) can be written as

\[
\frac{1}{2}(R_{\xi \xi} \xi - \varphi R_{\xi \varphi \xi} \xi) = -\varphi^2 + h^2
\]

and since the operator \(R_{\xi \xi} \xi\) is parallel with respect to \(\xi\), \(\xi\) being a geodesic vector field, we get \(\nabla_{\xi} h^2 = 0\). Now, writing (6) as \(\nabla_{\xi} h = -\varphi - 2h - \varphi h^2 - \varphi(R_{\xi} \xi)\) and applying \(\nabla_{\xi}\), we obtain \(\nabla_{\xi}(\nabla_{\xi} h) = -2\nabla_{\xi} h\). Moreover, \(\nabla_{\xi} h^2 = 0\) implies \((\nabla_{\xi} h) \circ h + h \circ \nabla_{\xi} h = 0\), and applying \(\nabla_{\xi}\) to this equality, we get \((\nabla_{\xi} h)^2 = 0\). Hence, \(\nabla_{\xi} h = 0\), since one easily verifies that \(\nabla_{\xi} h\) is a symmetric operator.

**Theorem 3.** Let \((M^{2n+1}, \varphi, \xi, \eta, g)\) be a locally symmetric almost Kenmotsu manifold. Then, the following conditions are equivalent:

a) \(M^{2n+1}\) is a Kenmotsu manifold;

b) \(h = 0\).

Moreover, if any of the above conditions holds, \(M^{2n+1}\) has constant sectional curvature \(K = -1\).
Proof. Assuming that $M^{2n+1}$ is a Kenmotsu manifold, we have $\nabla \xi = -\varphi^2$ and, by (2), $h = 0$. Now, supposing $h = 0$, it follows that $\nabla \xi = -\varphi^2$, $\nabla \eta = g - \eta \otimes \eta$ and, by (4), $R_{XY} \xi = -\eta(Y)X + \eta(X)Y$. Then, we get

$$(\nabla_Z R)(X, Y, \xi) = g(Z, X)Y - g(Z, Y)X - R_{XY}Z.$$ 

Since $\nabla R = 0$, $M^{2n+1}$ has constant sectional curvature $K = -1$. Now, each integral manifold $M'$ of $D$ is an almost Kähler, totally umbilical submanifold and then it has constant sectional curvature ([7]). Computing its sectional curvature for orthonormal vectors $X, Y$ we get:

$$k'(X, Y) = k(X, Y) + \|\xi\|^2 = k(X, Y) + 1 = 0$$

and thus $M'$ is Kähler and flat. By Proposition 2, $M^{2n+1}$ is a Kenmotsu manifold. Hence, $a)$ and $b)$ are equivalent and each of them implies the value $K = -1$ for the curvature.

**Theorem 4.** An almost Kenmotsu manifold of constant curvature $K$ is a Kenmotsu manifold and $K = -1$.

Proof. Clearly, $M^{2n+1}$ is locally symmetric, so $\nabla \xi h = 0$. Comparing (4) and $R_{XY} \xi = K(\eta(Y)X - \eta(X)Y)$, we obtain

$$(K + 1)(\eta(Y)X - \eta(X)Y) - \eta(Y)\varphi hX + \eta(X)\varphi hY - (\nabla Y \varphi h)X + (\nabla X \varphi h)Y = 0.$$ 

Choosing $X = \xi$ and $Y \in D$, we get $-(K + 1)Y + 2\varphi hY - h^2 Y = 0$. Now, if $Y$ is an eigenvector of $h$ with eigenvalue $\lambda$, then $-(K + 1)Y + 2\lambda \varphi Y - \lambda^2 Y = 0$, which implies $\lambda = 0$ and $K = -1$, since $Y$ and $\varphi Y$ are linearly independent. Hence $h = 0$, $K = -1$ and we apply the previous theorem.

Now, we consider the rank of the locally symmetric almost Kenmotsu manifold $M^{2n+1}$. If the rank is equal to one, then $M^{2n+1}$ has constant curvature $K$, being of odd dimension, it is Kenmotsu, $K = -1$ and $h = 0$. If $M^{2n+1}$ does not have constant curvature then, its rank must be greater than one and $h \neq 0$.

**Proposition 7.** Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be a locally symmetric almost Kenmotsu manifold. If $M^{2n+1}$ has rank greater than one, then $\pm 1$ are eigenvalues of $h$.

Proof. The hypothesis on the rank implies that there exists a vector $X$ orthogonal to $\xi$ such that $R_{X\xi} \xi = 0$ and by (6) we get $\varphi X + 2hX + \varphi h^2X = 0$. Let $(\xi, e_1, \ldots, e_n, \varphi e_1, \ldots, \varphi e_n)$ be a local frame of eigenvectors of $h$ with corresponding eigenvalues $(0, \lambda_1, \ldots, \lambda_n, -\lambda_1, \ldots, -\lambda_n)$. Writing $X = \sum_{i=1}^n (X^i e_i + \bar{X}^i \varphi e_i)$, we obtain

$$\sum_{i=1}^n \left( (X^i - 2\bar{X}^i \lambda_i + X^i \lambda_i^2)\varphi e_i + (-\bar{X}^i + 2X^i \lambda_i - \bar{X}^i \lambda_i^2)\bar{X}^i \varphi e_i \right) = 0,$$

which implies

$$\begin{cases} (1 + \lambda_i^2)X^i - 2\lambda_i \bar{X}^i = 0 \\
2\lambda_i X^i - (1 + \lambda_i^2)\bar{X}^i = 0 \end{cases}$$
for each \( i \in \{1, \ldots, n\} \). Since \( X \neq 0 \), there exists \( j \in \{1, \ldots, n\} \) such that the corresponding system admits a non trivial solution and this implies \(- (1 + \lambda_j)^2 + 4 \lambda_j^2 = 0\) and then \( \lambda_j = \pm 1 \).

Let us consider the operator \( h' = h \circ \varphi \). This operator is symmetric and, if \( Y \) is an eigenvector with eigenvalue \( \mu \), then \( \varphi Y \) is an eigenvector with eigenvalue \(- \mu \). Moreover, if \( X \) is an eigenvector of \( h \) with eigenvalue \( \lambda \), then \( X + \varphi X \) is an eigenvector of \( h' \) with eigenvalue \(- \lambda \), while \( X - \varphi X \) is an eigenvector of \( h' \) with eigenvalue \( \lambda \). It follows that \( h \) and \( h' \) admit the same eigenvalues. Denoting by \( [\lambda] \) and \( [\lambda]' \) respectively the eigenspaces of \( h \) and \( h' \) with eigenvalue \( \lambda \), we have \( [\lambda] \oplus [-\lambda] = [\lambda]' \oplus [-\lambda]' \). Furthermore, \( \nabla_{\xi} \varphi = 0 \) implies that \( \nabla_{\xi} h' = 0 \) if and only if \( \nabla_{\xi} h = 0 \).

The operators \( h \) and \( h' \) are related to the curvature by the following proposition.

**Proposition 8.** Let \((M^{2n+1}, \varphi, \xi, \eta, g)\) be a locally symmetric almost Kenmotsu manifold. Then,

1) \( k(X, \xi) = -(1 + \lambda^2) \) for any unit \( h \)-eigenvector \( X \) with eigenvalue \( \lambda \),

2) \( k(X, \xi) = -(1 + \mu^2) \) for any unit \( h' \)-eigenvector \( X \) with eigenvalue \( \mu \).

Furthermore, \( \text{Ric}(\xi, \xi) < 0 \).

**Proof.** Since \( \nabla_{\xi} h = 0 \), from (5), we have \( R_{X\xi} \xi = -X + 2\lambda \varphi X - \lambda^2 X \), and \( k(X, \xi) = g(R_{X\xi} \xi, X) = -1 - \lambda^2 \) which proves 1).

Analogously, since \( \nabla_{\xi} h' = 0 \), applying (5), we have \( R_{X\xi} \xi = -X - 2h'(X) - h''(X) \) for any \( X \in D \), and \( k(X, \xi) = -(1 + \mu^2) \), for any unit eigenvector \( X \) of \( h' \) with eigenvalue \( \mu \).

**Proposition 9.** Let \((M^{2n+1}, \varphi, \xi, \eta, g)\) be a locally symmetric almost Kenmotsu manifold. Then, for any \( X, Y \in \mathcal{X}(M^{2n+1}) \), the curvature tensor satisfies:

\[
R_{Y\xi} \xi + R_{hY\xi} \xi + R_{\xi X} Y + R_{\xi X} Y h' Y = -g(X, Y + h' Y) \xi - \eta(Y)(Y + h' Y) + 2\eta(Y)(X + 2h' X + h'' X) \]
\[
+ 2(\nabla_Y h') X + (\nabla_Y h'' X) \tag{8}
\]

**Proof.** Since \( M^{2n+1} \) is locally symmetric, then \( \nabla_{\xi} h' = 0 \). Being \( h^2 = h'' \), from (5), we have

\[
R_{\xi \xi} \xi = X - \eta(\xi) \xi + 2h' X + h'' X \tag{9}
\]

for any \( X \in \mathcal{X}(M^{2n+1}) \). Derivating with respect to \( Y \in \mathcal{X}(M^{2n+1}) \), since \( \nabla R = 0 \), we get

\[
R_{\nabla_Y \xi \xi} \xi + R_{\xi \nabla_Y \xi} \xi + R_{\xi \xi \nabla_Y \xi} = \nabla_Y X - Y(\eta(\xi)) \xi - \eta(\xi) \nabla_Y \xi + 2\nabla_Y (h' X) + \nabla_Y (h'' X) \tag{10}
\]

Now, applying (9), \( R_{\xi \nabla_Y \xi} \xi = \nabla_Y X - \eta(\nabla_Y X) \xi + 2h'(\nabla_Y X) + h''(\nabla_Y X) \). Moreover, from (1), \( \nabla_Y \xi = Y - \eta(Y) \xi + h' Y \) and thus, \( Y(\eta(\xi)) = Y(g(X, \xi)) = g(\nabla_Y X, \xi) + g(X, Y - \eta(Y) \xi + h' Y) \). Substituting in (10), and using again (9), by a simple computation we obtain (8).
In the following, we denote by $[\mu]$ the distribution of the eigenvectors of $h'$ with eigenvalue $\mu$. We remark that the condition $R_{XY}\xi = 0$ for any $X,Y \in \mathcal{X}(M^{2n+1})$, which gives the local decomposition $\mathbb{R}^{n+1} \times S^n(4)$ in the context of locally symmetric contact metric manifolds, in our case has to be relaxed to $X, Y \in \mathcal{D}$, otherwise we get a contradiction with Proposition 8.

Proposition 10. Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be a locally symmetric almost Kenmotsu manifold and suppose $h' \neq 0$. Then,

1) $\nabla_Y \xi = 0$ and $[\xi, Y] \in [-1]$ for any $Y \in [-1]$, while $\nabla_Y \xi = 2Y$ and $[\xi, Y] \in [+1]$ for any $Y \in [+1]$,

2) the distribution $[-1]$ is integrable with totally geodesic leaves or, equivalently, for any $X, Y \in [-1]$, $R_{XY} \xi = 0$.

Proof. If $Y \in \mathcal{D}$ then we have $\nabla_Y \xi = Y + h'Y$ and this implies that $\nabla_Y \xi = 0$ for any eigenvector $Y$ of $h'$ with eigenvalue $-1$, $\nabla_Y \xi = 2Y$ for any eigenvector $Y$ with eigenvalue $+1$. Furthermore, $\nabla_Y h' = 0$ implies $\nabla_L [-1] \subset [-1]$, $\nabla_L [+1] \subset [+1]$ and (1) holds. From (8) and (4), if $X$ and $Y$ are orthogonal to $\xi$, we have, respectively,

$R_{(Y+h'Y),X} \xi + R_{\xi X} (Y + h'Y) = -g(Y, X + h'Y)\xi + 2(\nabla_Y h')X + (\nabla Y h'^2)X$, \hspace{1cm} (11)

$R_{XY} \xi = (\nabla_X h')Y - (\nabla_Y h')X$. \hspace{1cm} (12)

Supposing $X, Y \in [-1]$, (11) gives

$\nabla_Y X + 2h'(\nabla_Y X) + h'^2(\nabla_Y X) = 0$. \hspace{1cm} (13)

Let $\{0, +1, -1, \lambda_i, -\lambda_i\}$ be the spectrum of $h'$, where $\lambda_i > 0, \lambda_i \neq +1$. Now, $\nabla_Y X$ decomposes as $\nabla_Y X = A_0 + A_1 + A_{-1} + \sum_i A_{\lambda_i} + \sum_i A_{-\lambda_i}$. Hence,

$h'(\nabla_Y X) = A_1 + \sum_i \lambda_i A_{\lambda_i} - \sum_i \lambda_i A_{-\lambda_i}$

$h'^2(\nabla_Y X) = A_1 + \sum_i \lambda_i^2 A_{\lambda_i} + \sum_i \lambda_i^2 A_{-\lambda_i}$.\hspace{1cm} (A)

Applying (13), we get $A_0 = A_1 = 0$ and, for any $i$, $(1+\lambda_i)^2 A_{\lambda_i} = 0$, $(1-\lambda_i)^2 A_{-\lambda_i} = 0$ which imply $A_{\lambda_i} = A_{-\lambda_i} = 0$. Thus $\nabla_Y X \in [-1]$. Being also $\nabla_Y Y \in [-1]$, we deduce that $[X, Y] \in [-1]$ and the distribution $[-1]$ is integrable with totally geodesic leaves. From (12), it follows that the integrability of the distribution $[-1]$ is equivalent to $R_{XY} \xi = 0$ for any $X, Y \in [-1]$.

Theorem 5. Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be a locally symmetric almost Kenmotsu manifold such that $h' \neq 0$ and $R_{XY} \xi = 0$ for any $X,Y \in \mathcal{D}$. Then, the spectrum of $h'$ is $\{0, +1, -1\}$, with 0 as simple eigenvalue. Moreover, choosing $Y \in [-1]$ and $X \in [+1]$ one has $\nabla_Y X \in [+1]$, $\nabla_X Y \in [-1]$ and the distribution $[+1] \oplus [\xi]$ is totally geodesic.

Proof. We know that $0, +1, -1$ are eigenvalues of $h'$. First we prove that for any unit eigenvector $X \in [\lambda]$, with $\lambda \neq -1$, and for any unit $Y \in \mathcal{D}$, orthogonal to $X$, we have

$k(X, Y) = k(\xi, Y)$. \hspace{1cm} (14)
Namely, since \( R_{XY}\xi = 0 \), covariantly derivating with respect to \( X \), we get
\[
0 = R_{\nabla_X Y} Y\xi + R_X \nabla_Y Y\xi + R_{XY} \nabla_X \xi \\
= g(\nabla_X X, \xi) R_{\xi Y} + g(\nabla_X Y, \xi) R_X \xi + (1 + \lambda) R_{XY} X \\
= -(1 + \lambda) R_{\xi Y} \xi + (1 + \lambda) R_{XY} X.
\]

Hence \( R_{\xi Y} \xi = R_{XY} X \) and, taking the scalar product with \( Y \), we get (14). Now, we suppose that there exists a unit eigenvector \( X \in [\lambda] \) with \( \lambda \neq \pm 1 \) and applying (14) to \( X \) and \( \varphi X \) we get \( k(X, \varphi X) = k(\xi, \varphi X) = -(1 - \lambda)^2 \). Again, applying (14) to \( \varphi X \in [-\lambda] \) and choosing \( Y = X \), we have \( k(\varphi X, X) = k(\xi, X) = -(1 + \lambda)^2 \). It follows that \( (1 - \lambda)^2 = (1 + \lambda)^2 \) so that \( \lambda = 0 \) and \( Sp(h') = \{0, +1, -1\} \). Finally, let us suppose that \( dim[0] > 1 \) and let \( X \) be a unit eigenvector orthogonal to \( \xi \) such that \( h'(X) = 0 \). Applying (14) to \( X \) and to a unit \( Y \in [+1] \), we get \( k(X, Y) = k(\xi, Y) = -4 \) and \( k(Y, X) = k(\xi, X) = -1 \), which is a contradiction.

Now, let be \( Y \in [-1] \) and \( X \in [+1] \). Since \([-1]\) is totally geodesic, then \( \nabla_Y X \in [+1] \).
Applying (12) it follows that \( 0 = R_{XY} \xi = -\nabla_X Y - h'(\nabla_X Y) \) so that \( \nabla_X Y \in [-1] \) and \([+1] + [\xi]\) is totally geodesic.

**Theorem 6.** Let \((M^{2n+1}, \varphi, \xi, \eta, g)\) be a locally symmetric almost Kenmotsu manifold such that \( h' \neq 0 \) and \( R_{XY} \xi = 0 \) for any \( X, Y \in D \). Then, \( M^{2n+1} \) is locally isometric to the Riemannian product of an \((n+1)\)-dimensional manifold of constant curvature \(-4\) and a flat \( n\)-dimensional manifold.

**Proof.** As proved in Proposition 10 and Theorem 5, the distributions \([\xi] + [+1], [-1]\) are integrable and totally geodesic. It follows that \( M^{2n+1} \) is locally isometric to the Riemannian product of an integral manifold \( M^{n+1} \) of \([\xi] + [+1]\) and an integral manifold \( M^n \) of \([-1]\). Therefore, we can choose coordinates \((u^0, \ldots, u^{2n})\) such that \( \partial/\partial u^0 \in [\xi] \), \( \partial/\partial u^1, \ldots, \partial/\partial u^n \in [+1] \) and \( \partial/\partial u^{n+1}, \ldots, \partial/\partial u^{2n} \in [-1] \). Now, we set \( X_i = \partial/\partial u^i \) for any \( i \in \{1, \ldots, n\} \), so that the distribution \([-1]\) is spanned by the vector fields \( \varphi X_1, \ldots, \varphi X_n \). We notice that \( [X_i, \varphi X_j] \in [-1] \) for any \( i, j \in \{1, \ldots, n\} \). Taking the scalar product with any \( Z \in [+1] \), since \( \nabla_X, \varphi X_j \in [-1] \), we get \( g(\nabla_{\varphi X_j} X_i, Z) = 0 \) and then \( \nabla_{\varphi X_j} X_i = 0 \). Applying (3), we have \((\nabla_{\varphi} X_j - \varphi(\nabla_{\varphi} X_j)) = 0\), which implies
\[
\nabla_{\varphi X_j} X_i = 0, \quad (\nabla_{\varphi} X_j) = 0,
\]
since the two addenda belong to \([-1]\) and \([+1]\), respectively. The first condition implies that \( M^n \) is flat. We compute the curvature of \( M^{n+1} \). Applying \( \varphi \) to \((\nabla_{\varphi} X_j) = 0\), we have
\[
\nabla_{X_i} X_j + \varphi(\nabla_{\varphi} X_j) = -2g(X_i, X_j)\xi.
\]

Derivating with respect to \( X_k \), we obtain:
\[
\nabla_{X_k} \nabla_{X_i} X_j + (\nabla_{X_k} \varphi)(\nabla_{X_i} \varphi X_j) + \varphi(\nabla_{X_k} \nabla_{\varphi} X_j) = -2X_k(g(X_i, X_j))\xi - 4g(X_i, X_j)X_k
\]
and, by scalar product with \( X_l \),
\[
g(\nabla_{X_k} \nabla_{X_i} X_j, X_l) - g(\nabla_{X_k} \nabla_{\varphi} X_j, \varphi X_l) = -4g(X_i, X_j)g(X_k, X_l),
\]
since \( g((\nabla_{X_k}\varphi)(\nabla_{X_l}\varphi X_j), X_i) = -g(\nabla_{X_i}\varphi X_j, (\nabla_{X_k}\varphi)X_l) = 0. \)

Now, we interchange \( i \) and \( k \), subtract and, being \( [X_i,X_k] = 0 \), obtain
\[
g(R_{X_k,X_l}X_j,X_i) - g(R_{X_k,X_l}\varphi X_j, \varphi X_i) = -4g(X_i,X_j)g(X_k,X_l) + 4g(X_k,X_j)g(X_i,X_l).
\]

Since \( \nabla_{\varphi X_j}X_j = 0 = [\varphi X_i, \varphi X_j] \), then \( g(R_{X_k,X_l}\varphi X_j, \varphi X_i) = g(R_{\varphi X_j,\varphi X_k}X_i, X_j) = 0 \), and thus
\[
g(R_{X_k,X_l}X_j,X_i) = -4g(X_i,X_j)g(X_k,X_l).
\]

Moreover, we recall that \( g(R_{X_k,X_l}\xi, X_k) = 0 \) and, by (5), \( g(R_{X_k,\xi}\xi, X_j) = -4g(X_i,X_j) \).

We conclude that \( M^{n+1} \) is a space of constant curvature \(-4\).

Now, we provide an example of an almost Kenmotsu manifold which is locally isometric to the Riemannian product \( \mathbb{H}^{n+1}(-4) \times \mathbb{R}^n \).

Let \( \{\xi, X_1, \ldots, X_n\} \) be the standard basis of \( \mathbb{R}^{n+1} \) and let us denote by \( \mathfrak{h} \) the Lie algebra obtained by defining:
\[
[X_i, X_j] = -2X_i, \quad [X_i, \xi] = 2X_i, \quad [X_i, X_j] = 0,
\]
for any \( i, j \in \{1, \ldots, n\} \). Let \( \{Y_1, \ldots, Y_n\} \) be the standard basis of \( \mathbb{R}^n \); we consider on \( \mathbb{R}^n \) the structure of abelian Lie algebra, denoted by \( \mathfrak{t} \). On the Lie algebra \( \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{t} \), define the endomorphism \( \varphi : \mathfrak{g} \to \mathfrak{g} \) such that
\[
\varphi(\xi) = 0, \quad \varphi(X_i) = Y_i, \quad \varphi(Y_i) = -X_i,
\]
for any \( i \in \{1, \ldots, n\} \). Let \( \eta : \mathfrak{g} \to \mathbb{R} \) be the 1-form defined by
\[
\eta(\xi) = 1, \quad \eta(X_i) = \eta(Y_i) = 0,
\]
for any \( i \in \{1, \ldots, n\} \). We denote by \( g \) the inner product on \( \mathfrak{g} \) such that the basis \( \{\xi, X_i, Y_i\} \) is orthonormal.

Let \( G, H \) and \( K \) be connected Lie groups with Lie algebras \( \mathfrak{g}, \mathfrak{h} \) and \( \mathfrak{t} \) respectively. Being \( \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{t} \), we have \( G = H \times K \). The vectors \( \xi, X_i, Y_i \) determine left-invariant vector fields on \( G \), which we denote in the same manner. Analogously, we denote by \( \varphi, \eta \) and \( \varphi \) the left-invariant tensor fields determined by the corresponding tensors. It can be easily seen that \( (\varphi, \xi, \eta, g) \) is an almost contact metric structure on \( G \). We prove that it is an almost Kenmotsu structure.

Indeed, for any \( X, Y \in \mathfrak{g} \), \( \eta(X) \) and \( \eta(Y) \) are constant, \( [X, Y] \) is orthogonal to \( \xi \) and then \( d\eta(X,Y) = 0 \) follows. It remains to prove that \( d\Phi = 2\eta \wedge \Phi \). Since \( \Phi(X,Y) \) is constant for any \( X, Y \in \mathfrak{g} \), it follows that for any \( X, Y, Z \in \mathfrak{g} \),
\[
d\Phi(X,Y,Z) = -\frac{1}{3} \left\{ \Phi([X,Y],Z) + \Phi([Y,Z],X) + \Phi([Z,X],Y) \right\}.
\]

On the other hand,
\[
2(\eta \wedge \Phi)(X,Y,Z) = \frac{2}{3} \left\{ \eta(X)\Phi(Y,Z) + \eta(Y)\Phi(Z,X) + \eta(Z)\Phi(X,Y) \right\}.
\]

Now, if \( X, Y \) and \( Z \) are orthogonal to \( \xi \), then \( \eta(X) = \eta(Y) = \eta(Z) = 0 \) and \( [X,Y] = [Z,X] = [X,Y] = 0 \). Hence, \( d\Phi(X,Y,Z) = 2(\eta \wedge \Phi)(X,Y,Z) = 0 \). Let
Almost Kenmotsu manifolds and local symmetry

us suppose that $X = \xi$ and $Y, Z$ orthogonal to $\xi$. Using (15) and (16), we have to verify that

$$-\Phi([\xi, Y], Z) - \Phi([Z, \xi], Y) = 2\Phi(Y, Z).$$

If $Y, Z \in \mathfrak{k}$, then $[\xi, Y] = [Z, \xi] = 0$; moreover, $\varphi Z \in \mathfrak{h}$ and thus $\Phi(Y, Z) = g(Y, \varphi Z) = 0$. Let us suppose that $Y, Z \in \mathfrak{h}$. Then, $[\xi, Y] = -2Y$ and $[Z, \xi] = 2Z$ imply $-\Phi([\xi, Y], Z) - \Phi([Z, \xi], Y) = 4\Phi(Y, Z)$ and, since $\varphi Z \in \mathfrak{k}$, we have $\Phi(Y, Z) = g(Y, \varphi Z) = 0$. Finally, we suppose $Y \in \mathfrak{h}$ and $Z \in \mathfrak{k}$. Since $[\xi, Y] = -2Y$ and $[Z, \xi] = 0$, we have $-\Phi([\xi, Y], Z) - \Phi([Z, \xi], Y) = 2\Phi(Y, Z)$.

Furthermore, it can be easily verified that, for any $X, Y \in \mathfrak{h}$, we have $[X, Y] = l(X)Y - l(Y)X$, where $l : \mathfrak{h} \to \mathbb{R}$ is the linear mapping such that $l(\xi) = -2$ and $l(X_i) = 0$ for any $i \in \{1, \ldots, n\}$. It follows that $H$ is a space of constant sectional curvature $k = -\|l\|^2 = -4$ (see Example 1.7 in [12]). Hence, $H$ is locally isometric to the hyperbolic space of dimension $n + 1$ and curvature $-4$, which implies that $G$ is locally isometric to the Riemannian product $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$.

References


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