Dominant lax embeddings of polar spaces

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Abstract

It is known that every lax projective embedding $e : \Gamma \to PG(V)$ of a point-line geometry Γ admits a *hull*, namely a projective embedding $\tilde{e} : \Gamma \to PG(\tilde{V})$ uniquely determined up to isomorphisms by the following property: V and $\tilde{V}$ are defined over the same skewfield, say K, there is morphism of embeddings $\tilde{f} : \tilde{e} \to e$ and, for every embedding $e' : \Gamma \to PG(V')$ with V' defined over K, if there is a morphism $g : e' \to e$ then a morphism $f : \tilde{e} \to e'$ also exists such that $\tilde{f} = gf$. If $e = \tilde{e}$ then we say that e is *dominant*. Clearly, hulls are dominant. Let now Γ be a non-degenerate polar space of rank $n \geq 3$. We shall prove the following: A lax embedding $e : \Gamma \to PG(V)$ is dominant if and only if, for every geometric hyperplane H of Γ, $e(H)$ spans a hyperplane of $PG(V)$. We shall also give some applications of the above result.

1 Introduction

In the first part of this introduction we will recall the essentials on embeddings, their morphisms and hulls. In the second part, we shall state our main results.

1.1 Basics on embeddings and their morphims

A *projective embedding* of a connected point-line geometry $\Gamma = (P, \mathcal{L})$ is an injective mapping e from the point-set P of Γ to the point-set of a desarguesian projective space Σ such that

(E1) the image $e(P)$ of P spans Σ,

(E2) for every line L of Γ, $e(L)$ spans a line of Σ,

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(E3) no two distinct lines of Γ are mapped by e into the same line of Σ.

If moreover $e(L)$ is a line of Σ for every line L of Γ, then e is said to be *full*. If e is non-full, or we don't know if it is full or we don't care of that, then we say that e is *lax*.

By assumption, Σ is desarguesian, namely $\Sigma = \mathrm{PG}(V)$ for a vector space V. The underlying skewfield K of V will be called the *underlying* skewfield of e. We also say that e is *defined* over K, also that e is a K-*embedding*, for short. As all embeddings considered in this paper are projective, henceforth we simply call them *embeddings*, omitting the word "projective".

Given a subskewfield K_0 of K, a K-embedding $e : \Gamma \to \mathrm{PG}(V)$ and a K_0-embedding $e_0 : \Gamma \to \mathrm{PG}(V_0)$, a *morphism* from e_0 to e is a semilinear mapping $f : V_0 \to V$, where V is regarded as a K_0-vector space and f is such that $e = \mathrm{PG}(f) \cdot e_0$, where $\mathrm{PG}(f)$ stands for the mapping from $\mathrm{PG}(V_0) \setminus \mathrm{Ker}(f)$ to $\mathrm{PG}(V)$ induced by f. Note that f is uniquely determined modulo the choice of a scalar and an embedding of K_0 in K (Granai [3, section 2.4]; compare [5, proposition 9]).

If the morphism f is injective and sends K_0-bases of V_0 to K-bases of V, then we say that f is a *scalar extension*. On the other hand, when $K_0 = K$ then f is surjective, by property (E1) of embeddings and the equality $e = \mathrm{PG}(f) \cdot e_0$. In this case we call f a *projection*, also saying that e is a *projection* of e_0. An *isomorphism* of embeddings is a bijective projection.

Every morphism $f : e_0 \to e$ splits as the composition $f = f_{\mathrm{proj}} f_{\mathrm{ext}}$ of a scalar extension $f_{\mathrm{ext}} : e_0 \to e_1$ and a projection $f_{\mathrm{proj}} : e_1 \to e$. The intermediate embedding e_1 is uniquely determined up to isomorphisms. Accordingly, both f_{ext} and f_{proj} are uniquely determined, modulo isomorphisms.

By [4, Section 3] (see also Ronan [6]), for every K-embedding e of Γ there exists a K-embedding $\tilde{e}$ of Γ such that e is a projection of $\tilde{e}$ and, for every K-embedding e' of Γ, if e is a projection of e' then e' is a projection of $\tilde{e}$. The embedding $\tilde{e}$ is uniquely determined up to isomorphisms and is called the *hull* of e. Following Tits [14, 8.5.2], we say that e is *dominant* if it is its own hull. (Some authors call dominant embeddings 'relatively universal'; see Shult [8], for instance.)

In the literature (see Shult [8], for instance) a full embedding $\tilde{e}_0$ of Γ is said to be absolutely universal if every full embedding e_0 of Γ is a projection of $\tilde{e}_0$. However, in this paper we prefer to call these embeddings *universal full embeddings*, giving the words "absolutely universal" a stronger meaning: We say that a lax embedding $\tilde{e}_0$ of Γ is *absolutely universal* if every embedding e of Γ is a projection of a scalar extension of $\tilde{e}_0$. Clearly, absolutely universal embeddings, when they exist, are uniquely determined up to isomorphisms. Also, the absolutely universal embedding (if it exists) is dominant, but not all dominant embeddings are absolutely universal. Moreover, if Γ admits the absolutely universal embedding $\tilde{e}_0$ and also admits a full embedding, then $\tilde{e}_0$ is full (whence it is the universal full embedding). However, it might happen that Γ admits the universal full embedding but no absolutely universal embedding.

1.2 Main results

All polar spaces to consider in this paper are assumed to be non-degenerate and with at least three points on each line. We recall that a *geometric hyperplane* of a polar space $\Gamma = (P, \mathcal{L})$ is a proper subspace H of Γ such that $L \cap H \neq \emptyset$ for every line $L \in \mathcal{L}$ (hence either $|H \cap L| = 1$ or $L \subseteq H$). We also recall that a polar space is said to be *classical* if it arises from a non-degenerate sesquilinear or non-singular pseudoquadratic form. Every classical polar space Γ admits a universal full embedding $e : \Gamma \to \Sigma = PG(V)$, and the image $e(\Gamma) := (e(P), \{e(L)\}_{L \in \mathcal{L}})$ of Γ is the system of 1- and 2-dimensional linear subspaces of V that are totally isotropic (or totally singular) for a suitable sesquilinear (pseudoquadratic) form of V (Tits [14, chapter 8]).

The following has been proved by Cohen and Shult [1]: given a classical polar space Γ of rank $n \geq 3$, let $e : \Gamma \to \Sigma$ be the universal full embedding of Γ. Then $e(H)$ spans a hyperplane of Σ, for every geometric hyperplane H of Γ. In this paper we generalize that result, proving the following:

Theorem 1.1. *Given a polar space Γ of rank $n \geq 3$ and a lax projective embedding $e : \Gamma \to \Sigma$, the embedding e is dominant if and only if:*

(H) *For every geometric hyperplane H of Γ, $e(H)$ spans a hyperplane of Σ.*

We shall prove this theorem in Section 2. We will prove its "only if" part first. Next, we shall show that the "only if" part of Theorem 1.1 implies the following:

Corollary 1.2. *A polar space Γ of rank $n \geq 3$ admits a lax embedding only if it is classical.*

With the help of this corollary and a few lemmas on subspaces of a classical polar space (to be stated in Section 2), we will be able to prove the "if" part of Theorem 1.1, thus finishing the proof of that theorem. Moreover, by combining our Theorem 1.1 with Theorem 5.1.1 of Steinbach and Van Maldeghem [9] and recalling that all classical polar spaces admit the universal full embedding, one easily obtains the following (see Section 3):

Theorem 1.3. *Every classical polar space of rank $n \geq 3$ admits the absolutely universal embedding.*

As we will notice in Section 3, the following can also be obtained from the proof of the "if" part of Theorem 1.1, as a byproduct:

Theorem 1.4. *Let Γ be a classical generalized quadrangle and $e : \Gamma \to \Sigma$ be a lax projective embedding of Γ. If e satisfies condition* (H) *of Theorem* 1.1*, then e is dominant.*

Remarks. The converse of Theorem 1.4 is false in general. Universal full embeddings of classical generalized quadrangles admitting non-classical ovoids (as $H(3, q^2)$ and $Q(4, 2^{2h+1})$, $h \geq 1$) are obvious counterxamples. More counterexamples can be found among exceptional embeddings of generalized quadrangles. For instance, the generalized quadrangles $Q^-(5,2)$, $Q(4,3)$ and $H(3,2^2)$ admit exceptional embeddings, respectively in $\mathrm{PG}(5,p)$ (p odd prime), $\mathrm{PG}(4,q)$ ($q \equiv 1 \bmod 3$) and $\mathrm{PG}(3,p)$ ($p > 5$, prime), which are dominant but do not satisfy condition (H) of Theorem 1.1 (see Thas and Van Maldeghem [12], [13]). Needless to say, the above mentioned dominant embeddings have nothing to do with the natural (universal) full embeddings of $Q^-(5,2)$, $Q(4,3)$ and $H(3,2^2)$. So, these generalized quadrangles do not admit the absolutely universal embedding. (This makes it clear that the assumption $n \geq 3$ cannot be dropped from Theorem 1.3.)

The case of $W(2)$ is possibly even more interesting. It is well known that $W(2)$ can be embedded in $\mathrm{PG}(2,4)$ as the geometry of exterior points and secant lines of a hyperoval of $\mathrm{PG}(2,4)$. This embedding is dominant (Van Maldeghem [16]), but (H) clearly fails to hold for it. Moreover, $W(2)$ admits a dominant embedding in $\mathrm{PG}(4,K)$, for any skewfield K of characteristic $\neq 2$ (Thas and Van Maldeghem [11], [12]), but these embeddings do not satisfy (H) (Van Maldeghem [16]). Clearly, the above mentioned lax embedding of $W(2)$ cannot arise from the universal full embedding of $W(2)$ (in $\mathrm{PG}(4,2)$). So, $W(2)$ does not admit the absolutely universal embedding.

2 Proof of Theorem 1.1 and Corollary 1.2

2.1 Proof of the “only if” part of Theorem 1.1

This subsection is devoted to the proof of the following proposition, corresponding to the “only if” part of Theorem 1.1

Proposition 2.1. *Let $\Gamma = (P, \mathcal{L})$ be a non-degenerate polar space of rank $n \geq 3$, where all lines have at least three points. Given a lax projective embedding $e : \Gamma \to \Sigma$ of Γ, suppose that e is dominant. Then e satisfies condition* (H) *of Theorem* 1.1.

We recall that the hyperplanes of Γ are maximal subspaces of Γ (Shult [7]). Therefore, given a lax embedding $e : \Gamma \to \Sigma$ and a hyperplane H of Γ, $e(H)$ spans either a hyperplane of Σ or all of Σ. By way of contradiction, suppose that e is dominant but $e(H)$ spans Σ, for a hyperplane H of Γ. Denoted by V the underlying vector space of Σ and given a point $x_0 \in P \setminus H$, put $V_0 := e(x_0)$ (a 1-dimensional linear subspace of V) and $\widehat{\Sigma} = PG(\widehat{V})$ where $\widehat{V} = V \oplus \widehat{V}_0$ for a 1-dimensional vector space $\widehat{V}_0$ defined over the same skewfield as V. The linear subspaces V_0 and $\widehat{V}_0$, regarded as points of $\widehat{\Sigma}$, will be denoted by p_0 and $\hat{p}_0$, respectively. Let V_1 be a complement of V_0 in $V_0 \oplus \widehat{V}_0$, different from $\widehat{V}_0$ and π be the natural projection of $\widehat{V}$ onto $\widehat{V}/V_1$ with $\mathrm{Ker}(\pi) = V_1$. Clearly, $\widehat{V}/V_1 \cong V$ and we can chose an isomorphism $f : \widehat{V}/V_1 \to V$ in such a way that $f(v + V_1) = v$ for every vector $v \in V$. Thus, $\varphi := f\pi$ is a surjective morphism from $\widehat{V}$ to V inducing the identity on V and mapping $\widehat{V}_0$ onto V_0. In the sequel, we also denote by φ the mapping from $\widehat{\Sigma} \setminus \{p_1\}$ to Σ induced by φ, where p_1 stands for the point of $\widehat{\Sigma}$ corresponding to V_1. Thus, regarded Σ as a hyperplane of $\widehat{\Sigma}$ (as we may, as V is a hyperplane of $\widehat{V}$), we have $p_0 = \varphi(p_0) = \varphi(\hat{p}_0)$. We shall define an embedding $\hat{e} : \Gamma \to \widehat{\Sigma}$ such that $\varphi\hat{e} = e$ and $\hat{e}(x_0) = \hat{p}_0$, thus contradicting the assumption that $e = \tilde{e}$.

Regarded Σ as a hyperplane of $\widehat{\Sigma}$, we first put $\hat{e}(x) = e(x)$ for every $x \in H$. Next we extend $\hat{e}$ to $x_0^{\perp}$ as follows:

1) $\hat{e}(x_0) = \hat{p}_0$.
2) Given $x \in x_0^\perp \setminus \{x_0\}$, let L be the line of Γ through x and x_0 and put $x_H := L \cap H$. Let $\widehat{L}$ be the line of $\widehat{\Sigma}$ spanned by the points $\hat{p}_0$ and $e(x_H)$ ($\in \Sigma \subset \widehat{\Sigma}$). As $e(x_H) \neq p_0$, the line $\widehat{L}$ does not contain the point p_1. Hence φ induces a bijection between $\widehat{L}$ an the line $\widetilde{L}$ of Σ containing the image $e(L)$ of L. We denote by $\hat{e}(x)$ the point of $\widehat{L}$ mapped onto $e(x)$ by φ.

So far, we have only defined $\hat{e}$ on the set $P_0 := H \cup x_0^\perp$. Let Π be the subgeometry induced by Γ on P_0.

Lemma 2.2. *The mapping $\hat{e}$ is a lax embedding of* Π *in* $\widehat{\Sigma}$*.*

Proof. Obviously, $\hat{e}$ is injective and $\hat{e}(P_0)$ spans $\widehat{\Sigma}$. It remains to prove the following:

(i) for every line L of Γ, $\hat{e}(L \cap P_0)$ is contained in a line of $\widehat{\Sigma}$.

(ii) if L and M are distinct lines of Γ each of which meets P_0 in at least two points, then $\hat{e}(L \cap P_0)$ and $\hat{e}(M \cap P_0)$ span different lines of $\widehat{\Sigma}$.

We prove (i) first. Let L be a line of Γ meeting P_0 in at least two points. If $x_0 \in L$ then (i) holds by the definition of $\hat{e}$. On the other hand, if $L \cap P_0$ is not contained in $x_0^\perp$ then, according to the definition of P_0, either $L \subseteq H$ or $|L \cap P_0| = 2$. In the first case (i) follows from that fact that $\hat{e}$ induces e on P_0. In the latter case, there is nothing to prove. So, we may assume that $x_0 \notin L$ and $L \subset x_0^\perp$. For every $x \in L$, let x_H be the point of H collinear with x and x_0 and put $L_H = \{x_H\}_{x \in L}$. As $L \subseteq x_0^\perp$, $L \cup \{x_0\}$ spans a singular plane α of Γ. Moreover, $L_H = \alpha \cap H$. Hence L_H is a line of Γ. Therefore $\hat{e}(L_H) = e(L_H)$ spans a line $\widehat{L}_H$ of $\widehat{\Sigma}$, which is also a line of Σ. Morever, $\hat{e}(L)$ is contained in the plane $\hat{\alpha}$ of $\widehat{\Sigma}$ spanned by $\{\hat{p}_0\} \cup \widehat{L}$ whereas $e(\alpha)$ is contained in the plane $\tilde{\alpha}$ of Σ spanned by $\{p_0\} \cup \widehat{L}_H$. Clearly, φ induces an isomorphism from $\hat{\alpha}$ to $\tilde{\alpha}$. Moreover, $e(L)$ is contained in a line of $\tilde{\alpha}$. Hence $\hat{e}(L)$, being the preimage of $e(L)$ by φ, is contained in a line of $\hat{\alpha}$. Claim (i) is proved. Claim (ii) follows from the definition of $\hat{e}$ and the fact that the property analogous of (ii) holds for e. ■

Our next step is to define $\hat{e}$ on the subset $P_1 \subset P$ formed by the points $x \notin P_0 = H \cup x_0^\perp$ such that $x^\perp \cap H \neq x_0^\perp \cap H$. For such a point x, let $S(x)$ be the set of singular planes α of Γ such that $x \in \alpha$ and $\alpha \cap H \neq \alpha \cap x_0^\perp$. Given $\alpha \in S(x)$, it follows from Lemma 2.2 that the set $\hat{e}(\alpha \cap P_0) = \hat{e}(\alpha \cap H) \cup \hat{e}(\alpha \cap x_0^\perp)$ spans a plane $\hat{\alpha}$ of $\widehat{\Sigma}$. On the other hand, the set $e(\alpha \cap P_0) = e(\alpha \cap H) \cup e(\alpha \cap x_0^\perp)$ spans a plane $\tilde{\alpha}$ of $\widehat{\Sigma}$, we have $e(\alpha) \subseteq \tilde{\alpha}$ and φ induces an isomorphism from $\hat{\alpha}$ to $\tilde{\alpha}$. We denote by $\hat{e}_\alpha(x)$ the unique point of $\hat{\alpha}$ mapped onto $e(x)$ by φ.

Lemma 2.3. *For every $x \in P_1$, we have $\hat{e}_\alpha(x) = \hat{e}_\beta(x)$ for any two planes $\alpha, \beta \in S(x)$.*

Proof. Define a graph $\mathcal{S}$ on $S(x)$ by stating that two planes α and β of $S(x)$ are adjacent in $\mathcal{S}$ when they meet in a line. Note that $\mathcal{S}$ is isomorphic to the graph $\mathcal{G}$ the vertices of which are the lines of $H \cap x^\perp$ not contained in $x_0^\perp$, two such lines being adjacent precisely when they meet in a point. The polar space $H \cap x^\perp$ is non-degenerate of rank $n-1 \geq 2$ (indeed, it is isomorphic to the residue of x in Γ) and $x_0^\perp$ meets it in a hyperplane. As proved by Shult [7], every hyperplane complement in a non-degenerate polar space of rank $n-1 \geq 2$ is connected. So, $(H \cap x^\perp) \setminus x_0^\perp$ is connected. Accordingly, the graph $\mathcal{G}$ is also connected. Hence $\mathcal{S}$ is connected, too. Suppose now that $\alpha, \beta \in S(x)$ are adjacent in $\mathcal{S}$. Then $\hat{e}_\alpha(x) = \hat{e}_\beta(x)$, as they are preimages of $e(x)$ by the mapping induced by φ on $\hat{\alpha} \cap \hat{\beta}$. ∎

For $x \in P_1$, we put $\hat{e}(x) = \hat{e}_\alpha(x)$ for $\alpha \in S(x)$. By Lemma 2.3, this definition does not depend on the particular choice of $\alpha \in S(x)$.

Lemma 2.4. *For $x \in P_1$ and every singular plane α of Γ on x, possibly $\alpha \notin S(x)$, let $L := \alpha \cap H$, $\tilde{\alpha}$ be the plane of Σ spanned by $e(L) \cup \{e(x)\}$ and $\hat{\alpha}$ be the plane of $\widehat{\Sigma}$ spanned by $\hat{e}(L) = e(L)$ and $\hat{e}(x)$. Then φ induces an isomorphism φ_α from $\hat{\alpha}$ to $\tilde{\alpha}$. Moreover, φ_α maps $\hat{e}(x)$ onto $e(x)$ and induces the identity mapping on the line spanned by $\hat{e}(L) = e(L)$.*

(Obvious, by considering a plane $\beta \in S(x)$ meeting α in a line.)

Lemma 2.5. *Let Π' be the geometry induced by Γ on $P_0 \cup P_1$. Then $\hat{e}$ is a lax embedding of Π' in $\widehat{\Sigma}$.*

Proof. As $\hat{e}(P_0)$ already spans $\widehat{\Sigma}$, $\hat{e}(P_0 \cup P_1)$ also spans $\widehat{\Sigma}$. Moreover, by definition, $\varphi\hat{e}$ is equal to the restriction $e_{|P_0 \cup P_1}$ of e to $P_0 \cup P_1$. Hence $\hat{e}$ is injective, as such is e. So, the followings remain to be proved:

(i) for every line L of Γ, $\hat{e}(L \cap (P_0 \cup P_1))$ is contained in a line of $\widehat{\Sigma}$.

(ii) if L and M are distinct lines of Γ each of which meets $P_0 \cup P_1$ in at least two points, then $\hat{e}(L \cap (P_0 \cup P_1))$ and $\hat{e}(M \cap (P_0 \cup P_1))$ span different lines of $\widehat{\Sigma}$.

Claim (ii) follows from the analogous property of e and the equality $\varphi\hat{e} = e_{|P_0 \cup P_1}$. We shall prove (i). Let L be a line of Γ with at least two (hence, all but at most one) points in $P_0 \cup P_1$. In view of Lemma 2.2, we may assume that $L \not\subseteq P_0$. Given a plane α on L, the conclusion immediately follows from Lemma 2.4 on α. ∎

Put $L_0 := (H \cap x_0^\perp)^\perp$ and $P_2 := L_0 \setminus (L_0 \cap (P_0 \cup P_1))$. If $P_2 = \emptyset$ then $\Pi' = \Gamma$ and $\hat{e}$ is an embedding of Γ in $\widehat{\Sigma}$ such that $\varphi\hat{e} = e$, which is a contradiction with the hypothesis that e is relatively universal. So, $P_2 \neq \emptyset$.

Let $x \in P_2$. Note that L_0 is either a set of mutually non-collinear points (in fact, a hyperbolic line of Γ) or a line of Γ. In the first case we denote by $L(x)$ the set of lines of Γ through x. In the second case, $L(x)$ stands for the set of lines of Γ through x different from L_0. In any case, given $L \in L(x)$, we have $L \cap P_2 = \{x\}$. Hence L contains at least two points $y, z \in P_0 \cup P_1$. As $\hat{e}(y)$ and $\hat{e}(z)$ have already been defined, and $\hat{e}(y) \neq \hat{e}(z)$, we can consider the line $\widehat{L}$ of $\widehat{\Sigma}$ spanned by $\{\hat{e}(y), \hat{e}(z)\}$ and the line $\widetilde{L}$ of Σ spanned by $\{e(y), e(z)\}$. Clearly, $\widetilde{L}$ does not depend on the particular choice of $y, z \in L \cap (P_0 \cup P_1)$. In view of Lemma 2.5, the line $\widehat{L}$ neither depends on that choice. By definition of $\hat{e}$, φ maps $\widehat{L}$ onto $\widetilde{L}$, hence it induces a bijection from $\widehat{L}$ onto $\widetilde{L}$. We denote by $\hat{e}_L(x)$ the preimage of $e(x)$ by that bijection.

Lemma 2.6. *For every $x \in P_2$, we have $\hat{e}_L(x) = \hat{e}_M(x)$ for any two lines $L, M \in L(x)$.*

Proof. Let $\mathcal{L}$ be the graph with $L(x)$ as the set of vertices, where two lines $L, M \in L(x)$ are adjacent when they are coplanar in Γ. Clearly, $\mathcal{L}$ is connected. So, in the statement of the lemma, we may assume that L and M are coplanar, and let α be the plane of Γ containing L and M. Then either $\alpha \cap L_0 = \{x\}$ or $L_0 \subset \alpha$, the latter case occurring only if L_0 is a line of Γ. In any case, $\alpha \cap (P_0 \cup P_1)$ contains at least one non-collinear triple of points and, exploiting Lemma 2.5, we see that $\alpha \setminus P_2$ is mapped by $\hat{e}$ into a plane $\hat{\alpha}$ of $\widehat{\Sigma}$ which, in its turn, is isomorphically mapped by φ onto a plane $\tilde{\alpha}$ of Σ containing $e(\alpha)$. Therefore, both $\hat{e}_L(x)$ and $\hat{e}_M(x)$ are equal to the preimage of $e(x)$ by the restriction of φ to $\hat{\alpha}$. In particular, $\hat{e}_L(x) = \hat{e}_M(x)$. ■

For $x \in P_2$, we put $\hat{e}(x) = \hat{e}_L(x)$ for $L \in L(x)$. By Lemma 2.6, this definition does not depend on the particular choice of $L \in L(x)$. Thus, we have defined $\hat{e}$ over the whole point-set P of Γ.

Lemma 2.7. *The mapping $\hat{e}$ is an embedding of Γ in $\widehat{\Sigma}$.*

Proof. For every $x \in P_2$ and $L \in L(x)$, all points of $L \setminus \{x\}$ belong to $P_0 \cup P_1$. Hence $\hat{e}(L)$ spans a line of $\widehat{\Sigma}$, by Lemma 2.5 and the definition of $\hat{e}(x)$. If no two points of L_0 are collinear, the above is sufficient to prove that $\hat{e}$ is an embedding. Suppose that L_0 is a line of Γ. Then it remains to prove that $\hat{e}(L_0)$ spans a line of $\widehat{\Sigma}$. We choose two planes α and β on L_0. Arguing as in the proof of Lemma 2.6, $\hat{e}(\alpha)$ and $\hat{e}(\beta)$ span planes $\hat{\alpha}$ and $\hat{\beta}$ of $\widehat{\Sigma}$ such that $\hat{\alpha} \cap \hat{\beta} \supseteq \hat{e}(L_0)$. By way of contradiction, suppose that no line of $\widehat{\Sigma}$ contains $\hat{e}(L_0)$. Then $\hat{e}(L_0)$ contains a non-collinear triple of points and, consequently, $\hat{\alpha} = \hat{\beta}$. However, denoted by $\tilde{\alpha}$ and $\tilde{\beta}$ the planes of Σ spanned by $e(\alpha)$ and $e(\beta)$, φ induces an isomorphism from $\hat{\alpha}$ to $\tilde{\alpha}$ as well as an isomorphism from $\hat{\beta}$ to $\tilde{\beta}$. The equality $\hat{\alpha} = \hat{\beta}$ forces $\tilde{\alpha} = \tilde{\beta}$. Let φ_α be the isomorphism induced by φ from $\hat{\alpha}$ $(= \hat{\beta})$ to $\tilde{\alpha}$ $(= \tilde{\beta})$. By definition of $\hat{e}$, φ_α maps $\hat{e}(L_0)$ onto $e(L_0)$. However, $\hat{e}(L_0)$ spans $\tilde{\alpha}$ whereas $e(L_0)$ spans a line of Σ. This forces $\varphi_\alpha(\tilde{\alpha})$ to be a line of Σ. We have reached a contradiction. ■

The conclusion of Lemma 2.7 is in contradiction with the hypothesis that e is dominant. Proposition 2.1 is proved.

2.2 Proof of Corollary 1.2

A proof of Corollary 1.2 can be found in [2], but we prefer to recall its main steps here. By contradiction, let Γ be a non-classical polar space of rank $n = 3$ admitting a lax embedding $e : \Gamma \to \Sigma$. Then, as proved by Ferrara Dentice, Marino and Pasini [2, Section 6], Σ is a plane and Γ is isomorphic to the grassmannian of lines of Σ_0, for a 3-dimensional projective space Σ_0 defined over a suitable non-commutative subskewfield of the (non-commutative) underlying skewfield of Σ. However, as every embedding admits a hull, we may assume that e is dominant. Then, by Proposition 2.1, e satisfies condition (H) of Theorem 1.1. This contradicts the fact that Σ is a plane.

2.3 On subspaces of classical polar spaces

This subsection is devoted to the proof of the following proposition, which will be used later to prove the "if" part of Theorem 1.1:

Proposition 2.8. *Let $\Gamma = (P, \mathcal{L})$ be a classical non-degenerate polar space of rank $n \geq 2$. Then every proper subspace of Γ of rank at least 2 is contained in a hyperplane of Γ.*

In order to prove Proposition 2.8, we need a couple of lemmas on proportionality between sesquilinear or pseudoquadratic forms (Lemmas 2.9 and 2.11 in the sequel). We presume these two lemmas are well known to everybody who is familiar with sesquilinear and pseudoquadratic forms. Nevertheless, as we have not found any explicit reference to them in the literature in the form we need, we shall prove them here. In either of these lemmas V is a right vector space over a given skewfield K, σ_1 and σ_2 are antiautomorphisms of K and ε_1, ε_2 are elements of K such that

$$\varepsilon_1^{\sigma_1} = \varepsilon_1^{-1}, \quad t^{\sigma_1^2} = \varepsilon_1 t \varepsilon_1^{-1} \text{ for all } t \in K;$$
$$\varepsilon_2^{\sigma_2} = \varepsilon_2^{-1}, \quad t^{\sigma_2^2} = \varepsilon_2 t \varepsilon_2^{-1} \text{ for all } t \in K.$$

Lemma 2.9. *For $i = 1, 2$, let f_i be a reflexive $(\sigma_i, \varepsilon_i)$-sesquilinear form over V, possibly degenerate. Suppose that the set of f_1-isotropic vectors of V contains a subset W with the following properties:*
(A) *W spans V;*
(B) *the totally f_1-isotropic subspaces of V contained in W form a non-degenerate polar space of rank at least 2 and, for any two vectors $v, w \in W$ (possibly, $v = w$), if $f_1(v, w) = 0$ then $\langle v, w\rangle \subseteq W$;*
(C) *for any two vectors $v, w \in W$ (possibly, $v = w$), if $f_1(v, w) = 0$ then $f_2(v, w) = 0$;*

Then $f_2 = \lambda f_1$ for a suitable scalar $\lambda \in K$. Moreover,

$$(*) \quad \lambda t = \varepsilon_1^{\sigma_2} t^{\sigma_1 \sigma_2} \lambda^{\sigma_2} \varepsilon_2$$

for every $t \in K$.

Proof. In the sequel we denote by $\perp$ the ortogonality relation with respect to f_1. We first prove that, for any three vectors $u, v_1, v_2 \in W$ such that $v_1 \perp v_2$ but $f_1(u, v_2) \neq 0$, we have

$$(1) \quad f_2(u, x) = \lambda_{u;v_1,v_2} f_1(u, x) \text{ for any } x \in \langle v_1, v_2\rangle$$

where $\lambda_{u;v_1,v_2} := f_2(u, v_2) f_1(u, v_2)^{-1}$. Indeed, $f_1(u, v_1 + v_2 t) = 0$ for $t = -f_1(u, v_2)^{-1}$ $f_1(u, v_1)$. By (C), we also have $f_2(u, v_1 + v_2 t) = 0$. That is, $f_2(u, v_1) + f_2(u, v_2)t = 0$. More explicitly,

$$f_2(u, v_1) - f_2(u, v_2) f_1(u, v_2)^{-1} f_1(u, v_1) = 0$$

which is (1) with $x = v_1$. Clearly, (1) holds for $x = v_2$, too. By the right-linearity of f_2 and f_1, (1) holds for any $x \in \langle v_1, v_2\rangle$. By (B), the graph induced by $\perp$ on $W \setminus u^\perp$

is connected. Therefore, $\lambda_{u;v_1,v_2} = \lambda_{u;w_1,w_2}$ for any two edges $\{v_1, v_2\}$ and $\{w_1, w_2\}$ of that graph. Hence $\lambda_{u;v_1,v_2}$ only depends on u. Accordingly,

$$(2) \quad f_2(u, x) = \lambda_u f_1(u, x)$$

for a suitable scalar $\lambda_u \in K$ and for any $x \in W$. By (A) and the right-linearity of f_1 and f_2, (2) holds for any $x \in V$. Given $u_1, u_2 \in W$ with $u_1 \perp u_2$, we have $u_1 + u_2 \in W$ by (B). So, we can apply (2) with $u = u_1 + u_2$, and we obtain $\lambda_{u_1+u_2}(f_1(u_1, x) + f_1(u_2, x)) = \lambda_{u_1+u_2} f_1(u_1 + u_2, x) = f_2(u_1 + u_2, x) = f_2(u_1, x) + f_2(u_2, x) = \lambda_{u_1} f_1(u_1, x) + \lambda_{u_2} f_1(u_2, x)$. Hence,

$$(3) \quad \lambda_{u_1+u_2}(f_1(u_1, x) + f_1(u_2, x)) = \lambda_{u_1} f_1(u_1, x) + \lambda_{u_2} f_1(u_2, x).$$

In (3) we can choose x such that $f_1(u_1, x) \neq 0 = f_1(u_2, x)$. Thus, we obtain $\lambda_{u_1+u_2} = \lambda_{u_1}$. Similarly, $\lambda_{u_1+u_2} = \lambda_{u_2}$. Consequently, $\lambda_{u_1} = \lambda_{u_2}$ when $u_1 \perp u_2$. However, by (B), the graph induced by $\perp$ on $W \setminus \{0\}$ is connected. It follows that λ_u does not depend on the choice of $u \in W$. So far,

$$(4) \quad f_2(y, x) = \lambda f_1(y, x)$$

for every $x \in V$ and every $y \in W$. We still must extend (4) to $y \in V \setminus W$. However, before that, we shall prove that λ satisfies $(*)$. By (4) with $x, y \in W$ we obtain the following:

$$\begin{aligned}&\lambda f_1(y, x) = f_2(y, x) = f_2(x, y)^{\sigma_2}\varepsilon_2 = (\lambda f_1(x, y))^{\sigma_2}\varepsilon_2 = \\ &f_1(x, y)^{\sigma_2}\lambda^{\sigma_2}\varepsilon_2 = (f(y, x)^{\sigma_1}\varepsilon_1)^{\sigma_2}\lambda^{\sigma_2}\varepsilon_2 = \varepsilon_1^{\sigma_2} f_1(y, x)^{\sigma_1\sigma_2}\lambda^{\sigma_2}\varepsilon_2.\end{aligned}$$

Hence

$$(5) \quad \lambda f_1(y, x) = \varepsilon_1^{\sigma_2} f_1(y, x)^{\sigma_1\sigma_2}\lambda^{\sigma_2}\varepsilon_2.$$

However, if $f_1(y, x) \neq 0$, by multiplying x for arbitrary scalars we can force $f(y, x)$ to take any value. So, (5) entails $(*)$. Suppose now $y \in V \setminus W$. Then $f_2(x, y) = \lambda f_1(x, y)$ for any $x \in W$. Therefore:

$$\begin{aligned}&f_2(y, x) = f_2(x, y)^{\sigma_2}\varepsilon_2 = (\lambda f_1(x, y))^{\sigma_2}\varepsilon_2 = f_1(x, y)^{\sigma_2}\lambda^{\sigma_2}\varepsilon_2 = \\ &(f_1(y, x)^{\sigma_1}\varepsilon_1)^{\sigma_2}\lambda^{\sigma_2}\varepsilon_2 = \varepsilon_1^{\sigma_1} f_1(y, x)^{\sigma_1\sigma_2}\lambda^{\sigma_2}\varepsilon_2 = \lambda f_1(y, x).\end{aligned}$$

(The last equality follows from $(*)$.) So, (4) holds for any $y \in V$. ■

Lemma 2.10. *If* $\lambda \neq 0$ *then condition* $(*)$ *of Lemma* 2.9 *is equivalent to the following pair of conditions:*

$$(1) \quad \lambda\varepsilon_1 = \lambda^{\sigma_2}\varepsilon_2,$$

$$(2) \quad t^{\sigma_2} = \lambda t^{\sigma_1}\lambda^{-1} \text{ for all } t \in K.$$

Moreover, if λ *satisfies the above two conditions* (1) *and* (2), *then*

$$(3) \quad \lambda t^{\sigma_1} st = t^{\sigma_2}\lambda st \text{ for all } s, t \in K,$$

$$(4) \quad \lambda(t - t^{\sigma_1}\varepsilon_1) = (\lambda t) - (\lambda t)^{\sigma_2}\varepsilon_2 \text{ for all } t \in K.$$

The proof of this lemma is straightforward. We leave it to the reader. Note that, in view of the above conditions (3) and (4), if q_1 is a $(\sigma_1, \varepsilon_1)$-pseudoquadratic form and λ satisfies (1) and (2) of Lemma 2.10, then the product λq_1 is defined and it is a $(\sigma_2, \varepsilon_2)$-pseudoquadratic form.

Lemma 2.11. *For $i = 1, 2$, let q_i be a non-singular $(\sigma_i, \varepsilon_i)$-pseudoquadratic form over V and suppose that every q_1-singular vector of V is also q_2-singular. Then $q_2 = \lambda q_1$ for a scalar $\lambda \neq 0$ satisfying conditions* (1) *and* (2) *of Lemma* 2.10.

Proof. For $i = 1, 2$, let f_i be the sesquilinearization of q_i and let W be the set of q_1-singular vectors of V. Then W, f_1 and f_2 are as in the hypotheses of Lemma 2.9. Therefore $f_2 = \lambda f_1$ for a scalar $\lambda \neq 0$ (note that f_2 is not the null form, as q_2 is non-singular by assumption). So, modulo replacing q_1 with λq_1, we may assume that q_2 and q_1 have the same sesquilinearization $f = f_1 = f_2$. Also, $\sigma_2 = \sigma_1 = \sigma$ and $\varepsilon_2 = \varepsilon_1 = \varepsilon$, say. Clearly, the difference $q_2 - q_1$ of q_2 and q_1 is a (σ, ε)-pseudoquadratic form and its sesquilinearization is the null form $f_2 - f_1 = f - f$. Let x and y be any two q_1-singular vectors of V. By assumption, x and y are q_2-singular, too. Hence they are singular for $q_2 - q_1$. Therefore $x + y$ is also singular for $q_2 - q_1$, as the sesquilinearization of $q_2 - q_1$ is the null form. It follows that any linear combination of q_1-singular vectors is singular for $q_2 - q_1$. However, the set of q_1-singular vectors spans V. Hence $q_2 - q_1$ is the null form. Consequently, $q_2 = q_1$. ■

Corollary 2.12. *Let $\Gamma = (P, \mathcal{L})$ be a classical non-degenerate polar space of rank $n \geq 2$ and S be a subspace of Γ such that the polar space induced by Γ on S is non-degenerate of rank at least* 2. *Let $e : \Gamma \to \Sigma$ be the universal full embedding of Γ and suppose that $e(S)$ spans Σ. Then $S = P$ (the improper subspace of Γ).*

Proof. Denoted by V the underlying vector space of Σ, let q be the non-singular pseudoquadratic form of V that defines the image $e(\Gamma)$ of Γ in Σ and, denoted by Γ_1 the polar space induced by Γ on S, let $e_1 : \Gamma_1 \to \Sigma$ be the full embedding induced by e on Γ_1. Assume first that e_1 is universal. Then $e(\Gamma_1)$ $(= e_1(\Gamma_1))$ is also defined by a non-singular pseudoquadratic form q_1 on V and all vectors of V that are q_1-singular are q-singular, too. By Lemma 2.11, q_1 and q are proportional. Hence $e(\Gamma_1) = e(\Gamma)$, namely $\Gamma_1 = \Gamma$.

We shall now prove that the hypothesis that e_1 is non-universal leads to a contradiction, thus finishing the proof Corollary 2.12. Assuming that e_1 is not universal, let $\tilde{e}_1 : \Gamma_1 \to PG(\widetilde{V})$ be the universal full embedding of Γ_1 (which is well known to exist). Then $\tilde{e}_1(\Gamma)$ is defined by a non-singular (σ, ε)-pseudoquadratic form $\tilde{q}_1$ on $\widetilde{V}$ and we have $e_1 = p\tilde{e}_1$ for a suitable projection $p : \widetilde{V} \to V$. Put $R := \mathrm{Ker}(p)$ and let $\tilde{f}_1$ be the sesquilinearization of $\tilde{q}_1$. Then $R \subseteq \mathrm{Rad}(\tilde{f}_1)$. Thus, we can define the projection f_1 of $\tilde{f}_1$ onto V, and we have $f_1(p(x), p(y)) = \tilde{f}_1(x, y)$ for any two vectors $x, y \in \widetilde{V}$. Denoted by f the sesquilinearization of q, we can apply Lemma 2.9 to f_1 and f, taking as W the set of all vectors of V representing points of $e(\Gamma_1)$ or equal to 0. We obtain that $f = \lambda f_1$ for a scalar $\lambda \neq 0$. Thus, modulo replacing q with λq, we may assume that $f_1 = f$. Accordingly, q is (σ, ε)-pseudoquadratic, for the same choice of σ and ε as for $\tilde{q}_1$. This allows us to 'lift' q to $\widetilde{V}$, as follows: given a complement U of R in $\widetilde{V}$, for $x \in V$ let x_R and x_U be the component of x in R and U, namely $x = x_R + x_U$. We put $\tilde{q}(x) := \tilde{q}_1(x_R) + q(p(x))$ $(= \tilde{q}_1(x_R) + q(p(x_U)))$ for every $x \in \widetilde{V}$. As the projection f_1 of $\tilde{f}_1$ onto V is equal to the sesquilinearization f of q, $\tilde{q}$ is indeed a non-singular (σ, ε)-pseudoquadratic form with the same sesquilinearization $\tilde{f}_1$ as $\tilde{q}_1$. Therefore $\tilde{q} = \tilde{q}_1$, by Lemma 2.11. Consequently, if $\tilde{q}_1(x) = 0$ then $\tilde{q}(x) = 0$, for every vector $x \in \widetilde{V}$. On the other hand, if $\tilde{q}_1(x) = 0$ and $x \neq 0$ then $p(x)$ spans a point of $e(\Gamma_1)$ and, as $e(\Gamma_1) \subseteq e(\Gamma)$, we also have $q(p(x)) = 0$. This and the equality $\tilde{q}(x) = 0$ imply $\tilde{q}_1(x_R) = 0$. Hence $x_R = 0$, as $\tilde{q}_1$ is non-degenerate. It follows that all $\tilde{q}_1$-singular vectors belong to U. However, the $\tilde{q}_1$-singular vectors span $\widetilde{V}$. Hence $R = 0$. Accordingly, $\tilde{e}_1 = e_1$, contrary to the hypothesis that e_1 is non-universal. ■

Proof of Proposition 2.8. We are now ready to prove Proposition 2.8. Let S be a proper subspace of Γ such that the polar space Γ_1 induced by Γ on it has rank at least 2. If Γ_1 is non-degenerate, then S is contained in the hyperplane $p^{\perp}$, for every point $p \in \mathrm{Rad}(\Gamma_1)$. In this case, we are done. Suppose that Γ_1 is non-degenerate. Then we can apply Corollary 2.12 to the universal full embedding $e : \Gamma \to \Sigma$ of Γ, obtaining that $e(S)$ does not span Σ. Therefore $e(S)$ is contained in a hyperplane $\overline{H}$ of Σ. Accordingly, S is contained in the hyperplane $H = e^{-1}(\overline{H} \cap e(\Gamma))$ of Γ.

2.4 End of the proof of Theorem 1.1

The "if" part of Theorem 1.1 remains to be proved. Given a non-degenerate polar space $\Gamma = (P, \mathcal{L})$ of rank at least 3, let $e : \Gamma \to \Sigma = PG(V)$ be a lax projective embedding of Γ satisfying condition (H) of Theorem 1.1 and $\tilde{e} : \Gamma \to \widetilde{\Sigma} = PG(\widetilde{V})$ be the hull of e. Then $e = f\tilde{e}$ for a semilinear mapping $f : \widetilde{V} \to V$. Put $R := \mathrm{Ker}(f)$. We shall prove that $R = 0$, thus obtaining that $e = \tilde{e}$.

Suppose that $R \neq 0$ and let $[R]$ be the subspace of $\widetilde{\Sigma}$ corresponding to R. Then $[R] \cap \tilde{e}(L) = \emptyset$ for every line L of Γ. Hence, given a line L of Γ, we can choose a complement U of R in $\widetilde{V}$ such that the subspace $[U]$ of $\widetilde{\Sigma}$ corresponding to U contains $\tilde{e}(L)$. Put $S := \tilde{e}^{-1}([U] \cap \tilde{e}(P))$. Then S is a proper subspace of Γ and the polar space induced by Γ on S has rank at least 2, as it contains L. On the other hand, Γ is classical by Corollary 1.2. (Note that we are free to apply that corollary, since only the "only if" part of Theorem 1.1 has been exploited in its proof.) Hence Proposition 2.8 can be applied and we obtain that S is contained in a hyperplane H of Γ. However, $e(S)$ spans Σ, as $U + R = \widetilde{V}$. Hence $e(H)$ also spans Σ, contrary to the hypothesis that e satisfies (H). Theorem 1.1 is proved.

3 Proof of Theorems 1.3 and 1.4

Proof of Theorem 1.3. Let Γ be a classical polar space of rank $n \geq 3$. Then Γ is embeddable and admits the universal full embedding $\tilde{e}_0 : \Gamma \to \widetilde{\Sigma}_0$. Every lax embedding $e : \Gamma \to \Sigma$ admits a hull $\tilde{e} : \Gamma \to \widetilde{\Sigma}$. By Theorem 1.1, $\tilde{e}$ is weak in the sense of [9]. By Steinbach and Van Maldeghem [9, 5.1.1], $\tilde{e}$ is a scalar extension of $\tilde{e}_0$.

Proof of Theorem 1.4. The "only if" part of the proof of Theorem 1.1 remains valid if Γ is a classical generalized quadrangle. Indeed, in that part of the proof, the hypothesis $n \geq 3$ has been exploited only when using Corollary 1.2 to conclude that Γ is classical.

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