Some properties of inductively minimal geometries

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Abstract

We study the properties $(2T)_1$, $(IP)_2$, PRI, RPRI, QPRI and RWPRI for inductively minimal pairs (Γ, G) consisting of a finite geometry and a group acting flag-transitively on it. For each of these properties, we characterize which inductively minimal pairs satisfy it.

1 Introduction and notation

This paper finds its origin in the systematic search of group-geometry pairs with specified properties for given almost simple groups provided in [4], [7], [8], [5], [9], [6] and the more theoretical paper [3]. The present work relies only on [3] as to needed results and uses concepts that were introduced in [1] and [7].

We recall notation and definitions used for finite diagram geometries. More details can be found in [2]. Let I be a finite set of $n \geq 1$ elements called *types*. A *geometry over* I is a triple $\Gamma = (X, *, t)$ where X is a set whose members are called *elements* of Γ, the symbol $*$ denotes a symmetric, reflexive binary relation on X, called *incidence relation* and t is a mapping of X onto I, called *type function*, such that $a * b$ and $t(a) = t(b)$ implies $a = b$. The *rank* of Γ is $|I| = n$. If $A \subseteq X$, the *type* of A is the set $t(A)$. A *flag* F of Γ is a complete subgraph of $(X, *)$. We assume that Γ is *firm* that is, every flag F with $t(F) \neq I$ is contained in at least two flags of type I.

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The *residue* Γ_F of a flag F, is the firm geometry over $I\backslash t(F)$ whose elements are all $a \in X\backslash F$ such that $\{a\} \cup F$ is also a flag, together with the restrictions of $*$ and t to these elements. For $J \subseteq I$, we define the *J-truncation* of Γ to be the geometry over J, whose element set is $t^{-1}(J)$, together with the restrictions of $*$ and t.

A geometry is said to satisfy property $(\mathbf{IP})_\mathbf{2}$ if for every residue of rank 2 we have that either every element of one type is incident with every element of the other type or that two different elements of the same type are incident with at most one element of the other type.

The *diagram* of Γ is the graph $(I, \sim)$ defined by $i \sim j$, for $i \neq j$ in I, if and only if there is a flag F of type $I\backslash\{i, j\}$ such that Γ_F is not a complete bipartite graph; this means that Γ_F has elements a, b with $t(a) = i$, $t(b) = j$ and $(a, b) \notin *$. We assume that the diagram $(I, \sim)$ is *connected.* A *subgraph* of $(I, \sim)$ is a subset J of I provided with the restriction of $\sim$ to J. From here on, adjacency in a graph is meant to be $\sim$. We call Γ *thin* if every flag F with $|t(F)| = n - 1$, has a residue containing exactly two elements. The geometry Γ is *residually connected* if for every residue Γ_F with $|t(\Gamma_F)| \geq 2$, the graph induced by $*$ on Γ_F is connected.

An *automorphism* of Γ is an automorphism α of the graph $(X, *)$ mapping every element x of X onto an element $\alpha(x)$ such that $t(x) = t(\alpha(x))$.
Let G be a group of automorphisms of Γ. If G acts transitively on the flags F with $t(F) = I$, we say that G acts *flag-transitively.*

We now consider pairs (Γ, G) where Γ is a geometry and G is a group of automorphisms acting flag-transitively. For $i \in I$, let G_i denote the stabilizer in G of an element of type i. A pair (Γ, G) is said to be *PRI* (resp. *WPRI*) provided for every (resp. at least one) type i, the subgroup G_i is maximal in G. We denote the element set of a residue Γ_F by X_F. The permutation group $G_F^{X_F}$ induced by the action of the stabilizer G_F of F in G on X_F is flag-transitive. In this work, the kernel of this action is always the identity and therefore we replace the symbol $G_F^{X_F}$ by G_F. The pair (Γ, G) is said to satisfy *RPRI* (resp. *RWPRI*) if (Γ_F, G_F) is PRI (resp. WPRI) for every residue in Γ. Another interesting property is QPRI. We recall that a subgroup H of G is *quasi-maximal* if there is a unique chain from H to G in the subgroup lattice of G. A pair (Γ, G) is called *QPRI* if every stabilizer G_i is quasi-maximal in G.

A pair (Γ, G) is said to satisfy $(\mathbf{2T})_\mathbf{1}$ if for each flag F of rank $|I| - 1$, the group G_F is 2-transitive on the residue Γ_F.

Let $(X, \sim)$ be a finite connected graph. A vertex e of X is called an *end* if the subgraph induced by $\sim$ on $X\backslash\{e\}$ is connected. In [10], the author defines what we would call nonends as *cutpoints.* It is known that every finite nontrivial connected graph has at least two ends (See for instance [10]). In a connected graph $(X, \sim)$, a nonend f such that all connected components of $X \setminus \{f\}$ have the same cardinality will be called a *middle* of that graph.

The definition for inductively minimal pair is given in [3] as follows. Let (Γ, G) be a pair consisting of a finite, firm geometry Γ of rank n with connected diagram, together with a group G of automorphisms acting flag-transitively on Γ. This pair is called *minimal* if $|G| \leq (n + 1)!$. We call (Γ, G) *inductively minimal* if for any connected subgraph J of $(I, \sim)$ and any flag F of Γ, with $t(F) = I\backslash J$, the pair (Γ_F, G_F) is minimal.

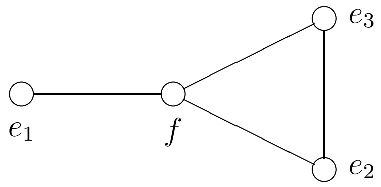
We recall the main theorem of [3].

Lemma 1 *If* (Γ, G) *is an inductively minimal pair and* Γ *has rank* n *and diagram* $(I, \sim)$*, then*

1. Γ *is thin and residually connected;*
2. $G \cong Sym(n+1)$ *and for each* $i \in I$ *such that* $I \setminus \{i\}$ *is connected,* Γ *has* $n+1$ *elements of type* i *on which* G *acts faithfully;*
3. $(I, \sim)$ *has no minimal circuits of length greater than* 3*;*
4. *Every edge of* $(I, \sim)$ *is on a unique maximal clique;*
5. *Each vertex of* $(I, \sim)$ *is on either one or two maximal cliques.*

In [3] it is also proved that for any connected finite graph $(I, \sim)$ satisfying the last three conditions of lemma 1, there is, up to isomorphism, exactly one inductively minimal pair (Γ, G) admitting $(I, \sim)$ for diagram.

For example, we can start with the following diagram which satisfies the conditions.



This diagram has exactly three ends e_1, e_2, e_3 and one nonend f. We now describe the inductively minimal pair (Γ, G) having this diagram. For details, we refer to [3].

First of all we have $G \cong Sym(5)$. The rank 4 geometry Γ has 5 elements of each end type e_1, e_2 and e_3 which can be seen as the intersection points of a (5×3)-grid. Two elements of end type are incident iff they are on different rows and on different columns of the grid. The elements of type f are pairs of orthogonal blocks in the grid where one block contains only elements of type e_1 and the other is its orthogonal complement. A given element of type f is incident with all elements of end type which are member of one of the blocks defining it.

A useful property of the diagram of an inductively minimal pair can easily be proved.

Lemma 2 *Let* $(I, \sim)$ *be the diagram of an inductively minimal geometry and* f *one of its vertices. The number of connected components of* $I \setminus \{f\}$ *is at most 2.*

Proof. Suppose the number of connected components of $I \setminus \{f\}$ is k. Then, by connectedness of $(I, \sim)$, there must be vertices $v_1, v_2, \ldots, v_k$ which are adjacent to f and all lie in different connected components. Since these vertices are mutually nonadjacent, no two of them can be member of the same (maximal) clique. We know that in $(I, \sim)$ every edge lies on exactly one maximal clique. Hence the edges $v_i \sim f$ for $1 \leq i \leq k$ define k different maximal cliques containing f. Since the vertex f can be on either one or two maximal cliques of the diagram, the number of connected components is also at most 2. ■

The main results of this paper can now be stated.

Theorem 1 *If* (Γ, G) *is inductively minimal then we have the following properties:*

1. (Γ, G) *satisfies RWPRI,* $(2T)_1$ *and* $(IP)_2$*;*
2. *The inductively minimal pair* (Γ, G) *satisfies PRI provided the rank of* Γ *is even;*
3. *If* (Γ, G) *is not PRI, then it is QPRI and this is the case if and only if the diagram* I *of* Γ *has a middle;*
4. *The pair* (Γ, G) *satisfies RPRI if and only if the diagram of* Γ *is a complete graph.*

2 Proof of the theorem

The definition of an inductively minimal pair (Γ, G) gives us the structure of the automorphism group induced on the residue of a flag F provided this residue has a connected diagram. Indeed, if $I \setminus t(F)$ is connected, we know that (Γ_F, G_F) is inductively minimal and hence that $G_F \cong Sym(m+1)$ where m is the rank of Γ_F. This yields the parts of the theorem concerning $(2T)_1$ and $(IP)_2$ which are shown in the next two lemmas.

Lemma 3 *An inductively minimal pair satisfies* $(2T)_1$.

Proof. since a residue of rank 1 has a connected diagram, we know that it has two elements and the group induced on it is the group of order 2. ∎

Lemma 4 *An inductively minimal pair satisfies* $(IP)_2$.

Proof. Consider a residue of rank two in an inductively minimal geometry. If its diagram is disconnected, it satisfies $(IP)_2$ because of the way we defined the diagram of a geometry. For a connected diagram, the residue is known to be inductively minimal. In such an inductively minimal geometry of rank 2, there are three elements of each type. Take two elements x and y of the same type. The residue being thin (See lemma 1), both of them are incident with two of the three elements of the other type. Suppose x and y are both incident with the *same* two elements of the other type. Then the third element z of this type is neither incident with x nor with y. But then z is incident with at most one element, contradicting the hypothesis stating that inductively minimal geometries must be firm. ∎

Knowledge of the residues with connected diagram enabled us to prove properties of inductively minimal pairs. It could be fruitful to know the structure of residues whose diagram is not connected.

Let J be the type of a flag F in an inductively minimal pair (Γ, G) with $I \setminus J$ disconnected. We denote the connected components of $I \setminus J$ by $I_1, \ldots, I_k$ where $k \geq 2$. The I_l-truncation of Γ_F will be denoted by $\Gamma_F^{I_l}$ for $1 \leq l \leq k$. By the direct sum theorem (see [2], p. 81), we know that every element of such a truncation $\Gamma_F^{I_l}$ is incident to every element of Γ_F whose type is not in I_l. Now we understand that if we complete the flag F to a flag F' of type $I \setminus I_l$, the residue $\Gamma_{F'}$ will exactly be $\Gamma_F^{I_l}$. Hence this truncation is the residue of a flag F' such that $I \setminus t(F') = I_l$ is connected.

We define the group G_F^l to be the stabilizer in G_F of the elements whose type is not in I_l. Again by the direct sum theorem we have $G_F^l \cong G_{F'}$. This means that for every $l \in \{1, \dots, k\}$, the pair $(\Gamma_F^{I_l}, G_F^l)$ is inductively minimal. An immediate consequence is that all G_F^l must be symmetric groups of degree $|I_l| + 1$.

Lemma 5 *Let F be a flag of type J in an inductively minimal pair (Γ, G) with $I \setminus J$ disconnected and having components $I_1, \dots, I_k$. Then G_F is the direct product of the groups G_F^l with $l = 1, \dots, k$.*

Proof. We give a proof by induction on the number k of connected components of $I \setminus J$. First we remark that the groups G_F^l are normal subgroups of G_F.

If $k = 2$, it is clear that $G_F^1 \cap G_F^2 = \{1_{\Gamma_F}\}$. For $g \in G_F$, we can decompose g to $g_1 g_2 \in G_F^1.G_F^2$. To achieve this, we define $g_1 : \Gamma_F^{I_1} \cup \Gamma_F^{I_2} \to \Gamma_F$ to be the identity on $\Gamma_F^{I_2}$ and the restriction of g on $\Gamma_F^{I_1}$. The automorphism g_2 is defined in the same way. We now have $G_F \cong G_F^1 \times G_F^2$.

Assume $k > 2$. Define G' to be the stabilizer in G_F of the elements whose type appears in I_k. By the direct sum theorem (see [2], p. 81), the group G' is isomorphic to $G_{F''}$ for a flag F'' of type $J \cup I_k$ containing F. Obviously, we have $G' \trianglelefteq G_F$ and $G' \cap G_F^k = \{1_{\Gamma_F}\}$. In the same way as for $k = 2$, one shows $G_F = G'.G_F^k$. These conditions are sufficient to show that $G_F \cong G' \times G_F^k$. Since $\Gamma_{F''}$ has a diagram with $k - 1$ connected components which are the components $I_1, \dots, I_{k-1}$ of Γ_F and for every $l \in \{1, \dots, k-1\}$ we have $G_F^l \cong G_{F''}^l$, the induction hypothesis provides that G' is the direct product of the groups G_F^l for $l < k$. ∎

A particular case of this lemma arises when we put $F = \{x\}$ for an element x of nonend type f. By lemma 2, we have exactly two connected components in $I \setminus \{f\}$. This implies that G_x is the direct product of two symmetric groups.

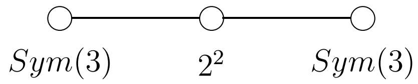
Corollary 1 *If an inductively minimal pair is not PRI, then it is QPRI. This is the case if and only if the diagram I of Γ has a middle.*

Proof. If an inductively minimal pair (Γ, G) of rank n does not satisfy PRI, there must be an $i \in I$ with G_i not maximal in $G \cong Sym(n+1)$. We know that i is certainly not an end since the residue of an element of end type yields an inductively minimal pair of rank $n-1$ and hence $G_i \cong Sym(n)$, which is maximal in $Sym(n+1)$ (See [11]). Let I_1 and I_2 be the two connected components of $I \setminus \{i\}$ and put $n_1 = |I_1|$ and $n_2 = |I_2|$. By lemma 5, we have that $G_i = Sym(n_1 + 1) \times Sym(n_2 + 1)$, where $n_1 + n_2 = n - 1$. Referring to [11] we can assert that the direct product G_i of such symmetric groups is not maximal in the symmetric group G if and only if $n_1 = n_2$, but then G_i is quasi-maximal in G. ∎

Corollary 2 *If (Γ, G) is an inductively minimal pair with Γ of even rank n, then (Γ, G) satisfies PRI.*

Proof. If the number of types n is even, there cannot be a type f such that $I \setminus \{f\}$ has two connected components of the same cardinality. ∎

Let us remark that there exist non-PRI inductively minimal pairs if n is odd. The smallest arises for $G = Sym(4)$ with the following diagram.



More generally, we can say that every inductively minimal pair of odd rank with a string diagram is not PRI.

Lemma 6 *If* (Γ, G) *is inductively minimal, then it satisfies RWPRI.*

Proof. Let (Γ, G) be an inductively minimal pair over I. We first have to show that there is at least one type i such that G_i is a maximal subgroup of G. Then we prove that condition WPRI holds in every nontrivial residue of (Γ, G). We use induction on the rank n of Γ.

1. For $n = 1$, the properties are clearly fulfilled.

2. Assume $n \geq 2$. Take an end e of the diagram of Γ. Like in the proof of corollary 1, we show that G_e is maximal in G.

3. Now take F to be a nonempty flag of Γ with $J = t(F) \neq I$. If $I \setminus J$ is connected, we apply induction. Assume now that $I \setminus J$ is disconnected and apply lemma 5 yielding $G_F \cong G_F^1 \times \cdots \times G_F^k$, where k is the number of connected components of $I \setminus J$. Take e to be an end of the connected component I_1 and complete F by an element x of type e. The stabilizer of x in the inductively minimal pair $(\Gamma_F^{I_1}, G_F^1)$ is then a maximal subgroup M of G_F^1. Applying lemma 5 in the residue $(\Gamma_F)_x = \Gamma_{F \cup \{x\}}$, we get $G_{F \cup \{x\}} \cong M \times G_F^2 \times \cdots \times G_F^k$, which is maximal in G_F. ■

We remark that the argument in part 3 of the proof of lemma 6 can be greatly simplified if J contains an end e of $(I, \sim)$. We then simply choose an element $x \in F$ of type e. Then Γ_x has rank $n - 1$ and, by induction, the pair (Γ_x, G_e) satisfies RWPRI. This means that (Γ_F, G_F) has the property WPRI, as a residue in Γ_x.

Finally, we characterize the inductively minimal pairs which are RPRI.

Lemma 7 *An inductively minimal pair* (Γ, G) *satisfies RPRI if and only if the diagram of* Γ *is a complete graph.*

Proof. If the diagram of Γ is a complete graph, then all types are ends and hence all G_i are maximal subgroups of G. It is also the case that all residues have a complete graph as diagram, so induction completes the proof of sufficiency.

If the diagram of Γ is not complete, then there is at least one pair of types i, j with $i \not\sim j$. Since the diagram is connected, we have a shortest path $(i = i_0 \sim i_1 \sim \cdots \sim i_k = j)$ whose length k is at least 2. Put $J = \{i_0, i_1, \ldots, i_k\}$; then a residue of type J has a string diagram and is inductively minimal of rank $k + 1$. If k is even, we have found a residue which is not PRI. In the other case, we take a residue of type $J \setminus \{j\}$. ■

Finally, the theorem results from putting together the lemmas 3, 4, 6, 7 and the corollaries 1 and 2.

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