# A partition of the Euclidean plane $\mathbb{R}^2$ into $k$ pairwise isometric connected subsets.

Pierre Sleewaegen *

**Abstract**

The purpose of this note is to construct, for any given $k \geq 1$ , a partition of the Euclidean plane $\mathbb{R}^2$ into $k$ pairwise isometric connected subsets.

It is easy to partition the Euclidean plane $\mathbb{R}^2$ into $k$ pairwise isometric subsets. For example, one may take the subsets

$$S_i^{(k)} = \{(x,y) \in \mathbb{R}^2 \,\Big|\, i-1 \ \leq \ y + c.k \ < \ i \ \text{ for some } \ c \in \mathbb{Z}\} \; i \in \{1, 2, \cdots, k\}$$

However, if in addition the subsets of the partition are required to be connected, we have not found any example of such a partition in the literature.

We will then prove the following
**Proposition :** *For any given* $k \in \mathbb{N}_0$ *, there exists a partition of the Euclidean plane into* $k$ *pairwise isometric connected subsets.*

*Proof :* We will construct such a partition explicitly...
For each pair of integers $(m,n) \in \mathbb{Z}^2$ , let $A_{m,n}$ be the 'vertical spring'

$$A_{m,n} = \left[ (x,y) \in \mathbb{R}^2 \,\Big|\, \left\{ \begin{array}{l} m < y \leq m+1 \\ x = \frac{1}{3}\sin(\frac{2\pi}{y-m}) + n \end{array} \right. \right]$$

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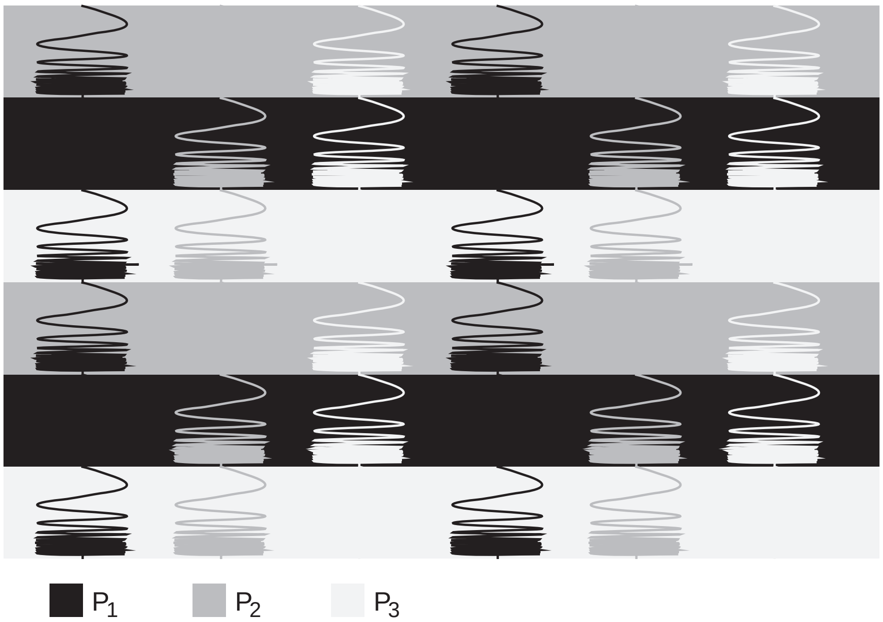
For $i \in \{1, 2, \cdots, k\}$ , let $S_i^{(k)}$ be defined as above. We now define the following subsets $P_j$ of $\mathbb{R}^2$ :

$$P_j = \left[ \left( \bigcup_{a,b \in \mathbb{Z}} (A_{a,j+k.b}) \right) \bigcup S_j^{(k)} \right] \setminus \left[ \bigcup_{\substack{d,e \in \mathbb{Z} \\ f \in \{1,2,\cdots,k\} \setminus \{j\}}} (A_{d,f+k.e}) \right] \qquad (j \in \{1, 2, \cdots, k\})$$

It is easy to check that the subsets $P_1, \cdots, P_k$ are connected, pairwise disjoint, pairwise isometric ($P_i$ is clearly the image of $P_j$ under a translation in $\mathbb{R}^2$), and that $\mathbb{R}^2 = \bigcup_{j \in \{1,2,\cdots,k\}} P_j$. Therefore $\left\{ P_j \,\middle|\, j \in \{1, 2, \cdots, k\} \right\}$ is a partition of $\mathbb{R}^2$ satisfying all our requirements... ■

To make the above construction more explicit, here is a figure describing the case when k = 3 :




**Remark :** Note that the subsets $P_1, \cdots, P_k$ constructed above are not arcwise-connected. As far as we know, the existence of a partition of $\mathbb{R}^2$ into $k$ pairwise isometric arcwise-connected subsets seems to be an unsolved problem...

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Department of Mathematics,
Free University of Brussels,
Campus Plaine, CP 218
Boulevard du Triomphe, B-1050 Brussels, Belgium.