Solutions to equations of p-Laplacian type in Lorentz spaces

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Abstract

We find solutions to the linear problem (1.1) and to the p-Laplacian type problem (1.2) in Lorentz spaces, improving the sumability of the solutions.

1 Introduction

We consider the linear problem

$$(1.1)\begin{cases} L(u) \equiv \operatorname{div}(M(x)\nabla u) & = & \operatorname{div} F & \text{in} & \Omega \\ u & = & 0 & \text{in} & \partial\Omega \end{cases}$$

where $\Omega \subset R^N$ is a bounded domain and $M(x)$ is a symmetric matrix in $L^\infty(\Omega)^{N\times N}$ satisfying the ellipticity condition: $M(x)\xi\cdot\xi \geq \alpha|\xi|^2$ for $x \in \Omega$, $\xi \in \mathbb{R}^N$ $(\alpha > 0)$, and the nonlinear problem

$$(1.2)\begin{cases} N(u) \equiv \operatorname{div}(a(x,u(x),\nabla u(x))) & = & \operatorname{div} F & \text{in} & \Omega \\ u & = & 0 & \text{in} & \partial\Omega \end{cases}$$

Specifically, let $A(u)$ be a monotone operator of Leray-Lions type ([LL],[Li]) : $A(u) = \operatorname{div}(a(x,u,\nabla u))$, with $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ a Caratheodory function verifying the following conditions:

i)There exist two constants $\alpha, \beta > 0$, and a function $d(x)$ in $L^{p'}(\Omega)$ $(\frac{1}{p}+\frac{1}{p'}=1)$, $1 < p < N$ such that:

$$a(x,s,\xi)\xi \geq \alpha|\xi|^p \ (1.3)$$

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$$|a(x,s,\xi)| \leq \beta(d(x) + |s|^{p-1} + |\xi|^{p-1})$$

ii) For $\xi, \eta \in R^N$, $\xi \neq \eta$, and a.e. for $x \in \Omega$: $[a(x,s,\xi) - a(x,s,\eta)] \cdot (\xi - \eta) > 0$

We recall that for $a(x,s,\xi) = |\xi|^{p-2}\xi$, the Leray-Lions type operator $A = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-laplacian (See [DJM],[G]).

Our main results are:

Theorem 1 Let $F \in L^{q,q^\#}(\Omega, \mathbb{R}^N)$ be, with $L^{q,q^\#}$ the Lorentz space, $2 < q < N$ and $q^\# = \frac{q(N-2)}{N-q}$. Then, there exists a unique solution $u \in H_0^1 \cap L^{q^*,q^\#}(\Omega)$ to (1.1). Morover we have the a-priori estimate:

$$\|u\|_{L^{q^*,q^\#}(\Omega)} \leq C\|F\|_{L^{q,q^\#}}$$

Theorem 2

Consider $F \in L^{q,q^\#}(\Omega, \mathbb{R}^N)$ and:

$$p' < q < \frac{N}{p-1}$$

$$q^\# = \frac{(N-p)q}{N-(p-1)q}$$

Then, there exists a unique solution $u \in W_0^{1,p} \cap L^{r,s}$ to (1.2) where:

$$r = \frac{N(p-1)q}{N-(p-1)q}$$

$$s = \frac{(N-p)(p-1)q}{N-(p-1)q}$$

Furthermore, we have the apriori estimate:

$$\|u\|_{L^{r,s}} \leq C\|F\|_{L^{q,q^\#}}^{p'}$$

Remark The weak formulation of problem (1.1) is: find $u \in H_0^1(\Omega)$ with

$$\int_\Omega M(x)\nabla u \cdot \nabla\varphi = \int_\Omega F \cdot \nabla\varphi \quad \forall\varphi \in H_0^1(\Omega)$$

if $F \in (L^2(\Omega))^N$, then $\operatorname{div}F \in H^{-1}(\Omega)$ and from Lax-Milgram's lemma we obtain the existence of a unique solution in $H_0^1(\Omega)$.

When $F \in L^q$ with $q > 2$, we obtain a better sumability of the solution u from the following theorem of G. Stampachia ([S2], Theorem 4.2)

Theorem Let $F \in L^q(\Omega)^N$ with $q > 2$, and supose that u is a weak solution to problem (1.1). Then we have:

i) If $2 \leq q < N$ then $u \in L^{q^*}(\Omega)$

ii) If $q = N$ then $u \in L^p(\Omega)$ for any $p < +\infty$

iii) If $q > N$ we have that $u \in L^\infty(\Omega)$

Here $q^* = \frac{qN}{N-q}$ is the Sobolev exponent.

As remarked by Boccardo ([B]) , a similar result holds in the non-linear case for operators of monotone type.

Remarks

i) In theorem 1, we have $q^{\#} > q$ hence $L^q \subset L^{q,q^{\#}}$, and $q^{\#} < q*$, so $L^{q^*,q^{\#}} \subset L^{q^*}$.
ii) Theorem 2 is a extension of theorem 1 to the nonlinear case.

2 Preliminaries: Lorentz spaces

The Lorentz spaces [Lo] are a generalization of the L^p spaces and are related to several topics of Harmonic Analysis, in connection with the Marcinkiewicz interpolation theorem (see [SW],[R]) and with the convolution operators (see [O]).

In this section we give some definitions and results [T] that we will need in order to prove theorems 1 and 2.

Definition 2.1 Let $(X, \mathcal{M}, \mu)$ a measure space and $u : X \to \mathbb{R}^k$ a measurable function. Suppose that $\mu(\{x \in X : u(x) > t\}) < \infty$ for any t. The distribution function of u, and the decreasing rearangement of u, u^*, are defined as:

$$d(t) = d_u(t) = \mu(\{x \in X : |u(x)| > t\})$$

and $u^*(s) = \min\{t \geq 0 : d_u(t) \leq s\}$.

Definition 2.2 For $1 < p < \infty$ we define the pseudo-norm

$$|u|_{L^{p,q}} = |u|_{L^{p,q}} = \left(\int_0^\infty (s^{1/p} u^*(s))^q \frac{ds}{s} \right)^{1/q}$$

and

$$|u|_{L^{p,\infty}} = \sup_{s>0} s^{1/p} u^*(s)$$

The Lorentz space $L^{p,q}(X, \mathbb{R}^k)$ is defined as the set of measurable functions with $|u|_{L^{p,q}} < \infty$.

We recall that $|u|_{L^{p,q}}$ is not a norm. A norm can be introduced defining

$$\|u\|_{L^{p,q}} = \left(\int_0^\infty (s^{1/p} u^{**}(s))^q \frac{ds}{s} \right)^{1/q}$$

where

$$u^{**}(s) = \frac{1}{s} \int_0^s u^*(t) dt$$

Proposition 2.3 (Equivalence between $\|u\|_{L^{p,q}}$ and $|u|_{L^{p,q}}$, see [T], (4.v))

1. If $p > 1$ then:
$$\|u\|_{L^{p,1}} = \frac{p}{p-1} \int_0^\infty s^{1/p} u^*(s) \frac{ds}{s}$$
2. If $p > 1$ and $1 < q \leq \infty$ then:
$$\left(1 - \frac{1}{p}\right) \|u\|_{L^{p,q}} \leq |u|_{L^{p,q}} \leq \|u\|_{L^{p,q}}$$

We recall that the Lorentz spaces $L^{p,p}$ are the classical spaces L^p:

Proposition 2.4 (see [O], lemma 2.2) Let $1 < p < \infty$ be and $\frac{1}{p} + \frac{1}{p'} = 1$, then we have:

$$\|f\|_{L^p} \leq \|f\|_{L^{p,p}} \leq p'\|f\|_{L^p}$$

From this result and the following one, we can compare the result obtained in L^p and the one in the Lorentz space.

Proposition 2.5 (see [T], 4. vii))
Let be $p > 1$ and $1 \leq q < r \leq \infty$. Then $L^{p,q} \subset L^{p,r}$ with continuous inclusion.

In the Lorentz spaces it is possible to improve the Sobolev inequality:

Theorem 2.6 ([T], Theorem 4.A) Let $1 \leq p < n$ then $W^{1,p}(\mathbb{R}^n) \subset L^{p^*,p}(R^n)$, where $p^* = \frac{np}{n-p}$, with continuous imbeding.

Using an standard density argument we obtain:

Corollary 2.7
Let $\Omega \subset R^n$, then $W_0^{1,p}(\Omega) \subset L^{p,p^*}(\Omega)$ with continuous imbeding, and for $\partial\Omega \in C^1$ we have the same result for $W^{1,p}(\Omega)$

We also have an inequality of the Hölder type:

Proposition 2.8 Let $f \in L^{p,q}(X)$ and $g \in L^{p',q'}(X)$ be, with $\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$. Then $f \cdot g \in L^1(X)$ and the following inequality holds:

$$\int_X |f \cdot g| d\mu \leq |f|_{p,q} |g|_{p',q'}$$

We recall that from ([T], Theorem 1.A) we also have:

$$\int_X |f \cdot g| d\mu \leq \int_0^\infty f^*(s) g^*(s) ds = \int_0^\infty s^{1/p} f^*(s) \frac{1}{s^{1/q}} s^{1/p'} g^*(s) \frac{1}{s^{1/q'}} ds$$

$$\leq \left\{ \int_0^\infty \left(s^{1/p} f^*(s) \right)^q \frac{ds}{s} \right\}^{1/q} \left\{ \int_0^\infty \left(s^{1/p'} f^*(s) \right)^{q'} \frac{ds}{s} \right\}^{1/q'} = |f|_{p,q} |g|_{p',q'}$$

where the last inequality holds from the usual Hölder inequality.

For $k > 0$ and $x \in R$ define the truncating function:

$$T_k(x) = \begin{cases} -k & \text{if} \quad x < k \\ x & \text{if} \quad -k \leq x \leq k \\ k & \text{if} \quad x > k \end{cases}$$

Remark 2.9 If

$$\|T_k(u)\|_{L^{(p,q)}(X)} \leq C$$

for any k, then $u \in L^{p,q}(X)$ and $\|u\|_{L^{(p,q)}(X)} \leq C$.

Remark 2.10 Let $f \in L^{p,q}$ and $m > 0$ be. Then $|f|^m \in L^{pm,qm}$

$$||f|^m|_{p,q} = |f|^m_{mp,mq}$$

In fact we have

$$d_{|u|^m}(t) = d_u(t^{1/m})$$

It follows that $(|u|^m)^* = (u^*)^m$, then,

$$||u|^m|_{p,q} = \left\{ \int_0^\infty (s^{1/pm} u^*(s))^{mq} \frac{ds}{s} \right)^{1/q} = |u|^m_{pm,qm}$$

3 The linear problem

Proof of theorem 1

We choose $\varphi = \frac{1}{2m+1} |T_k(u)|^{2m} T_k(u)$ as a test function in the weak formulation of the problem.

Hence, $\nabla\varphi = |T_k(u)|^{2m} \nabla(T_k u)$, $\varphi \in H_0^1$ and

$$\int_\Omega |T_k(u)|^{2m} M(x) \nabla u \cdot \nabla T_k(u) = \int_\Omega |T_k(u)|^{2m} F \cdot \nabla T_k(u) \tag{3.1}$$

Using the ellipticity condition, we can estimate the first term as:

$$\begin{aligned}\int_\Omega |T_k(u)|^{2m} M(x) \nabla u \cdot \nabla T_k(u) &= \int_\Omega |T_k(u)|^{2m} M(x) \nabla T_k(u) \cdot \nabla T_k(u) \\ &\geq \alpha \int_\Omega |\nabla T_k(u)|^2 |T_k(u)|^{2m}\end{aligned}$$

Regarding to the second term, we obtain:

$$\begin{aligned}\int_\Omega |T_k(u)|^{2m} F \cdot \nabla T_k(u) &\leq \int_\Omega |T_k(u)|^{2m} |F| \cdot |\nabla T_k(u)| \\ &= \int_\Omega (|T_k(u)|^m |F|) \cdot (|T_k(u)|^m |\nabla T_k(u)|) \\ &\leq \left(\int_\Omega |T_k(u)|^{2m} |F|^2 \right)^{1/2} \left(\int_\Omega |T_k(u)|^m |\nabla T_k(u)|^2 \right)^{1/2}\end{aligned}$$

From the Hölder inequality for Lorentz spaces,

$$\begin{aligned}\int_\Omega |T_k(u)|^{2m} |F|^2 &\leq \left| |F|^2 \right|_{L^{q/2, q^\#/2}} \left| |T_k(u)|^{2m} \right|_{L^{q/(q-2), q^\#/(q^\#-2)}} \\ &\leq |F|^2_{L^{q,q^\#}} |T_k u|^{2m}_{L^{2mq/(q-2), 2mq^\#/(q^\#-2)}}\end{aligned}$$

Then, the second term of (3.1) is smaller than

$$|F|_{L^{q,q^\#}} |T_k u|^m_{L^{2mq/(q-2), 2mq^\#/(q^\#-2)}} \left(\int_\Omega |T_k(u)|^{2m} |\nabla T_k(u)|^2 \right)^{1/2}$$

and writing all together,

$$\begin{aligned}&\alpha \int_\Omega |\nabla T_k(u)|^2 |T_k(u)|^{2m} \\ &\qquad \leq |F|_{L^{q,q^\#}} |T_k u|^m_{L^{2mq/(q-2), 2mq^\#/(q^\#-2)}} \left(\int_\Omega |T_k(u)|^{2m} |\nabla T_k(u)|^2 \right)^{1/2}\end{aligned}$$

or equivalently,

$$\alpha\left(\int_\Omega |\nabla T_k(u)|^2|T_k(u)|^{2m}\right)^{1/2} \leq |F|_{L^{q,q^\#}}|T_k u|^m_{L^{2mq/(q-2),2mq^\#/(q^\#-2)}}$$

On the other hand, we have that

$$|\nabla T_k(u)|^2|T_k(u)|^{2m} = \left|\nabla\left(\frac{|T_k(u)|^{m+1}}{m+1}\right)\right|^2$$

then,

$$\alpha\left\|\nabla\left(\frac{|T_k(u)|^{m+1}}{m+1}\right)\right\|_{L^2(\Omega)} \leq |F|_{L^{q,q^\#}(\Omega)}|T_k u|^m_{L^{2mq/(q-2),2mq^\#/(q^\#-2)}}$$

and from the Sobolev inequality in Lorentz spaces:

$$\||T_k(u)|^{m+1}\|_{L^{2^*,2}(\Omega)} \leq c|F|_{L^{q,q^\#}}|T_k u|^m_{L^{2mq/(q-2),2mq^\#/(q^\#-2)}}$$

where c is a constant depending on the ellipticity constant, on the Sobolev inequality constant and on m, but not on k.

Finally we have that

$$|T_k u|^{m+1}_{L^{2^*(m+1),2(m+1)}} \leq C|F|_{L^{q,q^\#}}|T_k u|^m_{L^{2mq/(q-2),2mq^\#/(q^\#-2)}}$$

Now we choose the exponents , in order to have the same norm in both sides of the inequallity:

i) $2^*(m+1) = \frac{2mq}{q-2}$

ii) $2(m+1) = \frac{2mq^\#}{q^\#-2}$.

Condition i) is the one in Stampachia's theorem. From i) we obtain: $m = \frac{N(q-2)}{2(N-q)}$.

With this value of m, we get

$$2^*(m+1) = \frac{2mq}{q-2} = \frac{2q}{q-2}\frac{N(q-2)}{2(N-q)} = \frac{qN}{N-q} = q^*$$

Finally, from ii) we obtain

$$q^\# = 2(m+1) = \frac{q(N-2))}{N-q}$$

and

$$|T_k u|_{L^{q^*,q^\#}} \leq c|F|_{L^{q,q^\#}}$$

with c independent of k. From remark 2.9 we conclude that $u \in L^{q^*,q^\#}(\Omega)$ and

$$|u|_{L^{q^*,q^\#}(\Omega)} \leq C|F|_{L^{q,q^\#}}$$

4 The nonlinear problem

Proof of theorem 2

First, we recall that for $F \in L^{p'}$, $\mathrm{div}F \in (W_0^{1,p}(\Omega))'$, by the Leray-Lions Theorem ([LL]) N is a monotone operator and there exists a unique solution to the problem:

$$(1.2) \begin{cases} N(u) \equiv \mathrm{div}(a(x,u(x),\nabla u(x))) & = & \mathrm{div}F & \text{in } \Omega \\ u & = & 0 & \text{in } \partial\Omega \end{cases}$$

in $W_0^{1,p}(\Omega)$.

In order to prove theorem 2, we choose

$$\varphi = \frac{|T_k(u)|^{mp}T_k(u)}{mp+1} \in W_0^{1,p}(\Omega)$$

as a test function in the weak formulation. Then,

$$\int_\Omega |T_k(u)|^{mp} a(x,u,\nabla u)\cdot\nabla T_k(u) = \int_\Omega |T_k(u)|^{mp} F\cdot\nabla T_k(u)$$

We can estimate the first term using (1.3) as:

$$\int_\Omega |T_k(u)|^{mp} a(x,u,\nabla u)\cdot\nabla T_k(u) = \int_{\{|u|\le k\}} |T_k(u)|^{mp} a(x,u,\nabla T_k(u))\cdot\nabla T_k(u)$$

$$= \int_\Omega |T_k(u)|^{pm} a(x,u,\nabla T_k(u))\cdot\nabla T_k(u) \ge \alpha \int_\Omega |\nabla T_k(u)|^p |T_k(u)|^{pm}$$

and using the Hölder inequality in the second term, we obtain:

$$\int_\Omega |T_k(u)|^{mp} F\cdot\nabla T_k(u) \le \int_\Omega (|T_k(u)|^{m(p-1)}|F|)|T_k(u)|^m|\nabla T_k(u)|$$

$$\le \left(\int_\Omega |T_k u|^{mp}|F|^{p'}\right)^{1/p'} \left(\int_\Omega |T_k(u)|^{mp}|\nabla T_k u|^p\right)^{1/p}$$

and then,

$$\int_\Omega |T_k u|^{mp}|\nabla T_k u|^p \le \frac{1}{\alpha^{p'}} \int_\Omega |T_k u|^{mp}|F|^{p'}$$

From the Hölder inequality in Lorentz spaces, and the fact that $F \in L^{q,q^\#}(\Omega)$, we get

$$\int_\Omega |T_k(u)|^{mp}|F|^{p'} \le \left\| |F|^{p'} \right\|_{L^{q/p',q^\#/p'}} \left\| |T_k(u)|^{mp} \right\|_{L^{q/(q-p'),q^\#/(q^\#-p')}}$$

or equivalently (Remark 2.10),

$$\int_\Omega |T_k(u)|^{mp}|F|^{p'} \le |F|^{p'}_{L^{q,q^\#}} \, |T_k(u)|^{mp}_{L^{r,s}}$$

where

$$r = \frac{mpq}{q-p'}, \qquad s = \frac{mpq^\#}{q^\# - p'}$$

On the other hand, by Sobolev inequallity,

$$\int_\Omega |T_k(u)|^{mp} |\nabla T_k u|^p = \int_\Omega \left| \nabla \left(\frac{|T_k(u)|^m T_k(u)}{m+1} \right) \right|^p$$

$$\geq c \, \| |T_k(u)|^m T_k(u) \|^p_{L^{p^*,p}} = \| T_k(u) \|^{(m+1)p}_{L^{p^*(m+1),p(m+1)}}$$

with $c = c(m, \Omega)$.

Then,

$$\| T_k u \|^{(m+1)p}_{L^{p^*(m+1),p(m+1)}} \leq c \, \| F \|^{p'}_{L^{q,q^\#}} \, \| T_k(u) \|^{mp}_{L^{r,s}}$$

Now we choose m and $q^\#$ such that:
$r = p^*(m+1) = \frac{mpq}{q-p'}$, and $s = p(m+1) = \frac{mpq^*}{q^*-p'}$.
Solving the first equation for m:

$$m = \frac{\frac{N}{N-p}}{\frac{q}{q-p'} - \frac{N}{N-p}}$$

From $p' < q < \frac{N}{p-1}$, we obtain $m > 0$. Hence,

$$r = p^*(m+1) = \frac{N(p-1)q}{N-(p-1)q}$$

$$s = p(m+1) = \frac{(N-p)(p-1)q}{N-(p-1)q}$$

and from the second equation

$$q^\# = \frac{(N-p)q}{N-(p-1)q}$$

From proposition 2.9 we get:

$$\| u \|_{L^{r,s}} \leq C \| F \|^{p'}_{L^{q,q^\#}}$$

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