Spectrum of a particular bounded self-adjoint linear operator

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Abstract

Let Ω be a connected bounded open set in $\mathbf{R}^N$, $N \geq 2$, with lipschitzian boundary. The best constant in the Poincaré type inequality:

$$| \, u \, |_2^2 \leq C(\Omega) \parallel grad\,(u) \parallel_{-1}^2, \; \forall u \in L^2(\Omega)/\mathbf{R}$$

is the inverse of the smallest spectral value of the the bounded self-adjoint linear operator $T = -div(-\Delta)^{-1}grad$ in $L^2(\Omega)/\mathbf{R}$ ([4]). In this paper we show that, in the case of an elliptical domain of $\mathbf{R}^2$, the point spectrum of this operator is the set $\sigma_p(T) = \{\, \lambda_n \,, \, \widetilde{\lambda}_n \,, \, 1 \,; \, n \in \mathbf{N^*}\}$, where 1 is an eigenvalue of infinite multiplicity and

$$\lambda_n = \frac{1}{2} - \frac{2ab(n+1)(a^2-b^2)^n}{(a+b)^{2n+2} - (a-b)^{2n+2}} \,, \qquad \widetilde{\lambda}_n = \frac{1}{2} + \frac{2ab(n+1)(a^2-b^2)^n}{(a+b)^{2n+2} - (a-b)^{2n+2}} \,.$$

If $a \neq b$, λ_n and $\widetilde{\lambda}_n$ are eigenvalues of multiplicity 1, they converge to $1/2$ when $n \to \infty$ and $\sigma(T) = \sigma_p(T) \cup \{1/2\}$. If $a = b$, $\lambda_n = \widetilde{\lambda}_n = 1/2$ is an eigenvalue of infinite multiplicity and $\sigma(T) = \sigma_p(T) = \{1/2\,,\,1\}$. Consequently, if $b \leq a$, $\dfrac{a^2+b^2}{b^2}$ is the best constant in the preceding Poincaré type inequality.

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1 Preliminaries

Let Ω be a bounded, open, connected domain in $\mathbf{R}^N$, $N \geq 2$, with regular boundary $\partial\Omega$. Throughout this paper, we use the usual product topology on the product spaces.

In $L^2(\Omega)$, the Hilbert norm and the scalar product are written $|\cdot|_2$ and $(\cdot,\cdot)_2$. Let $M(\Omega)$ be the closed subspace of $L^2(\Omega)$ of functions of zero mean :

$$M(\Omega) = \left\{ u \in L^2(\Omega) \, ; \, \int_\Omega u(x)dx = 0 \right\}.$$

$M(\Omega)$ is equipped with the norm induced by the Hilbert space $L^2(\Omega)$, and it is isometrically isomorphic to the quotient space $L^2(\Omega)/\mathbf{R}$.

The Sobolev space $H_0^1(\Omega)$ is equipped with the gradient norm. We denote by $H^{-1}(\Omega)$ the dual space of $H_0^1(\Omega)$ normed by :

$$\| f \|_{H^{-1}(\Omega)} = \text{Sup} \left\{ \frac{< f, v >}{\| v \|_{H_0^1(\Omega)}} \, ; \quad v \in H_0^1(\Omega) \, , \quad v \neq 0 \right\},$$

where $< \cdot, \cdot >$ denotes the duality between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. $(H_0^1(\Omega))^N$ is isomorphic to $(H^{-1}(\Omega))^N$ and $-\Delta$ is this isometric isomorphism. We shall write $\| \cdot \|_{-1}$ for the norm on $(H^{-1}(\Omega))^N$.

The important inequality which follow is proved in [5] :

Proposition 1. *There exists a constant $C(\Omega) \geq 1$, depending only on Ω, such that :*

$$| u |_2^2 \leq C(\Omega) \, \| \, grad\,(u) \, \|_{-1}^2 \, , \quad \forall u \in M(\Omega). \tag{1}$$

Notation. *In the remainder of this paper, the best value of constant $C(\Omega)$ in the inequality* (1) *is denoted by $P(\Omega)$:*

$$P(\Omega)^{-1} = \text{Inf} \left\{ \frac{\| \, grad\,(u) \, \|_{-1}^2}{| u |_2^2} \, ; \quad u \in M(\Omega) \, , \quad u \neq 0 \right\}.$$

From proposition 1, the operator $T = -div(-\Delta)^{-1} grad$ is an isomorphism from $M(\Omega)$ onto $M(\Omega)$. Moreover, for all $u \in M(\Omega)$, we have $(Tu, u)_2 = \| \, grad\,(u) \, \|_{-1}^2$. Consequently,

$$P(\Omega)^{-1} = \text{Inf} \{ \, (Tu, u)_2 \, ; \quad u \in M(\Omega) \, , \quad u \neq 0 \}$$

Important properties of this operator T are proved in [4] :

Theorem 1. *T is a self-adjoint and coercive operator. $Tu - u$ is a harmonic function, $\forall u \in M(\Omega)$. $\| T \| = 1$ and 1 is an eigenvalue of T of infinity multiplicity. If u is an eigenvector of T corresponding to an eigenvalue $\lambda \neq 1$, then u is a harmonic function.*

Consequently ([1]), *the spectrum $\sigma(T)$ of T is closed, $\sigma(T) \subset [P(\Omega)^{-1}, 1]$, the residual spectrum of T is empty and $P(\Omega)$ is the inverse of smallest spectral value of T.*

2 Case where Ω is an elliptical domain

In the particular case where Ω is an elliptical domain :

$$\Omega = \left\{(x,y) \in \mathbf{R}^2 \,;\; \frac{x^2}{a^2} + \frac{y^2}{b^2} < 1\right\},$$

we are able to give the spectrum $\sigma(T)$ of the operator T.

Proposition 2. *The point spectrum of the operator* $T = -div(-\Delta)^{-1}grad$ *is the set* $\sigma_p(T) = \left\{\lambda_n\,,\,\widetilde{\lambda}_n\,,1\,;\, n \in \boldsymbol{N}^*\right\}$ *where*

$$\lambda_n = \frac{1}{2} - \frac{2ab(n+1)(a^2-b^2)^n}{(a+b)^{2n+2}-(a-b)^{2n+2}} \qquad \text{and} \qquad \widetilde{\lambda}_n = \frac{1}{2} + \frac{2ab(n+1)(a^2-b^2)^n}{(a+b)^{2n+2}-(a-b)^{2n+2}}\,.$$

1 *is an eigenvalue of infinite multiplicity. If* $a \neq b$, λ_n *and* $\widetilde{\lambda}_n$ *are eigenvalues of multiplicity* 1*, they converge to* $1/2$ *when* $n \to \infty$ *and* $\sigma(T) = \sigma_p(T) \cup 1/2$*. If* $a = b$, $\lambda_n = \widetilde{\lambda}_n = 1/2$ *is an eigenvalue of infinite multiplicity and* $\sigma(T) = \sigma_p(T) = \{1/2, 1\}$*.*

Proof.- We are going to search harmonic polynomials of degree n such that they are eigenvectors of T. We shall write ∂_x for $\dfrac{\partial}{\partial x}$ and ∂_y for $\dfrac{\partial}{\partial y}$.

Let $u_n = \rho^n \cos(n\theta)$ be the harmonic homogeneous polynomial of degree $n = 2m$ (even) :

$$u_n = x^n - \binom{n}{2} x^{n-2}y^2 + \binom{n}{4} x^{n-4}y^4 + \ldots + (-1)^{m-1}\binom{n}{n-2} x^2y^{n-2} + (-1)^m y^n.$$

Let us calculate $Tu_n = -\partial_x(-\Delta)^{-1}\partial_x u_n - \partial_y(-\Delta)^{-1}\partial_y u_n$. The first step is to obtain $(-\Delta)^{-1}\partial_x u_n$. For this, we search a polynomial of the form

$$\alpha_0\, x^{n-1} + \alpha_2\, x^{n-3}y^2 + \ldots + \alpha_{n-4}\, x^3y^{n-4} + \alpha_{n-2}\, xy^{n-2} + P_{n-3}(x,y),$$

where $P_{n-3}(x,y)$ is a polynomial of degree $n-3$, such that

$$-\Delta\left[\Big(\alpha_0\, x^{n-1} + \alpha_2\, x^{n-3}y^2 + \ldots + \alpha_{n-2}\, xy^{n-2} + P_{n-3}(x,y)\Big)\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)\right] = \partial_x u_n.$$

We develop this expression and identifying the coefficients of the terms of degree $n-1$, we obtain the following system of m linear equations in m unknowns :

$$-\left(\frac{(n+1)n}{a^2} + \frac{2}{b^2}\right)\alpha_0 - \frac{2}{a^2}\,\alpha_2 = n,$$

$$-\frac{(n-j+1)(n-j)}{b^2}\,\alpha_{j-2} - \left(\frac{(n-j+1)(n-j)}{a^2} + \frac{(j+2)(j+1)}{b^2}\right)\alpha_j -$$

$$- \frac{(j+2)(j+1)}{a^2}\,\alpha_{j+2} = (-1)^{\frac{j}{2}}\, n\binom{n-1}{j}, \qquad j = 2,4,\ldots,n-4, \qquad (2)$$

$$-\frac{6}{b^2}\,\alpha_{n-4}-\left(\frac{6}{a^2}+\frac{n(n-1)}{b^2}\right)\alpha_{n-2}=(-1)^{\frac{n-2}{2}}\,n(n-1),$$

and the equation :

$$-\Delta\left[-(\alpha_0\,x^{n-1}+\alpha_2\,x^{n-3}y^2+\ldots+\alpha_{n-2}\,xy^{n-2})+P_{n-3}(x,y)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\right]=0. \tag{3}$$

We solve the system (2) by successive elimination of unknowns $\alpha_0,\,\alpha_2,\,\alpha_4,\ldots$ and we obtain

$$\alpha_{n-2}=\frac{(-1)^m\,b^2\left[n\binom{n+1}{0}a^{n+1}+(n-2)\binom{n+1}{2}a^{n-1}b^2+\ldots+2\binom{n+1}{n-2}a^3b^{n-2}\right]}{2\left[\binom{n+1}{0}a^{n+1}+\binom{n+1}{2}a^{n-1}b^2+\ldots+\binom{n+1}{n-2}a^3b^{n-2}+\binom{n+1}{n}ab^n\right]},$$

that is

$$\alpha_{n-2}=(-1)^{m-1}b^2\left[\frac{1}{2}-\frac{a(n+1)\,((a+b)^n+(a-b)^n)}{2\,((a+b)^{n+1}+(a-b)^{n+1})}\right]. \tag{4}$$

Hence, we easily compute $\alpha_{n-4},\ldots\,\alpha_2,\,\alpha_0$.

Similarly, to calculate $(-\Delta)^{-1}\partial_y u_n$ we search a polynomial of the form

$$\beta_0\,y^{n-1}+\beta_2\,y^{n-3}x^2+\ldots+\beta_{n-4}\,y^3x^{n-4}+\beta_{n-2}\,yx^{n-2}+Q_{n-3}(x,y),$$

where $Q_{n-3}(x,y)$ is a polynomial of degree $n-3$, such that

$$-\Delta\left[(\beta_0\,y^{n-1}+\beta_2\,y^{n-3}x^2+\ldots+\beta_{n-2}\,yx^{n-2})+Q_{n-3}(x,y)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\right]=\partial_y u_n.$$

As previously, we obtain the system

$$-\left(\frac{2}{a^2}+\frac{(n+1)n}{b^2}\right)\beta_0\;-\;\frac{2}{b^2}\,\beta_2=(-1)^{\frac{n}{2}}\,n,$$

$$-\frac{(n-j+1)(n-j)}{a^2}\beta_{j-2}-\left(\frac{(j+2)(j+1)}{a^2}+\frac{(n-j+1)(n-j)}{b^2}\right)\beta_j-$$

$$-\frac{(j+2)(j+1)}{b^2}\beta_{j+2}\;=\;(-1)^{\frac{n-j}{2}}n\binom{n-1}{j},\qquad j=2,4,\ldots,n-4, \tag{5}$$

$$-\frac{6}{a^2}\,\beta_{n-4}-\left(\frac{n(n-1)}{a^2}+\frac{6}{b^2}\right)\beta_{n-2}=-n(n-1),$$

and the equation :

$$-\Delta\left[-(\beta_0\,y^{n-1}+\beta_2\,y^{n-3}x^2+\ldots+\beta_{n-2}\,yx^{n-2})+Q_{n-3}(x,y)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\right]=0. \tag{6}$$

We solve the system (5) by successive elimination of unknowns β_{n-2}, β_{n-4}, $\beta_{n-6}, \ldots$ and we obtain

$$\beta_0 = \frac{(-1)^{m-1}ab^2}{2} \frac{(a+b)^n - (a-b)^n}{(a+b)^{n+1} - (a-b)^{n+1}}. \tag{7}$$

Hence, we easily compute $\beta_2, \ldots, \beta_{n-4}$, β_{n-2}.

Now we are going to determine the polynomials $P_{n-3}(x,y)$ and $Q_{n-3}(x,y)$ such that they verify the equations (3) and (6). For this, we get $P_{n-3}(x,y)$ of the form

$$P_{n-3}(x,y) = \gamma_0\, x^{n-3} + \gamma_2\, x^{n-5}y^2 + \ldots + \gamma_{n-6}\, x^3y^{n-6} + \gamma_{n-4}\, xy^{n-4} + P_{n-5}(x,y),$$

with $P_{n-5}(x,y)$ polynomial of degree $n-5$. Since the polynomial

$$(\alpha_0\, x^{n-1} + \alpha_2\, x^{n-3}y^2 + \ldots + \alpha_{n-4}\, x^3y^{n-4} + \alpha_{n-2}\, xy^{n-2}) - P_{n-3}(x,y)\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)$$

must be harmonic (equation (3)), we get $P_{n-3}(x,y)$ such that

$$\alpha_0\, x^{n-1} + \ldots + \alpha_{n-2}\, xy^{n-2} - (\gamma_0\, x^{n-3} + \ldots + \gamma_{n-4}\, xy^{n-4} + P_{n-5}(x,y))\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right) =$$

$$= \sigma_{n-1}\, u_{n-1} + \sigma_{n-3}\, u_{n-3} + \ldots + \sigma_3\, u_3 + \sigma_1\, u_1, \tag{8}$$

where u_j is the harmonic homogenous polynomial defined by $\rho^j cos(j\rho)$ (j odd) and $\sigma_j \in \mathbf{R}$.

Identifying the coefficients of the terms of degree $n-1$, we obtain the system :

$$-\frac{1}{a^2}\gamma_0 + \alpha_0 = \sigma_{n-1},$$

$$-\frac{1}{b^2}\gamma_{j-2} - \frac{1}{a^2}\gamma_j + \alpha_j = (-1)^{\frac{j}{2}}\binom{n-1}{j}\sigma_{n-1}\,, \qquad j = 2, 4, \ldots, n-4, \tag{9}$$

$$-\frac{1}{b^2}\gamma_{n-4} + \alpha_{n-2} = (-1)^{\frac{n-2}{2}}\sigma_{n-1}.$$

It is easy to solve this system and we have $\gamma_0, \gamma_2, \ldots, \gamma_{n-4}$ and σ_{n-1} (the values α_j are given by system (2)).

To calculate $P_{n-5}(x,y)$ we write

$$P_{n-5}(x,y) = \eta_0\, x^{n-5} + \eta_2\, x^{n-7}y^2 + \ldots + \eta_{n-8}\, x^3y^{n-8} + \gamma_{n-6}\, xy^{n-6} + P_{n-7}(x,y),$$

with $P_{n-7}(x,y)$ polynomial of degree $n-7$.

Introducing this expression in (8) and identifying the coefficients of the terms of degree $n-3$, we obtain a system similar to (9). Solving this system we obtain $\eta_0, \eta_2, \ldots, \eta_{n-6}$ and σ_{n-3}.

To calculate $P_{n-7}(x,y)$ we proceed similarly and so on. Thus, we can consider that $\sigma_{n-1}, \sigma_{n-3}, \ldots, \sigma_1$ and $P_{n-3}(x,y)$ are calculated.

Proceeding as previously, we obtain $Q_{n-3}(x,y)$ and $\tau_{n-1}, \tau_{n-3}, \ldots, \tau_1 \in \mathbf{R}$ such that

$$(\beta_0\, y^{n-1} + \ldots + \beta_{n-2}\, yx^{n-2}) - Q_{n-3}(x,y)\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right) = \tau_{n-1}\, v_{n-1} + \tau_{n-3}\, v_{n-3} + \ldots + \tau_1\, v_1,$$

where v_j is the harmonic homogeneous polynomial defined by par $\rho^j sin(j\theta)$ with j odd.

Let us return to Tu_n. We have

$$Tu_n = -\sum_{j=0,2,4,\ldots,n}\left(\frac{n+1-j}{b^2}\alpha_{j-2}+\frac{n+1-j}{a^2}\alpha_j+\frac{j+1}{a^2}\beta_{n-j-2}+\frac{j+1}{b^2}\beta_{n-j}\right)x^{n-j}y^j-$$
$$-\partial_x\left[-(\alpha_0\,x^{n-1}+\alpha_2\,x^{n-3}y^2+\ldots+\alpha_{n-2}\,xy^{n-2})+P_{n-3}(x,y)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\right]-$$
$$-\partial_y\left[-(\beta_0\,y^{n-1}+\beta_2\,y^{n-3}x^2+\ldots+\beta_{n-2}\,yx^{n-2})+Q_{n-3}(x,y)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\right].$$

Now, we are going to prove that there exits $\lambda_n\in\mathbf{R}$ such that

$$Tu_n=\lambda_n u_n-\,\partial_x\left[-(\alpha_0x^{n-1}+\ldots+\alpha_{n-2}xy^{n-2})+P_{n-3}(x,y)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\right]-$$
$$-\partial_y\left[-(\beta_0\,y^{n-1}+\beta_2\,y^{n-3}x^2+\ldots+\beta_{n-2}\,yx^{n-2})+Q_{n-3}(x,y)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\right].$$

For this, we must find λ_n satisfying

$$-\frac{n+1}{a^2}\,\alpha_0-\frac{1}{a^2}\,\beta_{n-2}=\lambda_n,$$
$$-(n+1-j)\left(\frac{\alpha_{j-2}}{b^2}+\frac{\alpha_j}{a^2}\right)-(j+1)\left(\frac{\beta_{n-j-2}}{a^2}+\frac{\beta_{n-j}}{b^2}\right)=(-1)^{\frac{j}{2}}\lambda_n\binom{n}{j},\qquad(10)$$
$$j=2,4,\ldots,n-2$$
$$-\frac{1}{b^2}\,\alpha_{n-2}-\frac{n+1}{b^2}\,\beta_0=(-1)^{\frac{n}{2}}\lambda_n.$$

Introducing α_{n-2} and β_0 given by (4) and (7) in the last equation of system (10), we obtain

$$\lambda_n=\frac{1}{2}-\frac{2ab(n+1)(a^2-b^2)^n}{(a+b)^{2n+2}-(a-b)^{2n+2}}.\qquad(11)$$

To show that this λ_n verifies the others equations of system (10), we add the last equation of (2) multiplied by par $\dfrac{1}{2}$ and the first equation of (5) multiplied by $\dfrac{n-1}{2}$. Thanks to the last equation of (10), we find the next to last equation of system (10). Repeating this procedure, we show that this λ_n verify all equations of system (10).

On the other hand, for the harmonic homogeneous polynomials $u_k=\rho^k cos(k\theta)$ and $v_k=\rho^k sin(k\theta)$, $k\geq 1$, we have $\partial_x u_k=ku_{k-1}$ and $\partial_y v_k=ku_{k-1}$, therefore

$$\partial_x\left[(\alpha_0\,x^{n-1}+\alpha_2\,x^{n-3}y^2+\ldots+\alpha_{n-2}\,xy^{n-2})-P_{n-3}(x,y)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\right]+$$

$$+\partial_y\left[(\beta_0\, y^{n-1}+\beta_2\, y^{n-3}x^2+\ldots+\beta_{n-2}\, yx^{n-2})-Q_{n-3}(x,y)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\right]=$$

$$=\partial_x(\sigma_{n-1}\, u_{n-1}+\sigma_{n-3}\, u_{n-3}+\ldots+\sigma_1\, u_1)+\partial_y(\tau_{n-1}\, v_{n-1}+\tau_{n-3}\, v_{n-3}+\ldots+\tau_1\, v_1)=$$

$$=(n-1)\sigma_{n-1}\, u_{n-2}+(n-3)\sigma_{n-3}\, u_{n-4}+\ldots+3\sigma_3\, u_2+\sigma_1+(n-1)\tau_{n-1}\, u_{n-2}+\ldots+3\tau_3\, u_2+\tau_1.$$

Hence,

$$Tu_n=\lambda_n u_n+(n-1)(\sigma_{n-1}+\tau_{n-1})u_{n-2}+(n-3)(\sigma_{n-3}+\tau_{n-3})u_{n-4}+\ldots+3(\sigma_3+\tau_3)u_2+\sigma_1+\tau_1.$$

Obsiously, we have a similar expresion for $u_{n-2}=\rho^{n-2}cos((n-2)\theta)$:

$$Tu_{n-2}=\lambda_{n-2}u_{n-2}+(n-3)(\mu_{n-3}+\nu_{n-3})u_{n-4}+\ldots+3(\mu_3+\nu_3)u_2+\mu_1+\nu_1.$$

Therefore

$$T\left(u_n+\frac{(n-1)(\sigma_{n-1}+\tau_{n-1})}{\lambda_n-\lambda_{n-2}}u_{n-2}\right)=\lambda_n u_n+$$

$$+\lambda_n\frac{(n-1)(\sigma_{n-1}+\tau_{n-1})}{\lambda_n-\lambda_{n-2}}u_{n-2}+\omega_{n-4}\, u_{n-4}+\ldots+\omega_2\, u_2+\omega_0.$$

Also

$$Tu_{n-4}=\lambda_{n-4}u_{n-4}+(n-5)(\rho_{n-5}+\delta_{n-5})u_{n-6}+\ldots+3(\rho_3+\delta_3)u_2+\rho_1+\delta_1$$

thus,

$$T\left(u_n+\frac{(n-1)(\sigma_{n-1}+\tau_{n-1})}{\lambda_n-\lambda_{n-2}}u_{n-2}+\frac{\omega_{n-4}}{\lambda_n-\lambda_{n-4}}u_{n-4}\right)=\lambda_n u_n+$$
$$+\lambda_n\frac{(n-1)(\sigma_{n-1}+\tau_{n-1})}{\lambda_n-\lambda_{n-2}}u_{n-2}+\lambda_n\frac{\omega_{n-4}}{\lambda_n-\lambda_{n-4}}u_{n-4}+\varepsilon_{n-6}\, u_{n-6}+\ldots+\varepsilon_2\, u_2+\varepsilon_0\,.$$

Finally, repeating this procedure, we show that λ_n is an eigenvalue of T.

If we take $v_n=\rho^n sin(n\theta)$ $(n=2m)$ and we repeat the same reasoning, we show that

$$\widetilde{\lambda}_n=\frac{1}{2}+\frac{2ab(n+1)(a^2-b^2)^n}{(a+b)^{2n+2}-(a-b)^{2n+2}} \tag{12}$$

is an eigenvalue of T of multiplicity 1.

Similarly, if $n=2m-1$, taking $u_n=\rho^n cos(n\theta)$ (resp. $v_n=\rho^n sin(n\theta)$) we show that $\widetilde{\lambda}_n$ (resp. λ_n) is an eigenvalue of T de multiplicity 1.

Finally, since $Tu-u$ is harmonic $\forall u\in L^2(\Omega)/\mathbf{R}$, the orthogonal in $L^2(\Omega)/\mathbf{R}$ of the space of harmonic functions is included in the eigenspace corresponding to the eigenvalue $\lambda=1$. On the other hand ([3]), the family of harmonic polynomials is a basis (in $L^2(\Omega)/\mathbf{R}$) of the subspace of harmonic functions. Thus λ_n, $\widetilde{\lambda}_n$ with $n\in\mathbf{N}$, and 1 are the only eigenvalues of T and the corresponding eigenvectors form a basis of $L^2(\Omega)/\mathbf{R}$. Consequently ([1]), like the limit of λ_n and $\widetilde{\lambda}_n$, as $n\to\infty$, is $1/2$, the spectrum of T is the set $\sigma(T)=\sigma_p(T)\cup 1/2$.

In the particular case where $b=a$, all eigenvalues λ_n and $\widetilde{\lambda}_n$ condense in $1/2$ and the spectrum of T only contain the eigenvalues of infinite multiplicity 1 and $1/2$ ([4]).

If $b < a$, $\{\lambda_n\}$ is an increasing sequence and the eigenvalue $\dfrac{b^2}{a^2+b^2}$ is the smallest spectral value of T. Thus, $\dfrac{a^2+b^2}{b^2}$ is the best constant in the inequality (1).

Remark 1. *In dimension 3, if Ω is the ellipsoid*

$$\Omega = \left\{ (x,y,z) \in \mathbf{R}^3 \, ; \, \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} < 1 \right\} \qquad \textit{with} \qquad c \le b \le a,$$

this problem is more complicated since an nth degree harmonic polynomial u_n contains $2n+1$ arbitrary constants and is a linear combination of $2n+1$ linearly independent harmonic polynomials. In this case, we obtain systems that cannot be solved explicitly. However, we conjecture that the eigenvalue $\dfrac{b^2c^2}{a^2b^2+a^2c^2+b^2c^2}$ corresponding to the eigenvector $u(x,y,z) = x$ is the smallest spectral value of T and thus, $\dfrac{a^2b^2+a^2c^2+b^2c^2}{b^2c^2}$ is the best constant in the Poincaré type inequality (1).

In the particular case where Ω is the sphere

$$\Omega = \left\{ (x,y,z) \in \mathbf{R}^3 \, ; \, x^2+y^2+z^2 < 1 \right\},$$

each harmonic homogeneous polynomial of degree $n \ge 1$ is an eigenvector of T corresponding to the eigenvalue $\dfrac{n}{2n+1}$. The point spectrum is $\sigma_p(T) = \{\dfrac{n}{2n+1}, 1 \, ; \, n \in \mathbf{N^*}\}$, where 1 is an eigenvalue of infinite multiplicity, $\dfrac{n}{2n+1}$ has finite multiplicity $(= 2n+1)$, and $\sigma(T) = \sigma_p(T) \cup \{1/2\}$. Consequently, 3 is the best constant in the Poincaré type inequality (1) ([4]).

Remark 2. *We note that the operator $T = -div(-\Delta)^{-1}grad$ appears in the static elasticity theory and that the best constant in the Poincaré inequality* (1) *is used in constructing and substantiating algorithms for solving equations like the Stokes and the Navier-Stokes equations* ([2]).

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