Recognizing $\mathcal{Q}_{p,0}$ Functions per Dirichlet Space Structure

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Abstract

Under $p \in (0, \infty)$ and Möbius map $\sigma_w(z) = (w-z)/(1-\bar{w}z)$, a holomorphic function on the unit disk $\triangle$ is said to be of $\mathcal{Q}_{p,0}$ class if $\lim_{|w|\to 1} E_p(f, w) = 0$, where

$$E_p(f, w) = \int_\triangle |f'(z)|^2[1 - |\sigma_w(z)|^2]^p dm(z),$$

and where dm means the element of the Lebesgue area measure on $\triangle$. In particular, $\mathcal{Q}_{p,0} = \mathcal{B}_0$, the little Bloch space for all $p \in (1, \infty)$, $\mathcal{Q}_{1,0} = VMOA$ and $\mathcal{Q}_{p,0}$ contains $\mathcal{D}$, the Dirichlet space. Motivated by the linear structure of $\mathcal{D}$, this paper is devoted to: first show that $\mathcal{Q}_{p,0}$ is a Möbius invariant space in the sense of Arazy-Fisher-Peetre; secondly identify $\mathcal{Q}_{p,0}$ with the closure of all polynomials; thirdly characterize the extreme points of the unit closed ball of $\mathcal{Q}_{p,0}$; and finally investigate the semigroups of the composition operators on $\mathcal{Q}_{p,0}$.

Introduction

Let $\triangle$ and $\partial\triangle$ be the unit disk and the unit circle in the finite complex plane $\mathbb{C}$. Denote by $\mathcal{H}$ the set of functions holomorphic on $\triangle$, endowed with the topology of the compact-open (i.e. the uniform convergence on compact subsets of $\triangle$). The

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symbol $Aut(\triangle)$ is employed to represent the group of all conformal automorphisms of $\triangle$, i.e. all Möbius maps of the form $\lambda\sigma_a$, where $\lambda \in \partial\triangle$, $a \in \triangle$ and

$$\sigma_a(z) = \frac{a-z}{1-\bar{a}z}$$

is the symmetry interchanging 0 and a. The Bloch space $\mathcal{B}$ is the class of all $f \in \mathcal{H}$ with the semi-norm

$$\|f\|_{\mathcal{B}} = \sup_{z\in\triangle}(1-|z|^2)|f'(z)| < \infty.$$

Moreover, the little Bloch space $\mathcal{B}_0$ is the family of functions $f \in \mathcal{H}$ satisfying

$$\lim_{|z|\to 1}(1-|z|^2)|f'(z)| = 0.$$

Recently, it has been found that $\mathcal{B}$ and $\mathcal{B}_0$ can be embedded into two new function families, i.e., the so-called $\mathcal{Q}_p$ and $\mathcal{Q}_{p,0}$, respectively (cf. [AuLa, AuXiZh, EsXi, NiXi]). Recall that for $p \in (0,\infty)$, $\mathcal{Q}_p$ resp. $\mathcal{Q}_{p,0}$ is the class of functions $f \in \mathcal{H}$ with

$$\|f\|_{\mathcal{Q}_p} = \sup_{w\in\triangle}[E_p(f,w)]^{\frac{1}{2}} < \infty \quad resp. \quad \lim_{|w|\to 1} E_p(f,w) = 0.$$

Here and throughout this paper,

$$E_p(f,w) = \int_{\triangle} |f'(z)|^2[1-|\sigma_w(z)|^2]^p dm(z),$$

where dm stands for the element of the Lebesgue area measure on $\triangle$. With respect to

$$|||f|||_{\mathcal{Q}_p} = |f(0)| + \|f\|_{\mathcal{Q}_p},$$

$\mathcal{Q}_p$ becomes a normed linear space and has $\mathcal{Q}_{p,0}$ as its subspace. Of particular interest is to point out that if $p \in (1,\infty)$ or $p = 1$ then $\mathcal{Q}_p = \mathcal{B}$; $\mathcal{Q}_{p,0} = \mathcal{B}_0$ or $\mathcal{Q}_p = BMOA$; $\mathcal{Q}_{p,0} = VMOA$.

Let $\mathcal{D}$ be the classical Dirichlet space consisting of functions $f \in \mathcal{H}$ with

$$\|f\|_{\mathcal{D}} = \left[\int_{\triangle} |f'(z)|^2 dm(z)\right]^{\frac{1}{2}} < \infty.$$

By some simple calculations involving power series, it is easy to establish that for $f \in \mathcal{H}$ and $w \in \triangle$,

$$F_p(f,w) \le E_p(f,w) \le 2^p F_p(f,w),$$

where

$$F_p(f,w) = p\int_0^1 (1-r)^{p-1}\left[\int_{|z|<r} |(f\circ\sigma_w)'(z)|^2 dm(z)\right] dr.$$

An important observation is that $(\mathcal{Q}_{p,0}, \|\cdot\|_{\mathcal{Q}_p})$ decreases to $(\mathcal{D}, \|\cdot\|_{\mathcal{D}})$ as $p \searrow 0$. In fact, if $f \in \mathcal{D}$ then for arbitrary $\epsilon > 0$, there is a $\delta \in (0,1)$ such that

$$\int_{|z|>\delta} |f'(z)|^2 dm(z) < \epsilon; \quad \int_{\delta}^1 (1-r)^{p-1} dr < \epsilon.$$

Thus,

$$\begin{aligned}\lim_{|w|\to 1} E_p(f,w) &\le \lim_{|w|\to 1} 2^p F_p(f,w)\\ &\le p2^p \lim_{|w|\to 1}\int_0^\delta (1-r)^{p-1}\Big[\int_{|z|<r} |(f\circ\sigma_w)'(z)|^2 dm(z)\Big]dr + \epsilon p 2^p \|f\|_{\mathcal{D}}^2\\ &\le \Big[2^p(1+p\|f\|_{\mathcal{D}}^2)\Big]\epsilon,\end{aligned}$$

which implies $f\in\mathcal{Q}_{p,0}$. On the other hand, the measure $p(1-r)^{p-1}dr$ (defined on $[0,1]$) converges weak-star to the point mass at 1 as $p\searrow 0$. Since $\int_{|z|<r}|(f\circ\sigma_w)'(z)|^2dm(z)$ is an increasing function of r, we have

$$\lim_{p\to 0} E_p(f,w) = \lim_{p\to 0} F_p(f,w) = \sup_{r\in(0,1)}\int_{|z|<r} |(f\circ\sigma_w)'(z)|^2 dm(z).$$

This leads to: $\mathcal{D}$ consists of those functions for which there is a constant $K(f)>0$ (depending on f) with $\|f\|_{\mathcal{Q}_p}\le K(f)$ for all $p>0$ and so, can be viewed as a limit space of $\mathcal{Q}_{p,0}$ as $p\searrow 0$.

The preceding observation appears to induce an idea: in order to solve problems regarding $\mathcal{Q}_{p,0}$, we may treat $\mathcal{Q}_{p,0}$ as a 'kind' of Dirichlet space and use the methods of functional analysis. This viewpoint has already be testified by [NiXi] where the closed graph theorem is an effective tool to discuss interpolation and projection problems from $\mathcal{Q}_{p,0}$. In the present article, we shall study the linear $\mathcal{D}$-like structure of $\mathcal{Q}_{p,0}$ in some details. The material we cover is as follows: 1) Möbius invariance of $\mathcal{Q}_{p,0}$ in the strict sense of Arazy-Fisher-Peetre; 2) density of the polynomials in $\mathcal{Q}_{p,0}$; 3) characterization of the extreme points in the unit ball of $\mathcal{Q}_{p,0}$; 4) semigroups of the composition operators on $\mathcal{Q}_{p,0}$.

1 Möbius Invariance

Since the norm $\|\!|\cdot|\!\|_{\mathcal{Q}_p}$ and the semi-norm $\|\cdot\|_{\mathcal{Q}_p}$ differ by one nonnegative constant, the first thing to do is to see how the semi-norm $\|\cdot\|_{\mathcal{Q}_p}$ affects $\mathcal{Q}_{p,0}$. We shall find that all the $\mathcal{Q}_{p,0}$ spaces are Möbius invariant in the sense of Arazy-Fisher-Peetre (cf.[ArFiPe]).

A semi-normed linear space $(X,\|\cdot\|_X)$ is called a *Möbius invariant space* provided the following conditions hold:

- $X\subset\mathcal{B}$ with $\|\cdot\|_X\le K\|\cdot\|_{\mathcal{B}}$ for some constant $K>0$;
- X is complete under the semi-norm $\|\cdot\|_X$;
- $Aut(\triangle)$-invariance: for each $\sigma\in Aut(\triangle)$ and each $f\in X$ the composition $C_\sigma(f)=f\circ\sigma$ belongs to X and $\|C_\sigma(f)\|_X=\|f\|_X$;
- Continuity of $Aut(\triangle)$-action: for every $f\in X$ the map $\sigma\to C_\sigma(f)$ is continuous from $Aut(\triangle)$ to X in the semi-norm $\|\cdot\|_X$.

It is well-known that $\mathcal{B}_0$ is Möbius invariant, and according to the above definition, $\mathcal{B}$ is not so in that $\mathcal{B}$ does not satisfy the strict continuity of $Aut(\triangle)$-action. There are some other Möbius invariant spaces such as $VMOA$ and the Besov p-space B_p. In [ArFi] it is proved that the unique Möbius invariant Hilbert space is

$(\mathcal{D}, \|\cdot\|_{\mathcal{D}})$. Nevertheless, as to a Möbius invariant Banach space which lies between $(\mathcal{D}, \|\cdot\|_{\mathcal{D}})$ and $(\mathcal{B}_0, \|\cdot\|_{\mathcal{B}})$, we have

Theorem 1.1. *Let $p \in (0, \infty)$. Then $(\mathcal{Q}_{p,0}, \|\cdot\|_{\mathcal{Q}_p})$ is a Möbius invariant space.*

Proof. Observe first that for any $f \in \mathcal{Q}_p$,

$$(1-|z|^2)|f'(z)| \leq \left(\frac{2^{p+1}}{\pi}\right)^{\frac{1}{2}} \|f\|_{\mathcal{Q}_p}, \quad z \in \triangle. \tag{1.1}$$

Thus $\mathcal{Q}_{p,0} \subset \mathcal{Q}_p \subset \mathcal{B}$ with $\|\cdot\|_{\mathcal{B}} \leq K\|\cdot\|_{\mathcal{Q}_p}$ where $K = (2^{p+1}/\pi)^{\frac{1}{2}}$.

Assume next that $\{f_n\}$ is a Cauchy sequence in $(\mathcal{Q}_{p,0}, \|\cdot\|_{\mathcal{Q}_p})$. Then for arbitrary $\epsilon > 0$ there is a positive integer n_0 such that as $m, n \geq n_0$, $\|f_m - f_n\|_{\mathcal{Q}_p} < \epsilon$. By the principle of normal family and (1.1), there exists some function $f \in \mathcal{Q}_p$ such that $\|f_{n_0} - f\|_{\mathcal{Q}_p} \leq \epsilon$. Since $f_{n_0} \in \mathcal{Q}_{p,0}$, it follows from the definition of $\mathcal{Q}_{p,0}$ that $f \in \mathcal{Q}_{p,0}$. So $(\mathcal{Q}_{p,0}, \|\cdot\|_{\mathcal{Q}_p})$ is complete.

Because $E_p(f\circ\sigma, w) = E_p(f, \sigma(w))$, each $\mathcal{Q}_{p,0}$ is $Aut(\triangle)$-invariant: $\|f\circ\sigma\|_{\mathcal{Q}_p} = \|f\|_{\mathcal{Q}_p}$ for all $f \in \mathcal{Q}_{p,0}$ and $\sigma \in Aut(\triangle)$.

To prove continuity of $Aut(\triangle)$-action on $\mathcal{Q}_{p,0}$, it suffices, by homogeneity, to verify that if $a \to 0$ in $\triangle$, then $f(-\sigma_a) \to f$ in $\mathcal{Q}_{p,0}$ whenever $f \in \mathcal{Q}_{p,0}$. Suppose $f \in \mathcal{Q}_{p,0}$, thus, for every $\epsilon > 0$ there exists a $\delta_1 \in (0,1)$ such that

$$\sup_{|w|>\delta_1} E_p(f, w) < \epsilon. \tag{1.2}$$

Without loss of generality, one may assume $|a| < 1/2$. Then there is a $\delta_2 \geq \delta_1$ such that as $|w| > \delta_2$ one has $|\sigma_a(w)| > \delta_1$ and by (1.2), $\sup_{|w|>\delta_2} E_p(f, \sigma_a(w)) < \epsilon$. Hence

$$\sup_{|w|>\delta_2} E_p(f\circ\sigma_a - f, w) < 4\epsilon. \tag{1.3}$$

In what follows, let $|w| \leq \delta_2$, and for $r \in (0,1)$ set

$$E_p(f\circ\sigma_a - f, w) = \left(\int_{|z|\leq r} + \int_{|z|>r}\right)(\cdots)dm(z) = I_1(r,a) + I_2(r,a).$$

Concerning $I_2(r,a)$, we apply the following basic inequality $|\sigma_a(w)| \leq (|a| + |w|)/(1+|a||w|)$ to get that $|\sigma_a(w)| < \delta_3 = (1+2\delta_2)/(2+\delta_2)$ for $|w| \leq \delta_2$ and $|a| \leq 1/2$. Since $f \in \mathcal{Q}_{p,0}$, f obeys $E_p(f, 0) < \infty$, and for the above $\epsilon > 0$ there is an $r_0 \in (0,1)$ such that

$$\int_{|z|>r_0} |f'(z)|^2(1-|z|^2)^p dm(z) < \epsilon.$$

Furthermore, some elementary calculations imply that for $r^2 = (2+r_0^2)/3$,

$$\begin{aligned} I_2(r,a) &\leq \left[\frac{2^{2+p}}{(1-\delta_2)^p} + \frac{2^{2+p}}{(1-\delta_3)^p}\right] \int_{|z|>r_0} |f'(z)|^2(1-|z|^2)^p dm(z) \\ &< \epsilon\left[\frac{2^{2+p}}{(1-\delta_2)^p} + \frac{2^{2+p}}{(1-\delta_3)^p}\right]. \end{aligned} \tag{1.4}$$

However, it is obvious that $\lim_{a\to 0} I_1(r,a) = 0$ for each $r \in (0,1)$. Therefore, from (1.3) and (1.4) we derive that $\lim_{a\to 0} \|f(-\sigma_a) - f\|_{\mathcal{Q}_p} = 0$. This concludes the proof. ∎

Due to Theorem 1.1, those general properties of Möbius invariant spaces (shown in Section 1 of [ArFiPe]) are valid for $\mathcal{Q}_{p,0}$. Specially, we have

Corollary 1.2. *Let $p \in (0,\infty)$. Then the $Aut(\triangle)$-invariant dual $\mathcal{Q}^*_{p,0}$ consists of all $f \in \mathcal{H}$ obeying* $\sup\{|\langle f,g\rangle| : g \in \mathcal{Q}_{p,0}, \|g\|_{\mathcal{Q}_p} \le 1\} < \infty$, *where*

$$\langle f,g\rangle = \int_{\triangle} f'(z)\overline{g'(z)}dm(z)$$

is the $Aut(\triangle)$-invariant pair.

Remark 1.3 a) Each $\mathcal{Q}_p$ has a weak continuity of $Aut(\triangle)$-action: for $f \in \mathcal{Q}_p$ the map $\sigma \to C_\sigma(f)$ is continuous from $Aut(\triangle)$ to $\mathcal{Q}_p$ with respect to the compact-open topology. For a discussion of the cases $p \ge 1$, refer to [ArFiPe].

b) The Hahn-Banach theorem can be used to establish that the second $Aut(\triangle)$-invariant dual of $\mathcal{Q}_{p,0}$ is isomorphic to $\mathcal{Q}_p$ under $\langle\cdot,\cdot\rangle$. It is worth pointing out that $\mathcal{D}^*$ is isomorphic to $\mathcal{D}$, moreover if $p=1$ resp. $p>1$ then $\mathcal{Q}^*_{p,0}$ isomorphic to the Hardy-Sobolev space $\mathcal{W}$ resp. the Besov space $\mathcal{M}$, which consists of all $f \in \mathcal{H}$ obeying

$$\|f\|_{\mathcal{W}} = \int_{\partial\triangle} |f'(z)||dz| < \infty \quad resp. \quad \|f\|_{\mathcal{M}} = \int_{\triangle} [|f'(z)| + |f''(z)|]dm(z) < \infty.$$

It would be interesting to provide a function-theoretic characterization of $\mathcal{Q}^*_{p,0}$ similar to that of $\mathcal{D}$, $\mathcal{W}$ or $\mathcal{M}$.

2 Polynomial Density

Although Theorem 1.1 actually tells us that $\mathcal{Q}_p$ and $\mathcal{Q}_{p,0}$ are Banach spaces under $\|\!|\cdot|\!\|_{\mathcal{Q}_p}$, since $\mathcal{Q}_{p,0}$ contains all polynomials, it is worth to consider the density of $\mathcal{P}$, the class of the polynomials, and hence to imply that $\mathcal{Q}_{p,0}$ is a closed subspace of $\mathcal{Q}_p$ with respect to the norm $\|\!|\cdot|\!\|_{\mathcal{Q}_p}$.

Theorem 2.1. *Let $p \in (0,\infty)$ and let $f \in \mathcal{Q}_p$ with $f_r(z) = f(rz)$ for $r \in (0,1)$. Then the following are equivalent:* (i) $f \in \mathcal{Q}_{p,0}$; (ii) $\lim_{r\to 1} \|\!|f_r - f|\!\|_{\mathcal{Q}_p} = 0$. (iii) f *belongs to the closure of $\mathcal{P}$ in the norm $\|\!|\cdot|\!\|_{\mathcal{Q}_p}$.* (iv) *For any $\epsilon > 0$ there is a $g \in \mathcal{Q}_{p,0}$ such that $\|\!|g - f|\!\|_{\mathcal{Q}_p} < \epsilon$.*

Proof. Since the implications: (ii)⇒(iii),(iii)⇒(iv) and (iv)⇒(i) are nearly obvious, it suffices to verify the implication: (i)⇒(ii). Let $f \in \mathcal{Q}_p$. An application of Poisson's formula to f_r gives

$$f_r(z) = \frac{1}{2\pi}\int_{\partial\triangle} f(z\zeta)\frac{1-r^2}{|1-r\bar\zeta|^2}|d\zeta|. \tag{2.1}$$

Derivating both sides of (2.1) with respect to z, integrating and using Minkowski's inequality, one has

$$\begin{aligned}[E_p(f_r,w)]^{\frac12} &\le \frac{1}{2\pi}\int_{\partial\triangle}\left[\int_{\triangle}|f'(z\zeta)|^2[1-|\sigma_w(z)|]^p dm(z)\right]^{\frac12}\frac{1-r^2}{|1-r\bar\zeta|^2}|d\zeta| \\ &= \frac{1}{2\pi}\int_{\partial\triangle}[E_p(f,\bar\zeta w)]^{\frac12}\frac{1-r^2}{|1-r\bar\zeta|^2}|d\zeta|. \end{aligned} \tag{2.2}$$

Consequently, $\|\!|f_r|\!\|_{\mathcal{Q}_p} \le \|\!|f|\!\|_{\mathcal{Q}_p}$. Furthermore, if $f \in \mathcal{Q}_{p,0}$, then by (2.2), $\lim_{|w|\to 1} E_p(f_r - f, w) = 0$ holds for a fixed $r \in (0,1)$. Also, for a given $\eta \in (0,1)$ it is not hard (by dividing the integral into two parts) to determine $\lim_{r\to 1}\sup_{|w|\le\eta} E_p(f_r - f, w) = 0$. Summing up, we see that $\lim_{r\to 1}\|\!|f_r - f|\!\|_{\mathcal{Q}_p} = 0$ and hence (i)⇒(ii) holds. ∎

Corollary 2.2. *Let $p \in (0,\infty)$. Then $(\mathcal{Q}_{p,0}, \|\!|\cdot|\!\|_{\mathcal{Q}_p})$ is a separable closed subspace of $(\mathcal{Q}_p, \|\!|\cdot|\!\|_{\mathcal{Q}_p})$.*

As for $f \in \mathcal{Q}_p$, denote by $d(f,\mathcal{Q}_{p,0})$ the distance of f to $\mathcal{Q}_{p,0}$, namely, $d(f,\mathcal{Q}_{p,0}) = \inf_{h\in\mathcal{Q}_{p,0}} \|f-h\|_{\mathcal{Q}_p}$. Meanwhile, put

$$\delta_{\mathcal{Q}_p}(f) = \limsup_{|w|\to 1}\Big[E_p(f,w)\Big]^{\frac{1}{2}}.$$

The argument for Theorem 2.1 can infer the following result.

Corollary 2.3. *Let $p \in (0,\infty)$ and let $f \in \mathcal{Q}_p$. Then*

$$\delta_{\mathcal{Q}_p}(f) \le d(f,\mathcal{Q}_{p,0}) \le 2\delta_{\mathcal{Q}_p}(f). \tag{2.3}$$

Proof. On the one hand, since $\lim_{|w|\to 1} E_p(h,w) = 0$ for any $h \in \mathcal{Q}_{p,0}$, by the triangle inequality of $\|\cdot\|_{\mathcal{Q}_p}$ one has

$$\Big[E_p(f,w)\Big]^{\frac{1}{2}} \le \Big[E_p(f-h,w)\Big]^{\frac{1}{2}} + \Big[E_p(h,w)\Big]^{\frac{1}{2}};$$

consequently,

$$\delta_{\mathcal{Q}_p}(f) \le \sup_{w\in\triangle}\Big[E_p(f-h,w)\Big]^{\frac{1}{2}}$$

for every $h \in \mathcal{Q}_{p,0}$. In other words, the left-hand side estimate of (2.3) holds.

On the other hand, if $f \in \mathcal{Q}_p$ with $f_r(z) = f(rz)$, $r \in (0,1)$, then $d(f,\mathcal{Q}_{p,0}) \le \|f - f_r\|_{\mathcal{Q}_p}$, owing to $f_r \in \mathcal{Q}_{p,0}$. From the proof of Theorem 2.1 it is seen that for arbitrary $\epsilon > 0$ and a fixed $\eta \in (0,1)$, there is an $r_0 \in (0,1)$ such that as $r \in [r_0, 1)$,

$$\Big[E_p(f-f_r,w)\Big]^{1/2} \le (1-\eta)^{-p}\Big[E_p(f-f_r,0)\Big]^{1/2} < \epsilon.$$

Hence by (2.2),

$$d(f,\mathcal{Q}_{p,0}) \le \|f-f_r\|_{\mathcal{Q}_p} \le \epsilon + \sup_{|w|\ge\eta}\Big[E_p(f-f_r,w)\Big]^{1/2} \le \epsilon + 2\sup_{|w|\ge\eta}\Big[E_p(f,w)\Big]^{1/2}.$$

This implies that the right-hand side of (2.3) is true. ∎

Remark 2.4. a) Theorem 2.1 is an extension of the corresponding results on $\mathcal{B}_0$ and VMOA. See also [An, Theorem 1; Si4, p.236].

b) For (analytic and geometric) estimates of the distance to VMOA (related to Corollary 2.3), see also [AxSha, CaCu, StSt].

c) In the case $p \in (0,1)$, the density of the polynomials in $\mathcal{Q}_{p,0}$ doesn't mean that the disc algebra $\mathcal{A}$ (consisting of functions $f \in \mathcal{H}$ continuous on $\partial\triangle$) is a subset of $\mathcal{Q}_{p,0}$. Indeed, let $f_1(z) = \sum_{k=0}^{\infty} 2^{-k(1-p)/2} z^{2^k}$. Then from [AuXiZh, Theorem 6] it follows that $f_1 \in \mathcal{A} \setminus \mathcal{Q}_p$. This phenomenon distinguishes the cases $p \in (0,1)$ from the cases $p \in [1,\infty)$.

d) An example of an unbounded function in $\mathcal{Q}_{p,0}$ is easily constructed by using the Riemann mapping theorem. Let Ω be the inside domain of the curve $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3$, where

$$\begin{cases} \gamma_1 = \{(x,y) \in \mathbb{C} : \quad x^2 + y^2 = 1, \quad x \in [-1,0]\}; \\ \gamma_2 = \{(x,y) \in \mathbb{C} : \quad y - e^{-x} = 0, \quad x \in [0,\infty)\}; \\ \gamma_3 = \{(x,y) \in \mathbb{C} : \quad y + e^{-x} = 0, \quad x \in [0,\infty)\}. \end{cases}$$

Let f_2 be a conformal map of $\triangle$ onto Ω. Clearly, f_2 is unbounded, but in $\mathcal{D} \subset \mathcal{Q}_{p,0}$, owing to $\|f_2\|_{\mathcal{D}}^2 = (\pi^2 + 4)/2$.

3 Extreme Points

Given a norm $\|\cdot\|_X$ on a Banach space X. In studying $(X, \|\cdot\|_X)$, one problem of considerable interest is that of characterizing the geometry of the unit closed ball

$$(B_X, \|\cdot\|_X) = \{f \in X : \|f\|_X \leq 1\}.$$

In particular, we would like to find the extreme points of $(B_X, \|\cdot\|_X)$, namely, the points in $(B_X, \|\cdot\|_X)$ which are not a proper convex combination of two different points of $(B_X, \|\cdot\|_X)$. The problem addressed here deals with the extreme points of $(B_{\mathcal{Q}_{p,0}}, \|\cdot\|_{\mathcal{Q}_p})$ in order to better understand the linear structure of $\mathcal{Q}_{p,0}$.

The following result is the Proposition 1 in [CiWo].

Lemma 3.1. *Let X and Y be Banach spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$ respectively. Let $N : [0,\infty) \times [0,\infty) \to \mathbb{R}$ be a function so that $(x,y) \to N(|x|,|y|)$ is a norm on $\mathbb{R}^2$. Define a norm $\|\cdot\|_Z$ on $Z = X \oplus Y$ by*

$$\|x \oplus y\|_Z = N(\|x\|_X, \|y\|_Y), \quad for \quad x \in X, \quad y \in Y.$$

Then $x \oplus y$ is an extreme point of $(B_Z, \|\cdot\|_Z)$ if and only if the following three conditions hold: (i) *x is an extreme point of the closed ball of radius $\|x\|_X$ of X.* (ii) *y is an extreme point of the closed ball of radius $\|y\|_Y$ of Y.* (iii) *$(\|x\|_X, \|y\|_Y)$ is an extreme point of the unit closed ball of $\mathbb{R}^2$ with the norm N.*

Before stating our result, we still need another useful lemma whose hyperbolic version is presented in [SmZh, Lemma 2.3].

Lemma 3.2. *Let $p \in (0,\infty)$ and $f \in \mathcal{H}$. If $E_p(f,w)$ is finite for some $w \in \triangle$, then $E_p(f,\cdot)$ is a continuous function on $\triangle$.*

Proof. It is easy to figure out that for three points $z, w_1, w_2 \in \triangle$,

$$\frac{(1-|w_1|)(1-|w_2|)}{4} \leq \frac{1-|\sigma_{w_1}(z)|^2}{1-|\sigma_{w_2}(z)|^2} \leq \frac{4}{(1-|w_1|)(1-|w_2|)}. \tag{3.1}$$

This indicates that if $E_p(f,\cdot)$ is finite at some point of $\triangle$ then so is it at all points of $\triangle$. Let now $E_p(f,w) < \infty$. To prove the continuity of $E_p(f,\cdot)$, it suffices to verify that it is continuous at w. For this end, one assumes that $\{w_n\} \subset \triangle$ is convergent to w. Then there is a positive integer n_0 such that as $n \geq n_0$, $1-|w_n| \geq (1-|w|)/2$ and thus, by (3.1),

$$\frac{1-|\sigma_{w_n}(z)|^2}{1-|\sigma_w(z)|^2} \leq \frac{8}{(1-|w|)^2}.$$

Accordingly,

$$|E_p(f,w_n) - E_p(f,w)| \leq \left[1 + \frac{8^p}{(1-|w|)^{2p}}\right] E_p(f,w) < \infty.$$

An application of the Lebesgue Dominated Convergence Theorem implies that $E_p(f,w_n) \to E_p(f,w)$ as $n \to \infty$, i.e., $E_p(f,\cdot)$ is continuous at w. The proof is complete. ■

It is a classical result that for a Hilbert space X (certainly, including $\mathcal{D}$), the extreme points of $(B_X, \|\!|\cdot|\!\|_X)$ are precisely those on the unit sphere:

$$(S_X, \|\!|\cdot|\!\|_X) = \{f \in X : \|\!|f|\!\|_X = 1\}.$$

The following (which seems surprising to us) says that this is also valid for the non-Hilbert space $\mathcal{Q}_{p,0}$.

Theorem 3.3. *Let $p \in (0,\infty)$ and $f \in \mathcal{Q}_{p,0}$. Then f is an extreme point of $(B_{\mathcal{Q}_{p,0}}, \|\!|\cdot|\!\|_{\mathcal{Q}_p})$ if and only if either $f \equiv \lambda$ with $|\lambda| = 1$ or $f(0) = 0$ with $\|f\|_{\mathcal{Q}_p} = 1$.*

Proof. For $p \in (0,\infty)$ let $\mathcal{Q}^0_{p,0} = \{f \in \mathcal{Q}_{p,0} : f(0) = 0\}$. Notice that $\|\!|f|\!\|_{\mathcal{Q}_p} = |f(0)| + \|f\|_{\mathcal{Q}_p}$ for $f \in \mathcal{Q}_{p,0}$. So by Lemma 3.1, we need only to verify that a function $f \in \mathcal{Q}^0_{p,0}$ is an extreme point of the unit closed ball $B_{\mathcal{Q}^0_{p,0}}$ in $\mathcal{Q}^0_{p,0}$ if and only if $\|f\|_{\mathcal{Q}_p} = 1$.

The necessity is essentially trivial. The key is to argue the sufficiency. Now suppose that f lies in $\mathcal{Q}^0_{p,0}$ with $\|f\|_{\mathcal{Q}_p} = 1$. Since $\lim_{|w|\to 1} E_p(f,w) = 0$, there is an $r \in (0,1)$ such that $\sup_{|w|>r} E_p(f,w) \leq 1/2$. Consequently, we have

$$1 = \|f\|_{\mathcal{Q}_p} = \sup_{w\in\triangle} E_p(f,w) = \max_{|w|\leq r} E_p(f,w).$$

Applying Lemma 3.2 to this f, we see that $E_p(f,\cdot)$ is continuous on the compact set $\{w \in \triangle : |w| \leq r\}$, and thus there exists a $w_0 \in \triangle$ ($|w_0| \leq r$) to ensure $E_p(f,w_0) = 1$.

Let g be any function in $\mathcal{Q}^0_{p,0}$ such that $\|f+g\|_{\mathcal{Q}_p} \leq 1$ and $\|f-g\|_{\mathcal{Q}_p} \leq 1$. Then

$$E_p(f,w_0) + E_p(g,w_0) = 2^{-1}[E_p(f+g,w_0) + E_p(f-g,w_0)] \leq 1.$$

Therefore $E_p(g,w_0) = 0$. This implies that $g = 0$. We conclude that f is extreme. ■

The previous proof actually leads to a sufficient condition for $f \in \mathcal{Q}_p$ to be an extreme point of $(B_{\mathcal{Q}_p}, \|\!|\cdot|\!\|_{\mathcal{Q}_p})$.

Corollary 3.4. *Let $p \in (0,\infty)$ and $f \in \mathcal{Q}_p$. If f is an extreme point of $(B_{\mathcal{Q}_p}, \|\!|\cdot\|\!|_{\mathcal{Q}_p})$ then $\|\!|f\|\!|_{\mathcal{Q}_p} = 1$. Conversely, if either $f \equiv \lambda$ with $|\lambda| = 1$ or there exists a point $w_0 \in \triangle$ such that $E_p(f, w_0) = 1$, then f is an extreme point of $(B_{\mathcal{Q}_p}, \|\!|\cdot\|\!|_{\mathcal{Q}_p})$.*

Remark 3.5. a) Different norms produce different sets of the extreme points. This viewpoint is reflected by our Theorem 3.3, Cima-Wogen's Corollary 1 and Theorem 2 in [CiWo], and Axler-Shields' Theorem in [AxShi].

b) It would be interesting to give a full description of the extreme points of $(B_{\mathcal{Q}_p}, \|\!|\cdot\|\!|_{\mathcal{Q}_p})$.

4 Composition Semigroups

Let now $\{\psi_t : t \geq 0\}$ be a composition semigroup of the holomorphic self-maps ψ_t of $\triangle$, that is: $\psi_t \circ \psi_s = \psi_{t+s}$ for $t, s \geq 0$; $\psi_0(z) = z$; and $\psi_t(z)$ is continuous in two-parameters: t and z.

A composition semigroup always consists of univalent functions and all such semigroups can be classified in two classes Ψ_0 and Ψ_1, according to whether the common fixed point of ψ_t is in $\triangle$ or on $\partial\triangle$. Without loss of generality, one can assume that the fixed point is 0 for Ψ_0 and 1 for Ψ_1 (where the fixed point 1 is understood to be the Denjoy-Wolff point, namely, $\lim_{r\to1}\psi_t(r) = 1$ for any $\{\psi_t\} \in \Psi_1$). Hence

- $\{\psi_t\} \in \Psi_0$ is of the form $\psi_t(z) = h^{-1}(e^{-ct}h(z))$, where $\Re c \geq 0$ and $h \in \mathcal{H}$ with: $h(0) = 0$ and $w\exp(-ct) \in h(\triangle)$ for each $w \in h(\triangle)$.
- $\{\psi_t\} \in \Psi_1$ has the form $\psi_t(z) = h^{-1}(ct + h(z))$, where $\Re c \geq 0$, $h \in \mathcal{H}$ with: $h(0) = 0$ and $\Re(c^{-1}(z-1)^2h'(z)) \geq 0$ for each $z \in \triangle$.

It is clear that each semigroup $\{\psi_t\}$ induces a one-parameter operator semigroup by composition $\{C_{\psi_t}\} : C_{\psi_t}(f) = f \circ \psi_t$. As in the $\mathcal{D}$-setting [Si3], more is true:

Theorem 4.1. *Let $p \in (0,\infty)$. Then $\{C_{\psi_t}\}$ is strongly continuous on $\mathcal{Q}_{p,0}$. Moreover* (i) *The infinitesimal generator of $\{C_{\psi_t}\}$ is given by $\Gamma(f) = Gf'$ and its domain is $\{f \in \mathcal{Q}_{p,0} : Gf' \in \mathcal{Q}_{p,0}\}$, where $G = -c(h/h')$ or c/h' whenever $\{\psi_t\} \in \Psi_0$ or Ψ_1.* (ii) *$\{C_{\psi_t}\}$ is not continuous in the uniform topology unless it is trivial.* (iii) *The growth bound $\omega = \lim_{t\to\infty} t^{-1}\log\|C_{\psi_t}\| = 0$, where*

$$\|C_{\psi_t}\| = \inf\{M : \|\!|C_{\psi_t}(f)\|\!|_{\mathcal{Q}_p} \leq M\|\!|f\|\!|_{\mathcal{Q}_p}, \quad f \in \mathcal{Q}_{p,0}\}.$$

Proof. Notice that if a holomorphic map $\psi : \triangle \to \triangle$ is univalent then

$$E_p(f\circ\psi, w) \leq \int_{\psi(\triangle)} |f'(z)|^2\Big[1 - |\sigma_{\psi(w)}(z)|\Big]^p dm(z),$$

and so the composition $C_\psi(f) = f \circ \psi$ exists as a bounded linear operator on $\mathcal{Q}_p$ with

$$\|\!|C_\psi(f)\|\!|_{\mathcal{Q}_p} \leq \left[1 + \left(\frac{2^{p-1}}{\pi}\right)^{1/2}\log\frac{1+|\psi(0)|}{1-|\psi(0)|}\right]\|\!|f\|\!|_{\mathcal{Q}_p} = K_1\|\!|f\|\!|_{\mathcal{Q}_p}. \tag{4.1}$$

Moreover, if $f \in \mathcal{Q}_{p,0}$ and $\epsilon > 0$, then by Theorem 2.1, there exists a polynomial p_n such that $|||f - p_n|||_{\mathcal{Q}_p} < \epsilon$. Thus (4.1) implies $|||C_\psi(f) - C_\psi(p_n)|||_{\mathcal{Q}_p} < \epsilon K_1$. Owing to $C_\psi(p_n) \in \mathcal{Q}_{p,0}$, it follows that $C_\psi(f) \in \mathcal{Q}_{p,0}$. Therefore $C_\psi : \mathcal{Q}_{p,0} \to \mathcal{Q}_{p,0}$ exists as a bounded operator with $\|C_\psi\| \le K_1$.

In order to show that each semigroup $\{C_{\psi_t}\}$ is strongly continuous on $\mathcal{Q}_{p,0}$, it suffices to verify that $\lim_{t\to 0} |||C_{\psi_t}(f) - f|||_{\mathcal{Q}_p} = 0$ for every $f \in \mathcal{Q}_{p,0}$ and every $\{\psi_t\} \in \Psi_0 \cup \Psi_1$. Since the polynomials are dense in $(\mathcal{Q}_{p,0}, |||\cdot|||_{\mathcal{Q}_p})$ but also (4.1) infers that

$$|||C_{\psi_t}(f) - f|||_{\mathcal{Q}_p} \le K_1 |||f - P|||_{\mathcal{Q}_p} + |||C_{\psi_t}(P) - P|||_{\mathcal{Q}_p}$$

holds for any polynomial P, it is enough, by the properties of Ψ_0 and Ψ_1, to prove $\lim_{t\to 0} \|\psi_t - z\|_{\mathcal{Q}_p} = 0$. While, this is a simple thing in that $\lim_{t\to 0} \|\psi_t - z\|_{\mathcal{D}} = 0$ and $\|\psi_t - z\|_{\mathcal{Q}_p} \le \|\psi_t - z\|_{\mathcal{D}}$.

The infinitesimal generator Γ of C_{ψ_t} is determined by

$$\Gamma(f)(z) = \frac{\partial C_{\psi_t}(f)(z)}{\partial t}\Big|_{t=0} = G(z) f'(z),$$

where G is the generator of $\{\psi_t\}$:

$$G = \begin{cases} \frac{-ch}{h'}, & \{\psi_t\} \in \Phi_0 \\ \frac{c}{h'}, & \{\psi_t\} \in \Phi_1. \end{cases}$$

By definition, the domain of Γ is the following set:

$$D(\Gamma) = \Big\{ f \in \mathcal{Q}_{p,0} : \lim_{t\to 0} \frac{C_{\psi_t}(f) - f}{t} \quad exists \quad in \quad \mathcal{Q}_{p,0} \Big\}.$$

On the one hand, if $f \in D(\Gamma)$ then some calculations involving Ψ_0 and Ψ_1 deduce that f meets the requirements of the domain stated in Theorem 4.1 (ii). On the other hand, if f is in $\mathcal{Q}_{p,0}$ with $g = Gf' \in \mathcal{Q}_{p,0}$ and G being as above, then for $t > 0$ one has

$$\frac{1}{t}\int_0^t C_{\psi_s}(g)\,ds = \frac{C_{\psi_t}(f) - f}{t}.$$

Because $\{C_{\psi_t}\}$ is a strongly continuous semigroup, the left-hand side of the last equation has a limit g as $t \to 0$, with respect to $|||\cdot|||_{\mathcal{Q}_p}$. Accordingly, $f \in D(\Gamma)$.

Next, observe that the strongly continuity of $\{C_{\psi_t}\}$ is equivalent to the boundedness of $\Gamma : \Gamma(f) = Gf'$ (cf. [Si4, p.231]). So if Γ is bounded on $\mathcal{Q}_{p,0}$ then $Gf' \in \mathcal{Q}_{p,0}$ when $f \in \mathcal{Q}_{p,0}$. In particular,

$$|||\Gamma(f_{n,p})|||_{\mathcal{Q}_p} \le \|\Gamma\| \, |||f_{n,p}|||_{\mathcal{Q}_p}, \tag{4.2}$$

where $\|\Gamma\|$ means the norm of operator Γ, and for each integer $n \ge 1$,

$$f_{n,p}(z) = \begin{cases} z^n, & p \ge 1 \\ \frac{z^n}{n}, & p \in (0,1). \end{cases}$$

Now consider $p \in (0,1)$. Clearly, $|||f_{n,p}|||_{\mathcal{Q}_p} \le 2$. If $G(z) = \sum_{k=0}^\infty a_k z^k$, then through (4.2) and some elementary calculations, we can find out a constant $K_2 > 0$ depending only on $p \in (0,1)$ such that

$$\sum_{k=0}^\infty |a_k|^2 (k + n - 1)^{1-p} \le K_2 \|\Gamma\|.$$

This derives all $a_k = 0$, and so $G = 0$ which is impossible.

The $\mathcal{B}_0$ and VMOA settings may be similarly treated, using the facts: $\mathcal{Q}_{1,0} = VMOA$ and $\mathcal{Q}_{p,0} = \mathcal{B}_0$ for $p > 1$.

Finally, let us come to the proof of $\omega = 0$. Since all C_{ψ_t} keep 1 unchanged, one always has $\|C_{\psi_t}\| \geq 1$ and so $\omega \geq 0$. Further, if $\{\psi_t\} \in \Psi_0$, then $\omega = 0$ in that $\psi_t(0) = 0$ and thus $\|C_{\psi_t}\| \leq 1$ (thanks to the constant K_1 above). However, if $\{\psi_t\} \in \Psi_1$ then by [Si3,(3.3)],

$$\limsup_{t\to\infty} \frac{\log\log[1/(1-|\psi_t(0)|)]}{t} \leq 0,$$

which, together with $\|C_{\psi_t}\| \leq K_1$, implies $\omega \leq 0$ and hence $\omega = 0$. The proof is complete. ■

Remark 4.2. a) $\lambda - \Gamma$ is invertible on $\mathcal{Q}_{p,0}$ whenever $\Re\lambda > 0$, and

$$(\lambda - \Gamma)^{-1}(f) = \int_0^\infty e^{-\lambda t} C_{\psi_t}(f)dt.$$

In addition, the spectral radius of C_{ψ_t} (acting on $\mathcal{Q}_{p,0}$) is 1.

b) Suppose $\{\psi_t\} \in \Psi_0$ and n is a natural number. Then, as in [Si3, Corollary 2], the semigroup $S_t(f) = (\psi_t)^n C_{\psi_t}(f)$ is strongly continuous on $\mathcal{Q}_{p,0}$ with generator

$$\Gamma_n(f)(z) = -c[h(z)/h'(z)]f'(z) - cn[h(z)/(zh'(z))]f(z).$$

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