Fixed point and homotopy results in uniform spaces

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Abstract

A fixed point and homotopic invariance result is presented for set valued contractive type maps in complete gauge spaces.

1 Introduction

This paper presents new fixed point results for contractive type maps on complete (or sequentially complete) gauge spaces (i.e. complete uniform spaces). We begin with a local fixed point result for single valued maps and then this result is extended to multivalued closed maps. In addition we present a homotopy type result for contractive type maps in complete gauge spaces. Our results in particular extend those in [1, 3, 4]. It was noted in [5] that many generalized contractive type maps F considered in the literature are in fact contractive maps with respect to a suitable uniform structure associated with F.

In Section 2, $E = (E, \{d_\alpha\}_{\alpha\in\Lambda})$ (here Λ is a directed set) will denote a gauge space endowed with a complete gauge structure $\{d_\alpha : \alpha \in \Lambda\}$. We denote by D_α the generalized Hausdorff pseudometric induced by d_α; that is, for $Z, Y \subseteq E$,

$$D_\alpha(Z,Y) = \inf\{\epsilon > 0 : \quad \forall\, x \in Z,\ \forall y \in Y,\ \exists x^\star \in Z,\ \exists y^\star \in Y \text{ such that } d_\alpha(x, y^\star) < \epsilon,\ d_\alpha(x^\star, y) < \epsilon\},$$

with the convention that $\inf(\emptyset) = \infty$.

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2 Fixed point theory in gauge spaces

In this section $E = (E, \{d_\alpha\}_{\alpha\in\Lambda})$ (here Λ is a directed set) will denote a gauge space endowed with a complete gauge structure $\{d_\alpha : \alpha \in \Lambda\}$ (see Dugundji [2 pp. 198, 308]). For $r = \{r_\alpha\}_{\alpha\in\Lambda} \in (0,\infty)^\Lambda$ and $x \in X$, we define the pseudo–ball centered at x of radius r by

$$B(x, r) = \{y \in E : \ d_\alpha(x, y) \le r_\alpha \ \text{ for all } \ \alpha \in \Lambda\}.$$

We begin with a local theorem for single valued maps.

Theorem 2.1. *Let E be a complete gauge space, $r = \{r_\alpha\}_{\alpha\in\Lambda} \in (0,\infty)^\Lambda$, $x_0 \in E$ and $F : B(x_0, r) \to E$. Suppose for each $\delta \in \Lambda$ that there exists a continuous nondecreasing function $\phi_\delta : [0,\infty) \to [0,\infty)$ satisfying $\phi_\delta(z) < z$ for $z > 0$. Also assume there exists functions $\beta : \Lambda \to \Lambda$ and $\gamma : \Lambda \to \Lambda$ such that for each $\alpha \in \Lambda$ and x, $y \in B(x_0, r)$ we have*

$$d_\alpha(F\,x, F\,y) \le \phi_{\beta(\alpha)}(d_{\gamma(\alpha)}(x, y)). \tag{2.1}$$

Finally suppose for each $\alpha \in \Lambda$ that

$$\begin{cases} \sum_{n=1}^{\infty} \phi_{\beta(\alpha)}\, \phi_{\beta(\gamma(\alpha))} \,....\phi_{\beta(\gamma^{n-1}(\alpha))} \ \Big(d_{\gamma^n(\alpha)}\,(x_0, F\,x_0)\Big) \\ +d_\alpha(x_0, F\,x_0) \le r_\alpha \end{cases} \tag{2.2}$$

holds; here $\gamma^0(\alpha) = \alpha$ and $\gamma^n(\alpha) = \gamma\,(\gamma^{n-1}(\alpha))$ for $n \in \{1, 2, ...\}$. Then F has a fixed point (i.e. there exists $x \in B(x_0, r)$ with $x = F\,x$).

Remark 2.1. If for each $\alpha \in \Lambda$ we have

$$d_\alpha(x_0, F\,x_0) \le r_\alpha - \phi_{\beta(\alpha)}(r_\alpha) \tag{2.3}$$

and

$$\begin{cases} \sum_{n=1}^{\infty} \phi_{\beta(\alpha)}\, \phi_{\beta(\gamma(\alpha))} \,....\phi_{\beta(\gamma^{n-1}(\alpha))} \ \Big(r_{\gamma^n(\alpha)} - \phi_{\beta(\gamma^n(\alpha))}(r_{\gamma^n(\alpha)})\Big) \\ \le \phi_{\beta(\alpha)}(r_\alpha) \end{cases} \tag{2.4}$$

then (2.2) holds. This is immediate since for fixed $\alpha \in \Lambda$ we have

$$\begin{aligned} &\sum_{n=1}^{\infty} \phi_{\beta(\alpha)}\, \phi_{\beta(\gamma(\alpha))} \,....\phi_{\beta(\gamma^{n-1}(\alpha))} \ \Big(d_{\gamma^n(\alpha)}\,(x_0, F\,x_0)\Big) + d_\alpha(x_0, F\,x_0) \\ &\le \sum_{n=1}^{\infty} \phi_{\beta(\alpha)}\, \phi_{\beta(\gamma(\alpha))} \,....\phi_{\beta(\gamma^{n-1}(\alpha))} \ \Big(r_{\gamma^n(\alpha)} - \phi_{\beta(\gamma^n(\alpha))}(r_{\gamma^n(\alpha)})\Big) \\ &+[r_\alpha - \phi_{\beta(\alpha)}(r_\alpha)] \\ &\le \phi_{\beta(\alpha)}(r_\alpha) + [r_\alpha - \phi_{\beta(\alpha)}(r_\alpha)] = r_\alpha. \end{aligned}$$

Proof : Let $x_n = F\,x_{n-1}$ for $n \in \{1, 2, ...\}$. Fix $\alpha \in \Lambda$ and we claim

$$\begin{cases} \{x_n\}_1^\infty \ \text{ is a Cauchy sequence with respect to } \ d_\alpha \\ \text{and } \ x_{n+1} \in B_\alpha(x_0, r_\alpha) = \{y \in E : \ d_\alpha(x_0, y) \le r_\alpha\} \\ \text{for } \ n \in \{0, 1, ...\}. \end{cases} \tag{2.5}$$

Before we prove (2.5) we first notice for $n \in \{0, 1, ...\}$ and $\delta \in \Lambda$ that

$$d_\delta(x_{n+1}, x_n) = d_\delta(F\,x_n, F\,x_{n-1}) \le \phi_{\beta(\delta)}\,(d_{\gamma(\delta)}(x_n, x_{n-1}))$$

and as a result

$$\begin{aligned} d_\delta(x_{n+1}, x_n) &\le \phi_{\beta(\delta)}\,\phi_{\beta(\gamma(\delta))} \ldots\ldots \phi_{\beta(\gamma^{n-1}(\delta))}\,(d_{\gamma^n(\delta)}(x_1, x_0)) \\ &= \phi_{\beta(\delta)}\,\phi_{\beta(\gamma(\delta))} \ldots\ldots \phi_{\beta(\gamma^{n-1}(\delta))}\,(d_{\gamma^n(\delta)}(x_0, F\,x_0)). \end{aligned} \tag{2.6}$$

Notice for $\alpha \in \Lambda$ that $x_{n+1} \in B_\alpha(x_0, r_\alpha)$ since (2.6) and (2.2) imply

$$\begin{aligned} d_\alpha(x_{n+1}, x_0) &\le d_\alpha(x_0, x_1) + d_\alpha(x_1, x_2) + \ldots\ldots + d_\alpha(x_n, x_{n+1}) \\ &\le \sum_{k=1}^{\infty} \phi_{\beta(\alpha)}\,\phi_{\beta(\gamma(\alpha))} \ldots \phi_{\beta(\gamma^{k-1}(\alpha))}\,\left(d_{\gamma^k(\alpha)}\,(x_0, F\,x_0)\right) \\ &+ d_\alpha(x_0, F\,x_0) \le r_\alpha. \end{aligned}$$

Also $\{x_n\}_1^\infty$ is a Cauchy sequence with respect to d_α since if $n,\, p \in \{0, 1, ...\}$ we have

$$\begin{aligned} d_\alpha(x_{n+p}, x_n) &\le d_\alpha(x_{n+p}, x_{n+p-1}) + \ldots\ldots + d_\alpha(x_n, x_{n+1}) \\ &\le \sum_{k=n}^{\infty} \phi_{\beta(\alpha)}\,\phi_{\beta(\gamma(\alpha))} \ldots \phi_{\beta(\gamma^{k-1}(\alpha))}\,\left(d_{\gamma^k(\alpha)}\,(x_0, F\,x_0)\right), \end{aligned}$$

so (2.2) guarantees that $\{x_n\}_1^\infty$ is a Cauchy sequence with respect to d_α. Thus (2.5) is true for each $\alpha \in \Lambda$. As a result $x_n \in B(x_0, r)$ for each $n \in \{1, 2, ...\}$ and $\{x_n\}_1^\infty$ is a Cauchy sequence. Thus there exists $x \in B(x_0, r)$ with $x_n \to x$. We now claim

$$d_\alpha(x, F\,x) = 0 \quad \text{for each} \quad \alpha \in \Lambda. \tag{2.7}$$

If (2.7) is true then $x = F\,x$ and we are finished. To see (2.7) fix $\alpha \in \Lambda$ and notice

$$\begin{aligned} d_\alpha(x, F\,x) &\le d_\alpha(x, x_n) + d_\alpha(x_n, F\,x) \\ &\le d_\alpha(x, x_n) + \phi_{\beta(\alpha)}\,(d_{\gamma(\alpha)}(x_{n-1}, x)). \end{aligned}$$

Let $n \to \infty$ (note $d_\delta(x_n, x) \to 0$ for all $\delta \in \Lambda$) to obtain (note $\phi_{\beta(\alpha)}(0) = 0$) $d_\alpha(x, F\,x) = 0$. ∎

Remark 2.1. It is easy to combine the ideas in Theorem 2.1 together with those in [1] to obtain an analogue of Theorem 2.1 when the space E is also a gauge space endowed with a gauge structure $\{d'_\alpha :\ \alpha \in \Lambda'\}$.

If our map $F : B(x_0, r) \to E$ in Theorem 2.1 is replaced by $F : E \to E$ then the iterates x_n defined in Theorem 2.1 do not need to belong to $B(x_0, r)$ and so we have the following result.

Theorem 2.2. *Let E be a complete gauge space, $x_0 \in E$ and $F : E \to E$. Suppose for each $\delta \in \Lambda$ that there exists a continuous nondecreasing function ϕ_δ : $[0, \infty) \to [0, \infty)$ satisfying $\phi_\delta(z) < z$ for $z > 0$. Also assume there exists functions $\beta : \Lambda \to \Lambda$ and $\gamma : \Lambda \to \Lambda$ such that for each $\alpha \in \Lambda$ and $x,\, y \in E$ we have*

$$d_\alpha(F\,x, F\,y) \le \phi_{\beta(\alpha)}(d_{\gamma(\alpha)}(x, y)).$$

Finally suppose for each $\alpha \in \Lambda$ that

$$\sum_{n=1}^{\infty} \phi_{\beta(\alpha)}\, \phi_{\beta(\gamma(\alpha))} \, ...\phi_{\beta(\gamma^{n-1}(\alpha))} \, \Big(d_{\gamma^n(\alpha)}\,(x_0, F\,x_0)\Big) + d_\alpha(x_0, F\,x_0) \, < \, \infty.$$

Then there exists $x \in E$ with $x = F\,x$.

Next we present an analogue of Theorem 2.1 for multivalued maps with closed values.

Theorem 2.3. *Let E be a complete gauge space, $r = \{r_\alpha\}_{\alpha\in\Lambda} \in (0,\infty)^\Lambda$, $x_0 \in E$ and $F : B(x_0, r) \to C(E)$ (here $C(E)$ denotes the family of nonempty closed subsets of E). Suppose for each $\delta \in \Lambda$ that there exists a continuous strictly increasing function $\phi_\delta : [0,\infty) \to [0,\infty)$ satisfying $\phi_\delta(z) < z$ for $z > 0$. Also assume there exists functions $\beta : \Lambda \to \Lambda$ and $\gamma : \Lambda \to \Lambda$ such that for each $\alpha \in \Lambda$ and $x, y \in B(x_0, r)$ we have*

$$D_\alpha(F\,x, F\,y) \le \phi_{\beta(\alpha)}(d_{\gamma(\alpha)}(x, y)). \tag{2.8}$$

Now suppose the following two conditions hold:

$$\text{for each } \alpha \in \Lambda \text{ we have } dist_\alpha\,(x_0, F\,x_0) \, < \, r_\alpha \, - \, \phi_{\beta(\alpha)}(r_\alpha) \tag{2.9}$$

and

$$\begin{cases} \text{for every } x \in B(x_0, r) \text{ and every } \epsilon = \{\epsilon_\alpha\}_{\alpha\in\Lambda} \in (0,\infty)^\Lambda \\ \text{there exists } y \in F\,x \text{ with } d_\alpha\,(x, y) \le dist_\alpha\,(x, F\,x) + \epsilon_\alpha \\ \text{for every } \alpha \in \Lambda. \end{cases} \tag{2.10}$$

Finally assume for each $\alpha \in \Lambda$ that

$$\begin{cases} \sum_{n=1}^{\infty} \phi_{\beta(\alpha)}\, \phi_{\beta(\gamma(\alpha))} \, ...\phi_{\beta(\gamma^{n-1}(\alpha))} \, \Big(r_{\gamma^n(\alpha)} - \phi_{\beta(\gamma^n(\alpha))}(r_{\gamma^n(\alpha)})\Big) \\ \le \phi_{\beta(\alpha)}(r_\alpha). \end{cases} \tag{2.11}$$

Then there exists $x \in B(x_0, r)$ with $x \in F\,x$.

Proof: From (2.9) and (2.10) we may choose $x_1 \in F\,x_0$ with

$$d_\alpha(x_0, x_1) < r_\alpha - \phi_{\beta(\alpha)}(r_\alpha) \text{ for every } \alpha \in \Lambda. \tag{2.12}$$

Next fix $\alpha \in \Lambda$ and choose $\epsilon_\alpha > 0$ so that

$$\phi_{\beta(\alpha)}(d_{\gamma(\alpha)}(x_0, x_1)) \, + \, \epsilon_\alpha \, < \, \phi_{\beta(\alpha)}(r_{\gamma(\alpha)} - \phi_{\beta(\gamma(\alpha))}(r_{\gamma(\alpha)})) \tag{2.13}$$

(this is possible from (2.12) and the fact that ϕ_δ is strictly increasing for each $\delta \in \Lambda$). From (2.10) we may choose $x_2 \in F\,x_1$ so that for every $\alpha \in \Lambda$ we have

$$\begin{aligned} d_\alpha(x_1, x_2) \; &\le \; dist_\alpha(x_1, F\,x_1) + \epsilon_\alpha \le D_\alpha(F\,x_0, F\,x_1) + \epsilon_\alpha \\ &\le \; \phi_{\beta(\alpha)}\,(d_{\gamma(\alpha)}(x_0, x_1)) + \epsilon_\alpha \end{aligned}$$

and this together with (2.13) yields

$$d_\alpha(x_1, x_2) \, < \, \phi_{\beta(\alpha)}(r_{\gamma(\alpha)} - \phi_{\beta(\gamma(\alpha))}(r_{\gamma(\alpha)})) \;\; \forall\, \alpha \in \Lambda. \tag{2.14}$$

Next fix $\alpha \in \Lambda$ and choose $\delta_\alpha > 0$ so that

$$\phi_{\beta(\alpha)}(d_{\gamma(\alpha)}(x_1, x_2)) + \delta_\alpha < \phi_{\beta(\alpha)}\, \phi_{\beta(\gamma(\alpha))}(r_{\gamma^2(\alpha)} - \phi_{\beta(\gamma^2(\alpha))}(r_{\gamma^2(\alpha)})). \tag{2.15}$$

¿From (2.10) we may choose $x_3 \in F\, x_2$ so that for every $\alpha \in \Lambda$ we have

$$\begin{aligned} d_\alpha(x_2, x_3) &\leq dist_\alpha(x_2, F\, x_2) + \delta_\alpha \leq D_\alpha(F\, x_1, F\, x_2) + \delta_\alpha \\ &\leq \phi_{\beta(\alpha)}\left(d_{\gamma(\alpha)}(x_1, x_2)\right) + \delta_\alpha, \end{aligned}$$

and this together with (2.15) yields

$$d_\alpha(x_2, x_3) < \phi_{\beta(\alpha)}\, \phi_{\beta(\gamma(\alpha))}(r_{\gamma^2(\alpha)} - \phi_{\beta(\gamma^2(\alpha))}(r_{\gamma^2(\alpha)}))\ \ \forall\, \alpha \in \Lambda. \tag{2.16}$$

Continue this process to construct $x_{n+1} \in F\, x_n$ for $n \in \{2, 3, ...\}$ so that

$$d_\alpha(x_{n+1}, x_n) < \phi_{\beta(\alpha)}\phi_{\beta(\gamma(\alpha))}.....\phi_{\beta(\gamma^{n-1}(\alpha))}(r_{\gamma^n(\alpha)} - \phi_{\beta(\gamma^n(\alpha))}(r_{\gamma^n(\alpha)})) \tag{2.17}$$

for all $\alpha \in \Lambda$. Notice $x_{n+1} \in B(x_0, r)$ for each $n \in \{0, 1, 2, ...\}$ since for $\alpha \in \Lambda$ we have

$$\begin{aligned} d_\alpha(x_{n+1}, x_0) &\leq d_\alpha(x_0, x_1) + d_\alpha(x_1, x_2) + ... + d_\alpha(x_n, x_{n+1}) \\ &\leq \sum_{k=1}^{\infty} \phi_{\beta(\alpha)}\phi_{\beta(\gamma(\alpha))}....\phi_{\beta(\gamma^{k-1}(\alpha))}\left(r_{\gamma^k(\alpha)} - \phi_{\beta(\gamma^k(\alpha))}(r_{\gamma^k(\alpha)})\right) \\ &+ d_\alpha(x_0, x_1) \leq d_\alpha(x_0, x_1) + \phi_{\beta(\alpha)}(r_\alpha) \\ &< [r_\alpha - \phi_{\beta(\alpha)}(r_\alpha)] + \phi_{\beta(\alpha)}(r_\alpha) = r_\alpha. \end{aligned}$$

Also for each $\alpha \in \Lambda$ and $n,\, p \in \{0, 1, ...\}$ notice

$$d_\alpha(x_{n+p}, x_n) \leq \sum_{k=n}^{\infty} \phi_{\beta(\alpha)}\, \phi_{\beta(\gamma(\alpha))}\, ...\phi_{\beta(\gamma^{k-1}(\alpha))}\ \left(r_{\gamma^k(\alpha)} - \phi_{\beta(\gamma^k(\alpha))}(r_{\gamma^k(\alpha)})\right),$$

and so (2.11) guarantees that $\{x_n\}_1^\infty$ is a Cauchy sequence with respect to d_α. As a result $\{x_n\}_1^\infty$ is a Cauchy sequence so exists $x \in B(x_0, r)$ with $x_n \to x$. It remains to show $x \in F\, x$. Notice for each $\alpha \in \Lambda$ that

$$\begin{aligned} dist_\alpha(x, F\, x) &\leq d_\alpha(x, x_n) + dist_\alpha(x_n, F\, x) \leq d_\alpha(x, x_n) + D_\alpha(F\, x_{n-1}, F\, x) \\ &\leq d_\alpha(x, x_n) + \phi_{\beta(\alpha)}\left(d_{\gamma(\alpha)}(x_{n-1}, x)\right). \end{aligned}$$

Let $n \to \infty$ to obtain $dist_\alpha(x, F\, x) = 0$ for each $\alpha \in \Lambda$. Thus $x \in \overline{F\, x} = F\, x$ and we are finished. ∎

Next we obtain a homotopy result via Zorn's Lemma.

Theorem 2.4. *Let E be a complete metric space with U an open subset of E. Suppose $H : \overline{U} \times [0, 1] \to C(E)$ is a closed map (i.e. has closed graph) and assume the following conditions are satisfied:*

(a). $x \notin H(x, t)$ for $x \in \partial U$ and $t \in [0, 1]$;

(b). for each $\delta \in \Lambda$, there exists a continuous strictly increasing function $\phi_\delta : [0, \infty) \to [0, \infty)$ satisfying $\phi_\delta(z) < z$ for $z > 0$ and also assume there exists

functions $\beta : \Lambda \to \Lambda$ *and* $\gamma : \Lambda \to \Lambda$ *such that for each* $\alpha \in \Lambda$ *and for* $\forall\, t \in [0,1]$ *and* $x,\, y \in \overline{U}$ *we have*

$$D_\alpha(H(x,t), H(y,t)) \leq \phi_{\beta(\alpha)}(d_{\gamma(\alpha)}(x,y));$$

(c). for each $\delta \in \Lambda$, $\Phi_\delta : [0,\infty) \to [0,\infty)$ *is strictly increasing and* $\Phi_\delta^{-1}(a) + \Phi_\delta^{-1}(b) \leq \Phi_\delta^{-1}(a+b)$ *for* $a \geq 0$, $b \geq 0$ *(here* $\Phi_\delta(x) = x - \phi_\delta(x)$*);*

(d). for each $\alpha \in \Lambda$ *and for any* $s = \{s_\alpha\}_{\alpha\in\Lambda} \in (0,\infty)^\Lambda$ *we have*

$$\begin{cases} \sum_{n=1}^{\infty} \phi_{\beta(\alpha)}\, \phi_{\beta(\gamma(\alpha))} \ldots \phi_{\beta(\gamma^{n-1}(\alpha))} \left(s_{\gamma^n(\alpha)} - \phi_{\beta(\gamma^n(\alpha))}(s_{\gamma^n(\alpha)})\right) \\ \leq \phi_{\beta(\alpha)}(s_\alpha); \end{cases}$$

(e). for every $t \in [0,1]$ *and every* $\epsilon = \{\epsilon_\alpha\}_{\alpha\in\Lambda} \in (0,\infty)^\Lambda$ *there exists* $y \in H(x,t)$ *with* $d_\alpha(x,y) \leq dist_\alpha(x, H(x,t)) + \epsilon_\alpha$ *for every* $\alpha \in \Lambda$*;*

and

(f). there exists $M = \{M_\alpha\}_{\alpha\in\Lambda} \in (0,\infty)^\Lambda$ *and there exists a continuous increasing function* $\psi : [0,1] \to \mathbf{R}$ *such that* $D_\alpha(H(x,t), H(x,s)) \leq M_\alpha\, |\psi(t) - \psi(s)|$ *for all* $t,\ s \in [0,1]$ *and* $x \in \overline{U}$*, for every* $\alpha \in \Lambda$.

Then $H(\,.\,,0)$ *has a fixed point iff* $H(\,.\,,1)$ *has a fixed point.*

Proof : Suppose $H(\,.\,,0)$ has a fixed point. Consider

$$Q = \{(t,x) \in [0,1] \times U :\ x \in H(x,t)\}.$$

Now Q is nonempty since $H(\,.\,,0)$ has a fixed point. On Q define the partial ordering (see (c) for transitivity)

$$(t,x) \leq (s,y) \quad \text{iff} \quad t \leq s \quad \text{and} \quad d_\alpha(x,y) \leq \Phi_{\beta(\alpha)}^{-1}(2\, M_\alpha\, [\psi(s) - \psi(t)])$$

for every $\alpha \in \Lambda$. Let P be a totally ordered subset of Q and let

$$t^\star = \sup\{t :\ (t,x) \in P\}.$$

Take a sequence $\{(t_n, x_n)\}$ in P such that $(t_n, x_n) \leq (t_{n+1}, x_{n+1})$ and $t_n \to t^\star$. We have

$$d_\alpha(x_m, x_n) \leq \Phi_{\beta(\alpha)}^{-1}\left(2\, M_\alpha\, [\psi(t_m) - \psi(t_n)]\right) \quad \text{for all} \quad m > n$$

and $\alpha \in \Lambda$. Thus $\{x_n\}_1^\infty$ is a Cauchy sequence with respect to d_α for each $\alpha \in \Lambda$, so $\{x_n\}_1^\infty$ is a Cauchy sequence and it converges to some $x^\star \in \overline{U}$. Now since H is a closed map we have $(t^\star, x^\star) \in Q$ (note $x^\star \in H(x^\star, t^\star)$ by closedness and (a) implies $x^\star \in U$). It is also immediate from the definition of $t^\star$ and the fact that P is totally ordered that

$$(t,x) \leq (t^\star, x^\star) \quad \text{for every} \quad (t,x) \in P.$$

Thus $(t^\star, x^\star)$ is an upper bound of P. By Zorn's Lemma Q admits a maximal element $(t_0, x_0) \in Q$.

We claim $t_0 = 1$. Suppose our claim is false. First note since U is open that $\exists\, \delta_1, \ldots.., \delta_m \in (0,\infty)$ with $U(x_0, \delta_1) \cap \ldots. \cap U(x_0, \delta_m) \subseteq U$; here $U(x_0, \delta_i) =$

$\{x : d_{\alpha_i}(x, x_0) < \delta_i\}$ for $i \in \{1, ..., m\}$ and $\alpha_i \in \Lambda$ for $i \in \{1, ..., m\}$. Choose $\delta = \{\delta_\alpha\}_{\alpha\in\Lambda} \in (0, \infty)^\Lambda$ and $t \in (t_0, 1]$ with

$$B(x_0, \delta) \subseteq U \quad \text{and} \quad \delta_\alpha = \Phi^{-1}_{\beta(\alpha)}(2\, M_\alpha\, [\psi(t) - \psi(t_0)]).$$

Notice for every $\alpha \in \Lambda$ that

$$\begin{aligned} dist_\alpha(x_0, H(x_0, t)) &\leq dist_\alpha(x_0, H(x_0, t_0)) + D_\alpha(H(x_0, t_0)), H(x_0, t))) \\ &\leq 0 + M_\alpha\, [\psi(t) - \psi(t_0)] \\ &= \frac{\Phi_{\beta(\alpha)}(\delta_\alpha)}{2} < \Phi_{\beta(\alpha)}(\delta_\alpha) = \delta_\alpha - \phi_{\beta(\alpha)}(\delta_\alpha). \end{aligned}$$

Now Theorem 2.3 (applied to $H(\,.\,, t)$, note (d)) guarantees that $H(\,.\,, t)$ has a fixed point $x \in B(x_0, \delta)$. Thus $(x, t) \in Q$ and notice since

$$d_\alpha(x_0, x) \leq \delta_\alpha = \Phi^{-1}_{\beta(\alpha)}(2\, M_\alpha\, [\psi(t) - \psi(t_0)] \quad \text{and} \quad t_0 < t,$$

that we have $(t_0, x_0) < (t, x)$. This of course contradicts the maximality of (t_0, x_0).■

Remark 2.2. Of course the results in this section hold if E a complete gauge space is replaced by E a sequentially complete gauge space.

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