The Existence of Chaos for Ordinary Differential Equations with a Center Manifold*

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Abstract

Ordinary differential equations are considered consisting of two equations with nonlinear coupling where the linear parts of the two equations have equilibria which are, respectively, a saddle and a center. Perturbation terms are added which correspond to damping and forcing. A reduced equation is obtained from the hyperbolic equation by setting to zero the variable from the center equation. Melnikov theory is used to obtain a transverse homoclinic solution, and hence chaos, in the reduced equation. Conditions are then established such that the chaos for the reduced equation is shadowed by chaos for the full equation. The resonant case is also studied when the chaos of the full system is not detected from the reduced equation. The techniques make use of exponential dichotomies.

1 Introduction

To illustrate the ideas of this work consider the equations

$$\ddot{x} = x - 2x(x^2 + y^2) - 2\mu_2\dot{x} + \mu_1 \cos\omega t, \tag{1a}$$

$$\ddot{y} = (1-k)y - 2y(x^2 + y^\theta) - 2\mu_2\dot{y} + \mu_1 \cos p\omega t \tag{1b}$$

*This work supported by National Science Foundation grant INT9603054 and grant GA-SAV 2/1140/21

Received by the editors August 2002.

Communicated by J. Mawhin.

2000 *Mathematics Subject Classification :* 34C37, 37C29, 37D45.

Key words and phrases : ordinary differential equations, homoclinic solutions, bifurcations, center manifold.

where $\theta \in \mathbb{N} \setminus \{1\}$, $p \in \mathbb{N}$. This system consists of a (radially symmetric for $\theta = 2$) Duffing oscillator with an additional spring of stiffness k in the y equation along with damping and external forces added as perturbation terms.

Let us assume $k > 1$ in (1b). Then, for the unperturbed equation i.e., when $\mu_1 = \mu_2 = 0$, the linear part of (1a) has a hyperbolic equilibrium and the linear part of (1b) has a center. Furthermore, for small μ_2, the eigenvalues of $\ddot{y} = (1-k)y - 2\mu_2\dot{y}$ are complex functions, $\lambda(\mu_2)$, with $\Re(\lambda(\mu_2)) = -\mu_2$ so that we have $\Re(\lambda(0)) = 0$ and $\Re(\lambda'(0)) = -1$. Thus, for small $\mu_2 \neq 0$, the equilibrium of (1b) is weakly hyperbolic.

If we set $y = 0$ in (1a) we get the standard forced, damped Duffing equation. Using Melnikov theory one can show that for small $\mu_1 \neq 0$ and for $\mu_2 \neq 0$, within an appropriate range, this equation has a transverse homoclinic orbit and hence exhibits chaos. (These ideas are explained in detail below.) The first purpose of the present work is to show that if $\mu_1 \neq 0$, $\mu_2 \neq 0$ are chosen to produce chaos in (1a) when $y = 0$ and if $p\omega \neq \sqrt{k-1}$ then, as a consequence of the weak hyperbolicity in the y equation, there exists chaos in the full equation (1) which, in some sense, shadows the chaos obtained in (1a) with $y = 0$. Condition $p\omega \neq \sqrt{k-1}$ means non-resonance in (1b). We also study (1) when $p\omega = \sqrt{k-1}$ for $\theta \geq 3$, but we do not start from the reduced equation.

As an abstract version of (1) we consider differential equations of the form

$$\dot{x} = f(x, y, \mu, t) = f_0(x, y) + \mu_1 f_1(x, y, \mu, t) + \mu_2 f_2(x, y, \mu, t), \tag{2a}$$
$$\dot{y} = g(x, y, \mu, t) = g_0(x, y) + \mu_1 g_1(x, y, \mu, t) + \mu_2 g_2(x, y, \mu) \tag{2b}$$

with $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $\mu = (\mu_1, \mu_2) \in \mathbb{R}^2$.

We make the following assumptions about (2):

(i) Each f_i, g_i is $\mathcal{C}^4$ in all arguments.

(ii) f_1, f_2 and g_1 are periodic in t with period T.

(iii) $D_2 f_0(x, 0) = 0$.

(iv) The eigenvalues of $D_1 f_0(0, 0)$ lie off the imaginary axis.

(v) The equation $\dot{x} = f_0(x, 0)$ has a homoclinic solution γ.

(vi) $g_0(x, 0) = g_2(x, 0, \mu) = 0$, $D_{21} g_0(0, 0) = 0$ and $D_{22} g_0(0, 0) = 0$.

(vii) The eigenvalues of $D_2 g_0(0, 0)$ lie on the imaginary axis.

(viii) If $\mu_2 \to \lambda(\mu_2)$ is a function such that $\lambda(\mu_2)$ is an eigenvalue of the matrix $D_2 g_0(0, 0) + \mu_2 D_2 g_2(0, 0, 0)$ then $\Re(\lambda'(0)) < 0$.

(ix) $D_2 g_1(0, 0, 0, t) = 0$.

(x) $D_{222} g_0(0, 0) = 0$.

Hypothesis (viii) is based on the examples for which the μ_2 perturbation represents damping which cases all the eigenvalues of (2b) to move to the left of the imaginary axis. In fact, it is sufficient to assume that $\Re(\lambda'(0)) \neq 0$. In other words, (2b) is weakly hyperbolic. This more general assumption requires a little more work since it is necessary to include a nontrivial projection in Lemma 3 below.

In our analysis we shall encounter the phenomenon of resonance. Hypotheses (i)-(ix) are sufficient for the nonresonant case but (ix) is required to be replaced by (x) in order to deal with resonance.

2 Chaotic Dynamics on the Hyperbolic Subspace

In this section we consider the equation

$$\dot{x} = f(x, 0, \mu, t) = f_0(x, 0) + \mu_1 f_1(x, 0, \mu, t) + \mu_2 f_2(x, 0, \mu, t) \tag{3}$$

obtained by setting $y = 0$ in (2a). Equation (3) will be called the reduced equation obtained from (2). We apply to this equation some Melnikov theory from [3]-[6] which we summarize here.

By hypothesis, the equation $\dot{x} = f_0(x, 0)$ has a hyperbolic equilibrium and a homoclinic solution γ. Then (3) has a unique small hyperbolic T-periodic solution $p_\mu(t)$ for $|\mu|$ small [1]. Let $\{u_1, \ldots, u_d\}$ denote a basis for the vector space of bounded solutions to the variational equation $\dot{u} = D_1 f_0(\gamma, 0)u$ with $u_d = \dot{\gamma}$ and let $\{v_1, \ldots, v_d\}$ denote a basis for the vector space of bounded solutions to the adjoint equation $\dot{v} = -D_1 f_0(\gamma, 0)^t v$.

Now define the functions $a_{ij} : \mathbb{R} \to \mathbb{R}$, constants b_{ijk} and function

$$M : \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^{d-1} \to \mathbb{R}^d$$

by

$$\begin{aligned} a_{ij}(\alpha) &= \int_{-\infty}^{\infty} \langle v_i(t), f_j(\gamma(t), 0, 0, t + \alpha)\rangle \, dt, \quad \begin{cases} 1 \le i \le d \\ 1 \le j \le 2; \end{cases} \\ b_{ijk} &= \int_{-\infty}^{\infty} \langle v_i, D_{11} f_0(\gamma, 0) u_j u_k\rangle \, dt, \quad \begin{cases} 1 \le i \le d \\ 1 \le j, k \le d - 1; \end{cases} \\ M_i(\mu, \alpha, \beta) &= \sum_{j=1}^{2} a_{ij}(\alpha)\mu_j + \frac{1}{2}\sum_{j,k=1}^{d-1} b_{ijk}\beta_j\beta_k, \quad 1 \le i \le d. \end{aligned} \tag{4}$$

The function M is our bifurcation function and is used in Theorem 2 below. The integer d has a geometric interpretation. Let $P = \gamma(0)$ and let W^s, W^u denote the stable, unstable manifolds respectively of the origin for the unperturbed equation from (3). Then the entire orbit of γ lies in $W^s \cap W^u$ so that $P \in W^s \cap W^u$ and $\dot{\gamma}(0) \in T_P W^s \cap T_P W^u$. The vectors $\{u_1(0), \ldots, u_d(0)\}$ are a basis for $T_P W^s \cap T_P W^u$ and $d = \dim(T_P W^s \cap T_P W^u)$.

Suppose that $W^s \cap W^u$ has a connected component which is a manifold of dimension d and which contains the orbit of γ. Then in (4), all $b_{ijk} = 0$, the hypotheses of Theorem 2 below cannot be satisfied and an alternate bifurcation function is required.

Let W^h denote a homoclinic d-manifold containing γ, let U_0 be an open neighborhood of the origin in $\mathbb{R}^{d-1}$, let $\eta : U_0 \to W^h$ be a differentiable function denoted $\beta \to \eta(\beta)$ with $\eta(0) = P$, let $t \to \gamma_\beta(t)$ be the solution to the unperturbed equation (3) satisfying $\gamma_\beta(0) = \eta(\beta)$, and assume η is constructed so that $(\beta, t) \to \gamma_\beta(t)$

establishes local coordinates on W^h. In other words, the original orbit γ is embedded in a $(d-1)$-parameter family of homoclinic orbits.

For each fixed β we let $\{v_{\beta 1}, \dots, v_{\beta d}\}$ denote a basis for the vector space of bounded solutions to the adjoint equation $\dot{v} = -D_1 f_0(\gamma_\beta, 0)^t v$. Without loss of generality we can assume that each $v_{\beta i}$ depends differentially on β. Now define functions $a_{ij} : \mathbb{R} \times U_0 \to \mathbb{R}$ and $M : \mathbb{R}^2 \times \mathbb{R} \times U_0 \to \mathbb{R}^d$ by

$$\begin{aligned} a_{ij}(\alpha, \beta) &= \int_{-\infty}^{\infty} \langle v_{\beta i}(t), f_j(\gamma_\beta(t), 0, 0, t+\alpha)\rangle \, dt, \quad \begin{cases} 1 \le i \le d \\ 1 \le j \le 2; \end{cases} \\ M_i(\mu, \alpha, \beta) &= \sum_{j=1}^{2} a_{ij}(\alpha, \beta)\mu_j, \quad 1 \le i \le d. \end{aligned} \tag{5}$$

This is our bifurcation function for the homoclinic manifold case.

In this paper, the concept of exponential dichotomy is important so we state the definition for easy reference. For details see Coppel [2].

1. Definition. *We say that (U, P) is an exponential dichotomy with constants (K, a) on the interval $[t_1, t_2]$ for the linear differential equation $\dot{x} = A(t)x$ if U is the fundamental solution for the differential equation with $U(0) = I$; P is a projection; and K, a are two positive constants such that the following hold:*

i) $|U(t)PU(s)^{-1}| \le Ke^{a(s-t)}$ *for* $t_1 \le s \le t \le t_2$,

ii) $|U(t)(I-P)U(s)^{-1}| \le Ke^{a(t-s)}$ *for* $t_1 \le t \le s \le t_2$.

In this definition we allow for the possibilities $t_1 = -\infty$ and/or $t_2 = +\infty$ in which case the interval is open at the corresponding end(s). If both of these hold we say the differential equation has an exponential dichotomy on the whole line.

By combining results from [3]-[6] we now get the following result.

2. Theorem. *Let M be as in (4) or (5) and suppose $(\mu_0, \alpha_0, \beta_0)$ are such that $M(\mu_0, \alpha_0, \beta_0) = 0$ and $D_{(\alpha,\beta)}M(\mu_0, \alpha_0, \beta_0)$ is nonsingular. Then there exists $\xi_0 > 0$ such that if $0 < \xi < \xi_0$ the equation $\dot{x} = f(x, 0, \xi\mu_0, t)$ has a homoclinic solution γ_ξ to $p_{\xi\mu_0}$.*

Furthermore, $\gamma_\xi(t) \to p_{\xi\mu_0}$ at an exponential rate as $t \to \pm\infty$, γ_ξ depends continuously on ξ, $\lim_{\xi\to 0} \gamma_\xi(t) = \gamma(t)$ uniformly in t and the variational equation along γ_ξ has an exponential dichotomy for the whole line when $\xi \neq 0$.

We can use the preceding result to obtain chaos for (3). We make this notion precise. Let Σ_N denote the space of doubly infinite sequences of N symbols with the usual topology and let $\varphi : \Sigma_N \to \Sigma_N$ be the Bernoulli shift map. The topological space Σ_N is compact, perfect and totally disconnected (a Cantor set) and φ is continuous with periodic orbits of every period and a dense orbit. These ideas were popularized by the work of Smale [9] and can be found in Wiggins [10]. Following Palmer [8], Theorem 2 establishes a topological conjugacy between φ and some multiple of the period map of the flow for the differential equation $\dot{x} = f(x, 0, \xi\mu_0, t)$.

We remark that the constant K_ξ of the exponential dichotomy for the variational equation $\dot{u} = D_1 f(\gamma_\xi, 0, \xi\mu_0, t)u$ along $\gamma_\xi(t)$ tends to infinity as $\xi \to 0$. Indeed, let a_ξ, P_ξ, U_ξ be the corresponding constant, projection and fundamental solution from

Definition 1, respectively. The roughness theorem for exponential dichotomies [2] implies that we can take $a_\xi = a_0 > 0$ for some constant a_0. If $\sup_{\xi>0} K_\xi < \infty$, then there is a sequence $\{\xi_i\}_{i=1}^{\infty}$ such that $\xi_i \to 0$, $K_{\xi_i} \to K_0$, $P_{\xi_i} \to P_0$ and $U_{\xi_i}(t) \to U_0(t)$ point-wise. Clearly, P_0 is a projection and $U_0(t)$ is the fundamental solution of $\dot{u} = D_1 f_0(\gamma, 0)u$ creating an exponential dichotomy for this equation on the whole line $\mathbb{R}$ with constants (K_0, a_0). This contradicts the existence of a bounded solution $\dot{\gamma}$ for this equation. Consequently, $K_\xi \to \infty$ as $\xi \to 0$.

We finish this section with the next result.

3. Lemma. *There exist constants $b > 0$, $B > 0$ independent of ξ such that given $\mu_{0,2} > 0$ the variational equation*

$$\dot{v} = [D_2 g_0(\gamma(t), 0) + \xi\mu_{0,2} D_2 g_2(\gamma(t), 0, 0)]\, v$$

has an exponential dichotomy (V_ξ, I) on $\mathbb{R}$ with constants $(B, b\xi\mu_{0,2})$.

Proof. Write the given equation in the form $\dot{v} = Rv + S(t)v$ where

$$\begin{aligned} R &= D_2 g_0(0,0) + \xi\mu_{0,2} D_2 g_2(0,0,0), \\ S(t) &= D_2 g_0(\gamma(t),0) - D_2 g_0(0,0) + \xi\mu_{0,2}\left[D_2 g_2(\gamma(t),0,0) - D_2 g_2(0,0,0)\right]. \end{aligned}$$

Let V_ξ be the fundamental solution for $\dot{v} = Rv + S(t)v$ with $V_\xi(0) = I$. Then for $s \le t$ we have

$$V_\xi(t) = e^{(t-s)R} V_\xi(s) + \int_s^t e^{(t-\tau)R} S(\tau) V_\xi(\tau)\, d\tau.$$

Using (vii) and (viii) for (2) we can, for ξ_0 sufficiently small, find $K_1, b > 0$ so that $|e^{(t-s)R}| \le K_1 e^{b\xi\mu_{0,2}(s-t)}$ when $0 < \xi \le \xi_0$ and $s \le t$. Now define

$$x(t) = |V_\xi(t) V_\xi(s)^{-1}| e^{b\xi\mu_{0,2}(t-s)}.$$

Then from the preceding equation for V_ξ we get

$$x(t) \le K_1 + \int_s^t K_1 |S(\tau)| x(\tau)\, d\tau.$$

Hence, from Gronwall's inequality,

$$x(t) \le K_1 e^{K_1 \int_s^t |S(\tau)|\, d\tau} \le B$$

for a constant $B > 0$. ∎

3 Chaos in the Full Equation

We construct the bifurcation function M from (4) or (5), as in the preceding section, from the reduced equation (3). If M satisfies the hypotheses for Theorem 2 we have a transverse homoclinic solution and hence chaos for (3) when $\mu = \xi\mu_0$, $0 < \xi < \xi_0$. We now establish a condition for chaos to exist in the full equation (2). Since the exponential constant K_ξ of $\dot{u} = D_1 f(\gamma_\xi, 0, \xi\mu_0, t)u$ tends to infinity as $\xi \to 0$, as we showed in previous section, we have to deal with the full system (2). For this we consider the modification of (2) in the form

$$\begin{aligned} \dot{x} &= f(x, \lambda y, \mu, t)\,, \\ \dot{y} &= g_0(x,y) + \lambda\mu_1 g_1(x,y,\mu,t) + \mu_2 g_2(x,y,\mu) \\ &\qquad 0 \le \lambda \le 1\,. \end{aligned} \tag{6}$$

To solve (6), we follow [3], [4] and substitute

$$x = \gamma + \sum_{i=1}^{d-1} \xi\beta_i u_i + \xi^2 u, \quad y = \xi^2 v, \quad \mu = \xi^2\mu_0, \quad |\mu_0| = 1\,,$$

where $\{u_1, \dots, u_d\}$ is a basis for the vector space of bounded solutions for $\dot{u} = D_1 f_0(\gamma(t), 0)u$ with $u_d = \dot{\gamma}$ and μ_0 is to be determined. Introducing this change of variables into (6) yields

$$\begin{aligned} \dot{u} = D_1 f_0(\gamma, 0)u &+ \frac{1}{2}\sum_{i,j=1}^{d-1} D_{11} f_0(\gamma, 0)\beta_i\beta_j u_i u_j \\ &+ \mu_{0,1} f_1(\gamma, 0, 0, t+\alpha) + \mu_{0,2} f_2(\gamma, 0, 0, t+\alpha) + O(\xi) \end{aligned} \tag{7a}$$

and

$$\begin{aligned} \dot{v} &= [D_2 g_0(\gamma, 0) + \xi^2\mu_{0,2} D_2 g_2(\gamma, 0, 0)]v \\ &+ \Bigg[D_2 g_0\left(\gamma + \xi\sum_{i=1}^{d-1}\beta_i u_i + \xi^2 u, 0\right) - D_2 g_0(\gamma, 0) \\ &+ D_{22} g_0\left(\gamma + \xi\sum_{i=1}^{d-1}\beta_i u_i + \xi^2 u, 0\right)\xi^2 v + O(\xi^4 v^2)\Bigg]v + \lambda\mu_{0,1} g_1(0,0,0,t+\alpha) \\ &+ \lambda\mu_{0,1}\Bigg\{g_1\left(\gamma + \xi\sum_{i=1}^{d-1}\beta_i u_i + \xi^2 u, \xi^2 v, \xi^2\mu_0, t+\alpha\right) - g_1(0,0,0,t+\alpha)\Bigg\} \\ &+ \xi^2\mu_{0,2}\Bigg\{D_2 g_2\left(\gamma + \xi\sum_{i=1}^{d-1}\beta_i u_i + \xi^2 u, 0, \xi^2\mu_0\right) - D_2 g_2(\gamma, 0, 0) + O(\xi^2 v)\Bigg\}v\,. \end{aligned} \tag{7b}$$

We consider the Banach spaces

$$\begin{aligned} X_n &= \left\{x \in C(\mathbb{R}, \mathbb{R}^n) \;\middle|\; \sup_{t\in\mathbb{R}} |x| < \infty\right\}, \\ Y_n &= \Bigg\{y \in X_n \;\Bigg|\; \int_{-\infty}^{\infty} \langle y(t), v(t)\rangle\, dt \\ &\qquad\qquad \text{for every bounded solution } v \text{ to } \dot{v} = -Df_0(\gamma, 0)^t v\Bigg\} \end{aligned}$$

with the supremum norm $||x|| = \sup_{t\in\mathbb{R}} |x(t)|$. To solve (7a), we need the following results from [5].

4. Lemma. *Given $h \in Y_n$, the equation $\dot{u} = D_1 f_0(\gamma(t), 0)u + h$ has a unique solution $u \in X_n$ satisfying $\langle u(0), v(0))\rangle = 0$ for every bounded solution, v, to the adjoint equation $\dot{v} = -D_1 f_0(\gamma, 0)^t v$.*

5. Lemma. *There exists a projection* $\Pi : X_n \to X_n$ *such that* $\operatorname{Im}(I - \Pi) = Y_n$.

We define the linear map $\mathcal{K} : Y_n \to X_n$ by $\mathcal{K}h = u$ where h, u are as in Lemma 4. Using the projection Π and the exponential dichotomy V_ξ from Lemma 3, where we suppose $\mu_{0,2} > 0$, we can rewrite (7) as the fixed point problem

$$\begin{aligned} u = \mathcal{K}(I - \Pi)\Big(\frac{1}{2}\sum_{i,j=1}^{d-1} D_{11}f_0(\gamma, 0)\beta_i\beta_j u_i u_j \\ + \mu_{0,1}f_1(\gamma, 0, 0, , t+\alpha) + \mu_{0,2}f_2(\gamma, 0, 0, t+\alpha) + O(\xi)\Big), \end{aligned} \tag{8a}$$

$$\begin{aligned} v(t) = \int_{-\infty}^{t} V_\xi(t)V_\xi(s)^{-1}\Big\{\Big[D_2 g_0\left(\gamma(s) + \xi\sum_{i=1}^{d-1}\beta_i u_i(s) + \xi^2 u(s), 0\right) \\ + D_{22}g_0\left(\gamma(s) + \xi\sum_{i=1}^{d-1}\beta_i u_i(s) + \xi^2 u(s), 0\right)\xi^2 v(s) \\ - D_2 g_0(\gamma(s), 0) + O(\xi^4 v(s)^2)\Big]v(s) \\ + \lambda\mu_{0,1}g_1(0, 0, 0, s+\alpha) \\ + \lambda\mu_{0,1}\Big\{g_1\left(\gamma(s) + \xi\sum_{i=1}^{d-1}\beta_i u_i(s) + \xi^2 u(s), \xi^2 v(s), \xi^2\mu_0, s+\alpha\right) \\ - g_1(0, 0, 0, s+\alpha)\Big\} \\ + \xi^2\mu_{0,2}\Big\{D_2 g_2\left(\gamma(s) + \xi\sum_{i=1}^{d-1}\beta_i u_i(s) + \xi^2 u(s), 0, \xi^2\mu_0\right) \\ - D_2 g_2(\gamma(s), 0, 0) + O(\xi^2 v)\Big\}v(s)\Big\}ds \end{aligned} \tag{8b}$$

along with the system of bifurcation equations

$$\begin{aligned} \int_{-\infty}^{\infty}\Big\langle v_i(t), \frac{1}{2}\sum_{i,j=1}^{d-1} D_{11}f_0(\gamma(t), 0)\beta_i\beta_j u_i(t)u_j(t) + \mu_{0,1}f_1(\gamma(t), 0, 0, t+\alpha) \\ + \mu_{0,2}f_2(\gamma(t), 0, 0, t+\alpha) + O(\xi)\Big\rangle dt = 0, \quad i = 1, 2, \cdots, d \end{aligned} \tag{9}$$

where $\{v_1, \ldots, v_d\}$ is a basis for the space of bounded solutions to the adjoint equation.

Using (ix) we have

$$
\begin{aligned}
D_2g_0\left(\gamma+\xi\sum_{i=1}^{d-1}\beta_iu_i+\xi^2u,0\right)&-D_2g_0(\gamma,0)+D_{22}g_0\left(\gamma+\xi\sum_{i=1}^{d-1}\beta_iu_i+\xi^2u,0\right)\xi^2v\\
&=O(\xi^2|\gamma||v|)+O(\xi^4|u||v|)+O(\xi^2|\gamma||u|)+O\left(\xi\sum_{i=1}^{d-1}\beta_i|u_i|\right),\\
g_1\left(\gamma+\xi\sum_{i=1}^{d-1}\beta_iu_i+\xi^2u,\xi^2v,\xi^2\mu_0,t+\alpha\right)&-g_1(0,0,0,t+\alpha)\\
&=O(\xi^2|\gamma||v|)+O(\xi^4|u||v|)+O(\xi^2)+O(\xi^4|v|^2)\\
&\quad+O(\xi^2|u|)+O(|\gamma|)+O\left(\xi\sum_{i=1}^{d-1}\beta_i|u_i|\right),\\
D_2g_2\left(\gamma+\xi\sum_{i=1}^{d-1}\beta_iu_i+\xi^2u,0,\xi^2\mu_0\right)&-D_2g_2(\gamma,0,0)\\
&=O(\xi^2)+O(\xi^2|u|)+O\left(\xi\sum_{i=1}^{d-1}\beta_i|u_i|\right).
\end{aligned}
$$

We note that $|\gamma(t)|\le ce^{-a|t|}$ and $|u_i(t)|\le ce^{-a|t|}$, $i=1,2,\cdots,d$ for constants $c>0$, $a>0$. Moreover, it holds that

$$
\begin{aligned}
\int_{-\infty}^{t}e^{-b\xi^2\mu_{0,2}(t-s)}\,ds&=\frac{1}{b\xi^2\mu_{0,2}},\\
\int_{-\infty}^{t}e^{-b\xi^2\mu_{0,2}(t-s)-a|s|}\,ds&\le\int_{-\infty}^{\infty}e^{-a|s|}ds=2/a\,.
\end{aligned}
$$

Consequently, if we assume that

$$
\begin{aligned}
\sup_{0\le\alpha\le T}\sup_{\xi>0}\int_{-\infty}^{t}\left|V_\xi(t)V_\xi(s)^{-1}g_1(0,0,0,s+\alpha)\,ds\right|&<\infty\,,\\
\sup_{0\le\alpha\le T}\sup_{\xi>0}\int_{-\infty}^{t}\left|V_\xi(t)V_\xi(s)^{-1}D_4g_1(0,0,0,s+\alpha)ds\right|&<\infty
\end{aligned}
\tag{H}
$$

then we can apply the Banach fixed point theorem on a ball centered at 0 in the space $X_n\times X_m$ to solve (8a-b) for $\xi>0$ sufficiently small. Substituting this solution into (9) yields a system of bifurcation equations of the form

$$
M(\mu,\alpha,\beta)+O(\xi)=0 \tag{10}
$$

where M is as in (4) or (5). The case for (5) can be handled like above.

The assumptions of Theorem 2 imply the solvability of (10). This gives a transverse homoclinic orbit $\Gamma(\lambda,\xi^2\mu_0)(t)=\big(\Gamma_1(\lambda,\xi^2\mu_0)(t),\Gamma_2(\lambda,\xi^2\mu_0)(t)\big)$ of (6) near γ such that $\Gamma_1(\lambda,\xi^2\mu_0)(t)=\gamma(t)+O(\xi)$. The transversality follows exactly as in [3], [4], so we omit its proof. Moreover, we have $\Gamma(0,\xi^2\mu_0)=(\gamma_\xi,0)$ for γ_ξ from Theorem 2, and $\Gamma(1,\xi^2\mu_0)$ is a homoclinic solution for (2). The dichotomy constants of the linearized system of (6) along $\Gamma(\lambda, \xi^2\mu_0)(t)$ are uniform for $0 \le \lambda \le 1$ and fixed ξ. This follows from the roughness result of exponential dichotomies from [2]. Now we can follow directly Palmer's construction [8] of a Smale horseshoe along $\Gamma(\lambda, \xi^2\mu_0)(t)$ for fixed small ξ. Thus we have a continuous family Σ_λ of Smale horseshoes for (6). This gives us the lifting of the Smale horseshoe of the reduced system to the full one.

The conditions (H) are, in fact, ones of nonresonance. To see this consider the equations

$$\begin{aligned}\dot v &= [D_2 g_0(\gamma, 0) + \xi^2\mu_{0,2} D_2 g_2(\gamma, 0, 0)]v + h,\\ \dot w &= [D_2 g_0(0, 0) + \xi^2\mu_{0,2} D_2 g_2(0, 0, 0)]w + h\end{aligned}$$

where $v, w, h \in X_m$. Then we get

$$\begin{aligned}\frac{d}{dt}(v - w) &= [D_2 g_0(0, 0) + \xi^2\mu_{0,2} D_2 g_2(0, 0, 0)](v - w)\\ &\quad + \Big[D_2 g_0(\gamma, 0) - D_2 g_0(0, 0) + \xi^2\mu_{0,2}(D_2 g_2(\gamma, 0, 0) - D_2 g_2(\gamma, 0, 0))\Big]v\,.\end{aligned}$$

This gives

$$|v(t) - w(t)| \le ||v||K_1 \int_{-\infty}^{t} e^{-b\xi^2\mu_{0,2}(t-s) - a|s|}\, ds \le 2||v||K_1/a$$

for constants $K_1 > 0$, $a > 0$. Hence there is a constant $K_2 > 0$ such that

$$||w - v|| \le K_2||v||, \quad ||w - v|| \le K_2||w||\,.$$

These inequalities imply that assumption (H) is equivalent to the condition that when $\xi > 0$ the only bounded solution, $v_{\alpha,\xi}$, of

$$\dot v = [D_2 g_0(0, 0) + \xi^2\mu_{0,2} D_2 g_2(0, 0, 0)]v + g_1(0, 0, 0, t + \alpha) \tag{11}$$

satisfies $\sup_{0\le\alpha\le T}\sup_{\xi>0}||v_{\alpha,\xi}|| < \infty$. Then also $\sup_{0\le\alpha\le T}\sup_{\xi>0}||\dot v_{\alpha,\xi}|| < \infty$. Hence by the Arzela-Ascoli theorem, there is a sequence $\{\xi_i\}_{i=1}^\infty$, $\xi_i > 0$, $\xi_i \to 0$ such that $v_{\alpha,\xi_i} \to v_0$ and $\dot v_{\alpha,\xi_i} \to \dot v_0$ uniformly on compact intervals. Consequently, we get

$$\dot v_0 = D_2 g_0(0, 0)v_0 + g_1(0, 0, 0, t + \alpha)\,. \tag{12}$$

We note that $v_{\alpha,\xi}$, v_0 are T-periodic. We know [1] that (12) has a T-periodic solution if and only if

$$\int_0^T \langle w_i(t), g_1(0, 0, 0, t)\rangle\, dt = 0, \quad i = 1, 2, \cdots, d_1, \tag{13}$$

where $\{w_1, \cdots, w_{d_1}\}$ is a basis of T-periodic solutions of the adjoint variational equation $\dot w = -D_2 g_0(0, 0)^t w$. Hence assumption (H) implies the validity of (13).

Conversely, let (13) hold. Then (12) has a T-periodic solution and we put $v = v_0 + w$ in (11) to get

$$\dot w = [D_2 g_0(0, 0) + \xi^2\mu_{0,2} D_2 g_2(0, 0, 0)]w + \xi^2\mu_{0,2} D_2 g_2(0, 0, 0)v_0\,. \tag{14}$$

The above arguments and Lemma 3 give that the unique solution $w_{\alpha,\xi} \in X_m$ of (14) satisfies $\sup_{0\le\alpha\le T} \sup_{\xi>0} ||w_{\alpha,\xi}|| < \infty$. Summarizing, we see that assumption (H) is equivalent to condition (13).

To illustrate these ideas consider the example

$$\begin{aligned}\ddot{x} &= x - 2x^3 + y^2 + \mu_1 \cos t + \mu_2 \dot{x},\\ \ddot{y} &= -y + \mu_1 \cos t - \mu_2 \dot{y}.\end{aligned}$$

For this example, (12) becomes $\ddot{v} + v = \cos t$ which lacks a periodic solution due to resonance. Thus, (H) does not hold.

We might try to proceed anyway. The second equation for $\mu_2 \neq 0$ has the unique bounded solution $(\mu_1/\mu_2)\sin t$. If we substitute this solution into the first equation, then for $\mu_i = \xi^2 \mu_{i,0}$ with $\xi \to 0$, we do not get the original unperturbed problem. In the case of resonance (where (H) or equivalently (13) fails to hold) and where $\mu_{0,1} = 0$ for the reduced equation we are able to utilize an alternate scaling. This is illustrated in Theorem 7 below.

Now we can state our results in the form of the next two theorems.

6. Theorem. *Let (i)-(ix) hold. Let M be as in (4) or (5) and suppose $(\mu_0, \alpha_0, \beta_0)$ are such that $M(\mu_0, \alpha_0, \beta_0) = 0$ and $D_{(\alpha,\beta)} M(\mu_0, \alpha_0, \beta_0)$ is nonsingular. If condition (13) holds then there exist $\bar{\xi}_0 > 0$, $K > 0$ such that if $0 < \xi \le \bar{\xi}_0$ and if the parameters in (2) are given by $\mu = \xi\mu_0$, then there exists a continuous map $\phi : \Sigma_N \times [0,1] \to \mathbb{R}^{n+m}$ and $m_0 \in \mathbb{N}$ such that:*

(i) *$\phi_\lambda = \phi(\cdot, \lambda) : \Sigma_N \to \mathbb{R}^{n+m}$ is a homeomorphism of Σ_N onto a compact subset of $\mathbb{R}^{n+m}$ on which the m_0th iterate, $F_\lambda^{m_0}$, of the period map F_λ of (6) is invariant and satisfies $F_\lambda^{2m_0} \circ \phi_\lambda = \phi_\lambda \circ \varphi$ where φ is the Bernoulli shift on Σ_N.*

(ii) *$\phi_0 = \phi(\cdot, 0) : \Sigma_N \to \mathbb{R}^n \times \{0\}$ and $F_0 = (G_0, 0)$ for the period map G_0 of the reduced equation (3).*

(iii) *F_1 is the period map of the full system (2).*

(iv) *$|\phi(x,\lambda) - \phi(x,0)| \le K\sqrt{\xi}$ for any $(x,\lambda) \in \Sigma_N \times [0,1]$.*

In the case where (H) does not hold we can get a result analogous to the preceding if we have $\mu_{0,1} = 0$ for the reduced equation. In this case we use the scaling $\mu_1 = \xi^4 \mu_{0,1}$ and $\mu_2 = \xi^2 \mu_{0,2}$ and proceed as before. This yields the following result.

7. Theorem. *Let (i)-(ix) hold. Let M be as in (4) or (5) and suppose $(\mu_0, \alpha_0, \beta_0)$ with $\mu_0 = (0, \mu_{0,2})$ are such that $M(\mu_0, \alpha_0, \beta_0) = 0$ and $D_{(\alpha,\beta)} M(\mu_0, \alpha_0, \beta_0)$ is nonsingular. Then there exist $\bar{\xi}_0 > 0$, $K > 0$ such that if $0 < \xi \le \bar{\xi}_0$ and if the parameters in (2) are given by $\mu_1 = \xi^2 \mu_{0,1}$, $\mu_{0,1} \in \mathbb{R}$ and $\mu_2 = \xi \mu_{0,2}$, then the statement of Theorem 6 holds.*

Theorems 6 and 7 roughly state that the Smale horseshoe of the reduced equation (3) can be shadowed and continued to the full system (2).

Referring back to our previous example (1) we see that resonance holds when $p\omega = \sqrt{k-1}$ so Theorem 6 does not apply in that case. Further, $\mu_{0,1} \neq 0$ (see Example 1 below) and so Theorem 7 also does not apply in this case.

We now develop a result for (2) when (H) fails where we now use assumption (x) for (2) instead of (ix). We modify the preceding approach by putting

$$x = \gamma + \sum_{i=1}^{d-1} \xi\beta_i u_i + \xi^2 u, \quad y = \xi v, \quad \mu_1 = \xi^3\mu_{0,1}, \quad \mu_2 = \xi^2\mu_{0,2}\,,$$

in (2) to get

$$\begin{aligned} \dot u &= D_1 f_0(\gamma,0)u + \frac{1}{2}\sum_{i,j=1}^{d-1} D_{11}f_0(\gamma,0)\beta_i\beta_j u_i u_j \\ &\quad + \mu_{0,2}f_2(\gamma,0,0,t+\alpha) + \frac{1}{2}D_{22}f_0(\gamma,0)vv + O(\xi) \end{aligned} \tag{15a}$$

and

$$\begin{aligned} \dot v &= [D_2 g_0(\gamma,0) + \xi^2\mu_{0,2}D_2 g_2(\gamma,0,0)]v \\ &\quad + \xi^2\mu_{0,1}g_1(0,0,0,t+\alpha) + O(\xi^2|\gamma||u||v|) \\ &\quad + O\left(\xi\sum_{i=1}^{d-1}\beta_i|u_i|\right) + O(\xi^2|\gamma|) + O(\xi^3) + O(\xi\gamma|v|^3)\,. \end{aligned} \tag{15b}$$

Analogous to our preceding work, we rewrite (15) as the fixed point problem

$$\begin{aligned} u &= \mathcal{K}(I-\Pi)\Big\{D_1 f_0(\gamma,0)u + \frac{1}{2}\sum_{i,j=1}^{d-1} D_{11}f_0(\gamma,0)\beta_i\beta_j u_i u_j \qquad (16a) \\ &\quad + \mu_{0,2}f_2(\gamma,0,0,t+\alpha) + \frac{1}{2}D_{22}f_0(\gamma,0)vv + O(\xi)\Big\}\,, \\ v(t) &= \int_{-\infty}^{t} V_\xi(t)V_\xi(s)^{-1}\Big\{\xi^2\mu_{0,1}g_1(0,0,0,s+\alpha) \qquad (16b) \\ &\quad + O\big(\xi^2|\gamma(s)||u(s)|v(s)|\big) + O\left(\xi\sum_{i=1}^{d-1}\beta_i|u_i(s)|\right) \\ &\quad + O\big(\xi|\gamma(s)||v(s)|^3\big) + O\big(\xi^2|\gamma(s)|\big) + O(\xi^3)\Big\}\,ds\,, \end{aligned}$$

and the system of bifurcation equations

$$\begin{aligned} &\int_{-\infty}^{\infty}\Big\langle v_i(t), \frac{1}{2}\sum_{i,j=1}^{d-1} D_{11}f_0(\gamma(t),0)\beta_i\beta_j u_i(t)u_j(t) + \mu_{0,2}f_2(\gamma(t),0,0,t+\alpha) \\ &\qquad + \frac{1}{2}D_{22}f_0(\gamma(t),0)v(t)v(t) + O(\xi)\Big\rangle\,dt = 0, \quad i = 1,2,\cdots,d\,. \end{aligned} \tag{17}$$

By using the Banach fixed point theorem on a ball in $X_n \times X_m$ centered at 0, (16) has a solution $(u,v) \in X_n \times X_m$ for any sufficiently small ξ such that

$$v(t) = \xi^2\mu_{0,1}\int_{-\infty}^{t} V_\xi(t)V_\xi(s)^{-1}g_1(0,0,0,s+\alpha)\,ds + O(\xi)\,.$$

We remark that the function $v_\xi(t) = \xi^2\mu_{0,1}\int_{-\infty}^{t} V_\xi(t)V_\xi(s)^{-1}g_1(0,0,0,s+\alpha)\,ds$ satisfies $\sup_{0\le\alpha\le T}\sup_{\xi>0}|v_\xi(t)| < \infty$ along with the equation

$$\dot v_\xi = [D_2 g_0(\gamma,0) + \xi^2\mu_{0,2}D_2 g_2(\gamma,0,0)]v_\xi + \xi^2\mu_{0,1}g_1(0,0,0,t+\alpha)\,. \tag{18}$$

To solve (18), we take $v_{\xi,1}, v_{\xi,2} \in X_m$ satisfying

$$\begin{aligned}\dot v_{\xi,1} &= [D_2 g_0(0,0) + \xi^2 \mu_{0,2} D_2 g_2(0,0,0)] v_{\xi,1} \\ &\quad + \xi^2 \mu_{0,1} g_1(0,0,0,t+\alpha)\,, \end{aligned} \tag{19}$$

$$\begin{aligned}\dot v_{\xi,2} &= [D_2 g_0(\gamma,0) + \xi^2 \mu_{0,2} D_2 g_2(\gamma,0,0)] v_{\xi,2} \\ &\quad + [D_2 g_0(\gamma,0) - D_2 g_0(0,0)] v_{\xi,1}\,. \end{aligned} \tag{20}$$

Then $v_\xi = v_{\xi,1} + v_{\xi,2} + O(\xi^2)$. We note that

$$v_{\xi,1} \in X_m^T = \Big\{ v \in X_m \mid v \quad \text{is } T\text{-periodic} \Big\}\,.$$

Let

$$\begin{aligned} Y_m^T &= \Big\{ v \in X_m^T \;\Big|\; \dot w = D_2 g_0(0,0) w + v \quad \text{for a } C^1\text{-smooth function} \quad w \in X_m^T \Big\}\,, \\ Z_m^T &= \Big\{ v \in X_m^T \cap C^1(\mathbb{R}) \;\Big|\; \dot v = D_2 g_0(0,0) v \Big\}\,. \end{aligned}$$

There is a projection $\Pi_T : X_m \to X_m$ such that $\operatorname{Im} \Pi_T = Y_m^T$. Moreover, we can split $X_m = Z_m^T \oplus U_m^T$. Applying the Lyapunov-Schmidt method to (19), we split $v_{\xi,1} = z_\xi + u_\xi$, $z_\xi \in Z_m^T$, $u_\xi \in U_m^T$ to get

$$\begin{aligned} \dot u_\xi &= D_2 g_0(0,0) u_\xi \\ &\quad + \xi^2 \Pi_T \Big(\mu_{0,2} D_2 g_2(0,0,0)(z_\xi + u_\xi) + \mu_{0,1} g_1(0,0,0,t+\alpha) \Big)\,, \end{aligned} \tag{21}$$

$$0 = (I - \Pi_T) \Big(\mu_{0,2} D_2 g_2(0,0,0)(z_\xi + u_\xi) + \mu_{0,1} g_1(0,0,0,t+\alpha) \Big)\,. \tag{22}$$

Equation (21) can be solved and gives $u_\xi = O(\xi^2)$ which we substitute into (22) to obtain

$$\mu_{0,2}(I - \Pi_T) D_2 g_2(0,0,0) z_\xi + \mu_{0,1}(I - \Pi_T) g_1(0,0,0,\cdot + \alpha) = O(\xi^2)\,. \tag{23}$$

If the linear map

$$(I - \Pi_T) D_2 g_2(0,0,0) : Z_m^T \to \operatorname{Im}(I - \Pi_T) \tag{24}$$

is invertible then we can solve (23) to get $v_{\xi,1} = \frac{\mu_{0,1}}{\mu_{0,2}} z_\alpha + O(\xi^2)$ for

$$z_\alpha(t) = -\Big((I - \Pi_T) D_2 g_2(0,0,0) \Big)^{-1} (I - \Pi_T) g_1(0,0,0,\cdot + \alpha)\,.$$

Equation (20) gives

$$v_{\xi,2}(t) = \int_{-\infty}^{t} V_\xi(t) V_\xi(s) \Big[D_2 g_0(\gamma(s),0) - D_2 g_0(0,0) \Big] v_{\xi,1}(s)\, ds\,.$$

Since $D_2 g_0(\gamma(s),0) - D_2 g_0(0,0) = O(|\gamma(s)|)$ and $V_\xi(s) \to V_0(s)$, $v_{\xi,1}(s) \to \frac{\mu_{0,1}}{\mu_{0,2}} z_\alpha(s)$ point-wise as $\xi \to 0$, the Lebesgue dominated convergence theorem gives

$$v_{\xi,2}(t) \to \frac{\mu_{0,1}}{\mu_{0,2}} w_\alpha(t)$$

point-wise as $\xi \to 0$ where

$$w_\alpha(t) = \int_{-\infty}^{t} V_0(t)V_0(s)\Big[D_2 g_0(\gamma(s),0) - D_2 g_0(0,0)\Big] z_\alpha(s)\, ds.$$

Summarizing, we obtain that the solution $(u,v) \in X_n \times X_m$ of (16) satisfies $v(t) \to \frac{\mu_{0,1}}{\mu_{0,2}}\Big(z_\alpha(t) + w_\alpha(t)\Big)$ point-wise. By using this for $\xi \to 0$, our bifurcation equation takes the form

$$M_i(\mu,\alpha,\beta) + o(1) = 0, \quad i = 1,2,\dots,d$$

where

$$\begin{aligned} M_i(\mu,\alpha,\beta) = {} & \frac{1}{2}\sum_{j,k=1}^{d-1} b_{ijk}\beta_j\beta_k + a_{i2}(\alpha)\mu_2 \\ & + \frac{\mu_1^2}{2\mu_2^2}\int_{-\infty}^{\infty} \Big\langle v_i(t), D_{22} f_0(\gamma(t),0)(z_\alpha(t) + w_\alpha(t))^2 \Big\rangle\, dt\,. \end{aligned} \tag{25}$$

Let $\{z_i\}_{i=1}^{\infty}$ be a basis of T-periodic solutions of $\dot v = D_2 g_0(0,0)v$. Then the invertibility of the map (24) is equivalent to the condition

$$\det\left(\int_0^T \Big\langle w_j(t), D_2 g_2(0,0,0) z_i(t)\Big\rangle\, dt\right) \neq 0\,. \tag{26}$$

If for the map M in (25), there exists (μ_0,α_0,β_0) such that $M(\mu_0,\alpha_0,\beta_0) = 0$ and $D_{(\alpha,\beta)}M(\mu_0,\alpha_0,\beta_0)$ is nonsingular, then (17) can be solved when (α,β) is near (α_0,β_0) for $\xi > 0$ small. We get in this way a transversal homoclinic orbit of (2). The above results are summarized in the next theorem.

8. Theorem. *Let* (i)-$(viii)$, (x) *and condition* (26) *hold. Let* M *be as in* (25) *and suppose* (μ_0,α_0,β_0), $\mu_{0,2} \neq 0$ *are such that* $M(\mu_0,\alpha_0,\beta_0) = 0$ *and such that* $D_{(\alpha,\beta)}M(\mu_0,\alpha_0,\beta_0)$ *is nonsingular. Then there exists* $\bar\xi_0 > 0$, *such that if* $0 < \xi \le \bar\xi_0$ *and if the parameters in* (2) *are given by* $\mu_1 = \xi^{3/2}\mu_{0,1}$, $\mu_2 = \xi\mu_{0,2}$ *then* (2) *has a transverse homoclinic orbit near* $(\gamma,0)$.

We note that $w_\alpha(t)$ satisfies

$$w_\alpha(t) = \int_{-\infty}^{t} e^{D_2 g_0(0,0)(t-s)}\Big(D_2 g_0(\gamma(s),0) - D_2 g_0(0,0)\Big)\Big(w_\alpha(s) + z_\alpha(s)\Big)\, ds \tag{27}$$

which gives an iteration method for finding $w_\alpha(t)$.

4 Examples

We now illustrate the above theory with two examples. For convenience in our calculations let us denote $r(t) = \operatorname{sech} t$. Note that $\ddot{r} = r - 2r^3$ and $\dddot{r} = (1 - 6r^2)\dot{r}$.

4.1 Example 1

As our first example consider the equations from the introduction which we repeat here:

$$\ddot{x} = x - 2x(x^2 + y^2) - 2\mu_2\dot{x} + \mu_1 \cos\omega t,$$
$$\ddot{y} = (1 - k)y - 2y(x^2 + y^\theta) - 2\mu_2\dot{y} + \mu_1 \cos p\omega t$$

where $\theta \in \mathbb{N} \setminus \{1\}$, $p \in \mathbb{N}$. The reduced equation is

$$\ddot{x} = x - 2x^3 - 2\mu_2\dot{x} + \mu_1 \cos\omega t$$

which we consider as a first order system in the phase space $(x, \dot{x})$. Since this system is in $\mathbb{R}^2$ we necessarily have $d = 1$. A bounded solution to the adjoint equation is $v = (-\ddot{r}, \dot{r})$ and from this we compute

$$a_{11}(\alpha) = \int_{-\infty}^{\infty} \dot{r} \cos\omega(t + \alpha)\, dt = \pi\omega \operatorname{sech} \tfrac{\pi\omega}{2} \sin\omega\alpha,$$
$$a_{12} = \int_{-\infty}^{\infty} -2\dot{r}^2\, dt = -\tfrac{4}{3}.$$

The bifurcation equation obtained from (4) is

$$M(\alpha, \mu) = \left(\pi\omega \operatorname{sech} \tfrac{\pi\omega}{2} \sin\omega\alpha\right) \mu_1 - \tfrac{4}{3}\mu_2 = 0.$$

We can satisfy this equation by choosing $\alpha_0 \in [0, \pi/2]$ and then taking

$$\frac{\mu_{0,2}}{\mu_{0,1}} = \frac{3\pi\omega}{4} \operatorname{sech} \tfrac{\pi\omega}{2} \sin\omega\alpha_0.$$

Since in (4), $d = 1$ the transversality condition is

$$D_\alpha M(\mu_0, \alpha_0) = \frac{3\pi\omega^2}{4} \operatorname{sech} \tfrac{\pi\omega}{2} \cos\omega\alpha_0 \neq 0$$

which is satisfied for $\alpha_0 \in [0, \pi/2)$. Let $m_0 = (3\pi\omega/4) \operatorname{sech} \pi\omega/2$. By varying α_0 we see that the reduced equation exhibits chaos for all sufficiently small $|\mu_0|$ satisfying $-m_0 < \mu_{0,2}/\mu_{0,1} < m_0$. Theorem 6 gives the next result.

9. Theorem. *If $p\omega \neq \sqrt{k-1}$ then the full equation* (1) *exhibits chaos for all sufficiently small $|\mu_0|$ satisfying $|\mu_{0,2}/\mu_{0,1}| < m_0$.*

Now we consider the case $p\omega = \sqrt{k-1} = \eta > 0$ for $\theta \geq 3$. Then (18) has the form

$$\ddot{w}_{\xi,\alpha} = -(\eta^2 + 2r^2)w_{\xi,\alpha} - 2\xi^2\mu_{0,2}\dot{w}_{\xi,\alpha} + \xi^2\mu_{0,1}\cos\eta(t+\alpha)\,.$$

Clearly $z_\alpha(t) = (1/2\eta)\sin\eta(t+\alpha)$ and $w_\alpha(t)$ satisfies

$$\ddot{w}_\alpha = -(\eta^2 + 2r^2)w_\alpha - \frac{1}{\eta}r^2\sin\eta(t+\alpha)\,.$$

Since $\sin\eta(t+\alpha) = \sin\eta t\cos\eta\alpha + \cos\eta t\sin\eta\alpha$, we get $w_\alpha = (\cos\eta\alpha)w_o + (\sin\eta\alpha)w_e$ for w_o odd and w_e even. Now the map (25) has the form

$$M(\alpha,\mu) = -\frac{4}{3}\mu_2 - \frac{2\mu_1^2}{\mu_2^2}\int\limits_{-\infty}^{\infty}\dot{r}(t)r(t)\Big(z_\alpha(t) + w_\alpha(t)\Big)^2\,dt\,.$$

Since $r(t)$ is even and $\dot{r}(t)$ is odd, we obtain

$$\begin{aligned} M(\alpha,\mu) = {} & -\frac{4}{3}\mu_2 - \frac{2\mu_1^2}{\mu_2^2}\sin 2\eta\alpha \\ & \times \int_{-\infty}^{\infty}\dot{r}(t)r(t)\Big(\frac{1}{2\eta}\cos\eta t + \frac{\bar{w}_e(t)}{\eta}\Big)\Big(\frac{1}{2\eta}\sin\eta t + w_o(t)\Big)\,dt \end{aligned}$$

while

$$\begin{aligned} \bar{w}_e(t) &= -2\int\limits_{-\infty}^{t}\sin\eta(t-s)r^2(s)\left[\bar{w}_e(s) + \frac{1}{2}\cos\eta s\right]\,ds, \\ w_o(t) &= -2\int\limits_{-\infty}^{t}\sin\eta(t-s)r^2(s)\left[w_o(s) + \frac{1}{2\eta}\sin\eta s\right]\,ds\,. \end{aligned} \tag{28}$$

Summarizing we get the next result.

10. Theorem. *If* $p\omega = \sqrt{k-1}$, $\theta \geq 3$ *and*

$$0 \neq A_\eta = \int\limits_{-\infty}^{\infty}\dot{r}(t)r(t)\Big(\frac{1}{2\eta}\cos\eta t + \frac{\bar{w}_e(t)}{\eta}\Big)\Big(\frac{1}{2\eta}\sin\eta t + w_o(t)\Big)\,dt\,,$$

then (1) *has a chaos for parameters* $\mu_1 = \xi^3\mu_{0,1}$, $\mu_2 = \xi^2\mu_{0,2}$, $\mu_{0,2} \neq 0$ *with* $|\mu_{0,2}^3/\mu_{0,1}^2| < 3A_\eta/2$.

From (28) for $t \leq 0$ we get

$$\begin{aligned} |\bar{w}_e(t)| &\leq 2\Big(||\bar{w}_e|| + \frac{1}{2}\Big)\eta\int\limits_{-\infty}^{t}(t-s)r^2(s)\,ds\,, \\ |w_o(t)| &\leq 2\int\limits_{-\infty}^{t}\eta(t-s)r^2(s)\Big(||w_0|| - \frac{s}{2}\Big)\,ds\,. \end{aligned}$$

Since $\int_{-\infty}^{t}(t-s)r^2(s)\,ds < \infty$ and $\int_{-\infty}^{t}\eta(t-s)r^2(s)(-s)\,ds < \infty$, we get $||\bar{w}_e|| \to 0$ and $||w_o|| \to 0$ as $\eta \to 0$. By using Lebesgue dominated convergence theorem, we get

$$\lim_{\eta\to 0}\int\limits_{-\infty}^{\infty}\dot r(t)r(t)\Big(\frac{1}{2}\cos\eta t+\bar w_e(t)\Big)\Big(\frac{1}{2\eta}\sin\eta t+w_o(t)\Big)\,dt$$

$$=\int\limits_{-\infty}^{\infty}\dot r(t)r(t)\frac{1}{4}t\,dt>0\,.$$

Hence $A_\eta > 0$ for $\eta > 0$ sufficiently small. Of course, for a concrete numerical value of η, we can estimate the numerical value of A_η.

4.2 Example 2

As a generalization of the preceding example consider the equations

$$\begin{aligned}
\ddot x &= x-2x(x^2+y^2+z^2)-\mu_2(\dot x+\dot y)+\mu_1\cos\omega t,\\
\ddot y &= y-2y(x^2+y^2+z^2)-\mu_2(\dot x+\dot y),\\
\ddot z &= (1-k)z-2z(x^2+y^2+z^\theta)-\mu_2\dot z+\mu_1\cos p\omega t
\end{aligned}$$

where, as before, we assume $k > 1$ and $\theta \in \mathbb{N}\setminus\{1\}$, $p \in \mathbb{N}$. We consider these equations as a first order system in the phase space $(x,\dot x,y,\dot y,z,\dot z)$.

The reduced equations are

$$\begin{aligned}
\ddot x &= x-2x(x^2+y^2)-\mu_2(\dot x+\dot y)+\mu_1\cos\omega t,\\
\ddot y &= y-2y(x^2+y^2)-\mu_2(\dot x+\dot y).
\end{aligned}$$

The unperturbed motion of this system has a homoclinic 2-manifold with a family of homoclinic orbits given by $x = r(t)\cos\beta$, $y = r(t)\sin\beta$. Writing out the adjoint equation in $\mathbb{R}^4$ we obtain as a basis for the space of bounded solutions

$$\begin{aligned}
v_{\beta 1} &= (-\ddot r\cos\beta,\dot r\cos\beta,-\ddot r\sin\beta,\dot r\sin\beta),\\
v_{\beta 2} &= (-\dot r\sin\beta,r\sin\beta,\dot r\cos\beta,-r\cos\beta).
\end{aligned}$$

Next we compute

$$\begin{aligned}
a_{11}(\alpha,\beta) &= \int_{-\infty}^{\infty}\dot r\cos\beta\cos\omega(t+\alpha)\,dt=\pi\omega\operatorname{sech}\tfrac{\pi\omega}{2}\sin\omega\alpha\cos\beta,\\
a_{12}(\alpha,\beta) &= \int_{-\infty}^{\infty}-\dot r\cos\beta(\dot r\cos\beta+\dot r\sin\beta)-\dot r\sin\beta(\dot r\cos\beta+\dot r\sin\beta)\,dt\\
&= -\tfrac{2}{3}(\cos\beta+\sin\beta)^2,\\
a_{21}(\alpha,\beta) &= \int_{-\infty}^{\infty}r\sin\beta\cos\omega(t+\alpha)\,dt=\pi\operatorname{sech}\tfrac{\pi\omega}{2}\cos\omega\alpha\sin\beta,\\
a_{22}(\alpha,\beta) &= \int_{-\infty}^{\infty}r\sin\beta(\dot r\cos\beta+\dot r\sin\beta)+r\cos\beta(\dot r\cos\beta+\dot r\sin\beta)\,dt=0,
\end{aligned}$$

In (5), $d = 2$, β is a scalar and the bifurcation equations take the form

$$\begin{aligned} a_{11}(\alpha,\beta)\mu_1 + a_{12}(\alpha,\beta)\mu_2 &= 0 \\ a_{21}(\alpha,\beta)\mu_1 \qquad\qquad\quad &= 0. \end{aligned}$$

A sufficient condition for a nontrivial solution is $a_{21} = 0$ which is satisfied by taking $\omega\alpha_0 = \pi/2$. We then have

$$\frac{\mu_{0,2}}{\mu_{0,1}} = -\frac{a_{11}(\alpha_0,\beta_0)}{a_{12}(\alpha_0,\beta_0)} = \frac{3\pi\omega \operatorname{sech}\frac{\pi\omega}{2}\cos\beta_0}{2(\cos\beta_0+\sin\beta_0)^2}.$$

It is easy to see that by varying the parameter β_0 we obtain bifurcation curves through the origin in the μ_1-μ_2 plane of all slopes.

It remains to check the transversality condition which takes the form

$$\begin{aligned} &\det\Big(D_{(\alpha,\beta)}M(\alpha_0,\beta_0,\mu_0)\Big) \\ &= \begin{vmatrix} \frac{\partial a_{11}}{\partial\alpha}(\alpha_0,\beta_0)\mu_{0,1} + \frac{\partial a_{12}}{\partial\alpha}(\alpha_0,\beta_0)\mu_{0,2} & \frac{\partial a_{11}}{\partial\beta}(\alpha_0,\beta_0)\mu_{0,1} + \frac{\partial a_{12}}{\partial\beta}(\alpha_0,\beta_0)\mu_{0,2} \\ \frac{\partial a_{21}}{\partial\alpha}(\alpha_0,\beta_0)\mu_{0,1} + \frac{\partial a_{22}}{\partial\alpha}(\alpha_0,\beta_0)\mu_{0,2} & \frac{\partial a_{21}}{\partial\beta}(\alpha_0,\beta_0)\mu_{0,1} + \frac{\partial a_{22}}{\partial\beta}(\alpha_0,\beta_0)\mu_{0,2} \end{vmatrix} \\ &= -\frac{(\mu_{0,1})^2\pi^2\omega^2(\sin\beta_0 + 2\cos^3\beta_0)\sin\beta_0 \operatorname{sech}^2\frac{\pi\omega}{2}}{(\cos\beta_0+\sin\beta_0)^2} \neq 0 \end{aligned}$$

We see that this condition is satisfied for β_0 in the set

$$\Big\{\beta \in [0,2\pi] \mid \beta \notin \{0, \tfrac{3\pi}{4}, \pi, \tfrac{7\pi}{4}, 2\pi\}\Big\}.$$

Thus, the reduced equation exhibits chaos for all sufficiently small $|\mu_0|$ in the μ_1-μ_2 plane except along the lines of slope $m = \pm m_0$ where $m_0 = (3\pi\omega/2)\operatorname{sech}(\pi\omega/2)$. From Theorem 6, if $p\omega \neq \sqrt{k-1}$ then the full equation exhibits chaos for all sufficiently small $|\mu_0|$ lying except along the lines of slope $m = \pm m_0$. The case $p\omega = \sqrt{k-1}$ can be studied like in Example 1 but computations are rather tedious, so we omit them.

Bibliography

[1] E. A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations*, McGraw-Hill, New York, 1955;

[2] W. A. Coppel, *Dichotomies in Stability Theory*, Lecture Notes in Mathematics, v. 629, Springer-Verlag,, 1978;

[3] M. Fečkan, *Higher dimensional Melnikov mappings*, Math. Slovaca **49** (1999), 75-83;

[4] J. R. Gruendler, *Homoclinic solutions for autonomous dynamical systems in arbitrary dimension*, SIAM J. Math. Anal. **23(3)** (1992), 702-721;

[5] J. R. Gruendler, *Homoclinic solutions for autonomous ordinary differential equations with nonautonomous perturbations*, J. Differential Equations **122(1)** (1995), 1-26;

[6] J. R. Gruendler, *The existence of transverse homoclinic solutions for higher order equations*, J. Differential Equations **130(2)** (1996), 307-320;

[7] J. K. Hale, *Ordinary Differential Equations*, 2nd ed., Robert E. Krieger, New York, 1980;

[8] K.J. Palmer, *Exponential dichotomies and transversal homoclinic points*, J. Differential Equations **55** (1984), 225-256;

[9] S. Smale, *Differentiable dynamical systems*, Bull. Amer. Math. Soc. **73** (1967), 747-817;

[10] S. Wiggins, *Global Bifurcations and Chaos, Analytical Methods*, Applied Mathematical Sciences, vol. 73, Springer-Verlag, New York, Heidelberg, Berlin, 1988;

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