GRAPHS WITH SMALL BOOK THICKNESS

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Abstract. In an article published in 1979, Kainen and Bernhart [1] laid the groundwork for further study of book embeddings of graphs. They define an *n-book* as a line L in 3-space, called the *spine*, and n half-planes, called *pages*, with L as their common boundary. An *n-book embedding* of a graph G is an embedding of G in an n-book so that the vertices of G lie on the spine and each edge of G lies within a single page so that no two edges cross. The *book thickness* $bt(G)$ or *page number* $pg(G)$ of a graph G is the smallest n so that G has an n-book embedding.

Finding the book thickness of an arbitrary graph is a difficult problem. Even with a pre-specified vertex ordering, the problem has been shown to be *NP*-complete [6]. In this paper we will introduce book embeddings with particular focus on results for graphs with small book thickness.

1. Graphs With Book Thickness $\mathbf{bt(G) \leq 1}$. The only graphs with $bt(G) = 0$ consist entirely of isolated vertices, since each edge of a graph must be assigned to a page. Observing that the vertices of the components C_1, C_2, $\ldots$, C_k of a disconnected graph G can be grouped by components along the spine, it follows that $bt(G) = \max\{bt(C_1), bt(C_2), \ldots, bt(C_k)\}$. Hence, from this point forward, all graphs are assumed to be connected. Loops and multiple edges also do not complicate the book embedding problem. In a book embedding, a loop can be placed next to the spine and a single edge can be replaced by multiple copies without causing edge crossings. For simplicity, we will also restrict our discussion to simple graphs.

It is easy to see that the set of one-page embeddable graphs includes paths. We embed the vertices along the spine according to the natural ordering of the path, v_0, v_1, $\ldots$, v_n. Now all edges $\{v_i, v_{i+1}\}$ can be placed on a single page without crossing (see Figure 1). Not only paths, but all trees admit one-page embeddings. This is shown by induction on the number of vertices in the tree.

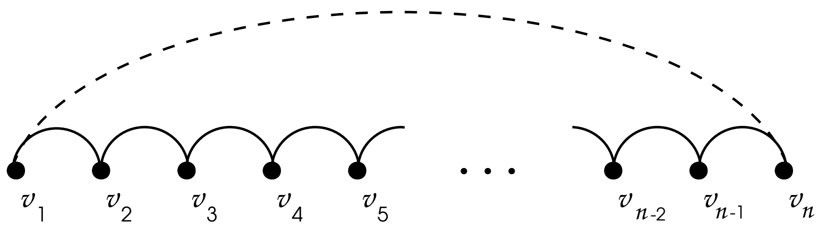


Figure 1.
One-page book embedding of the path of length n.

Theorem 1. If T is a tree, then $bt(T) \leq 1$.

Proof. Let T be a tree. If $|V(T)| = 1$, place the single vertex on the spine.

Now suppose the theorem holds for all trees with $|V(T)| = 1, 2, \ldots, k-1, k \geq 2$. Consider tree T with $|V(T)| = k$. Since $k \geq 2$, T must have at least one leaf, v. Removing v and its adjoining edge e results in a tree $T - v$ with $k-1$ vertices. By induction, we may now embed $V - v$ in a book with one or fewer pages. Let u be the unique vertex of T adjacent to v. Then u must lie on the spine in the book embedding of $V - v$. Place v on the spine to the immediate right of u. Since edges of $V - v$ lie only on the pages, this placement will not conflict with the existing book embedding of $V - v$. We may now draw edge e between u and v below any edges on the page, avoiding crossings with other edges adjacent to u. This gives the desired one-page embedding of T (see Figure 2).

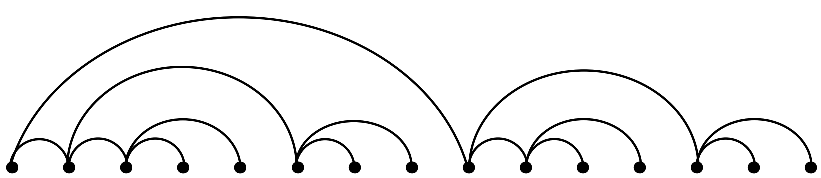


Figure 2.
One-page book embedding of the height three binary tree.

Figure 2 shows a one-page embedding of the complete binary tree of height three. Theorem 1 guarantees that graphs without cycles are one-page embeddable. However, there are clearly graphs with cycles that also have book thickness one. We can embed the cycle of length n in a one-page book just as we embedded the path of length n. In the case of the cycle, the additional edge $\{v_0, v_n\}$ can be placed above the other edges in the page, without crossing, as illustrated by the dotted line segment in Figure 1.

Now suppose the vertices of an arbitrary graph are ordered $v_1, v_2, \ldots, v_n$ along the spine of a book. The edges of the cycle $v_1, v_2, \ldots, v_n$ can be added to any page of the book without causing edge crossings in a simple way. We place the edges of the path $v_1, v_2, \ldots, v_n$ close to the spine and the edge $\{v_0, v_n\}$ above the other edges on the page. Every edge on a particular page lies within or is on this outer cycle.

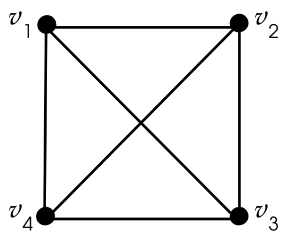
If we stretch this cycle into a circle, the problem of determining whether a given graph can be embedded in a k-page book can be viewed in terms of a circular embedding problem. Embedding a graph G in a k-book is equivalent to placing the vertices in a circle and coloring the edges (represented by chords of the circle) with k colors so that no two edges of the same color cross. With this circular view of the spine, it is also now clear that if G is embeddable in a k-page book with vertex-ordering $v_1, v_2, \ldots, v_n$ along the spine, then any cyclic permutation of the vertices along the spine also gives a k-book embedding of G.

The circular realization of the spine allows us to give an alternate description of graphs with book thickness k. A graph G is called *outerplanar* if it can be drawn in the plane so that all vertices of G lie on the same face. Equivalently, G is outerplanar if all the vertices of G can be placed in a circle in such a way that all edges of G are non-crossing chords of the circle. This leads to the following results [1, 7].

Theorem 2. A graph G has a k-page embedding with vertex ordering $v_1, v_2, \ldots, v_n$ if and only if $G = G_1 \cup G_2 \cup \cdots \cup G_k$, where each G_i is an outerplanar graph embedded with vertex-ordering $v_1, v_2, \ldots, v_n$.

Theorem 3. $bt(G) \leq 1$ if and only if G is outerplanar.

Large classes of outerplanar, and thus one-page embeddable, graphs are known [14]. There are many examples of graphs that are planar but not outerplanar. A simple one-page embeddable graph with n vertices can have at most $2n-3$ edges, since it can have at most n edges for a completed outer n-cycle and at most $n-3$ edges (corresponding to a complete triangulation) in the interior of that n-cycle. The graph K_4 is the smallest example of a graph that is not outerplanar. K_4 has $n = 4$ vertices and 6 edges, which exceeds the upper bound of $2(4)-3 = 5$ edges. Although it is not one-page embeddable, K_4 does admit a two-page embedding (see Figure 3).



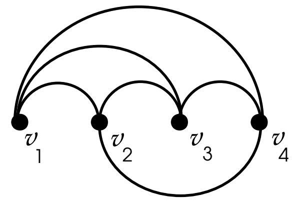


Figure 3.
Two-page book embedding of K_4.

2. Book Thickness of Planar Graphs. A two-page book consists of two half-planes that meet at the spine. This may be realized by drawing a straight line L in the plane for the spine. The two pages correspond to the half-planes above and below L. Thus, it is clear that any two-page embeddable graph is planar. Is the converse true? Does every planar graph have a two-page embedding? The following characterization of two-page embeddable graphs helps answer this question [1].

Theorem 4. $bt(G) \leq 2$ if and only if G is a subgraph of a planar Hamiltonian graph.

Proof. Let G be a graph with $bt(G) \leq 2$. Consider a two-page book embedding of G. The desired Hamiltonian cycle is found by following the natural ordering of the vertices along the spine, adding any missing edges to form the outer cycle. With the added edges, we now have a planar Hamiltonian graph.

Conversely, suppose G is a subgraph of a planar Hamiltonian graph G'. Draw G' in the plane and trace out a Hamiltonian cycle C in G'. The cycle C together with the edges inside C form one page and the edges outside C form the second page. Now we have a two-page embedding of G' which induces the desired two-page book embedding of G.

The 3-dimensional hypercube Q_3 is a bipartite planar graph that is not outerplanar [14]. Hence, Q_3 is not embeddable in a one-page book. Figure 4 shows an optimal two-page book embedding of Q_3 using a Hamiltonian ordering of the vertices.

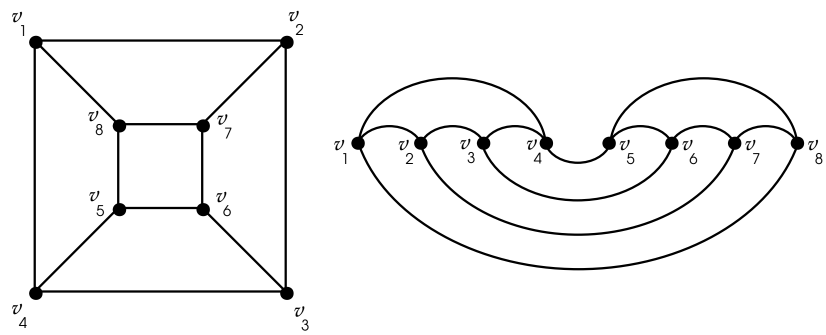


Figure 4.
Two-page book embedding of Q_3.

Graphs that are subgraphs of planar Hamiltonian graphs are called *subhamiltonian*. Planar Hamiltonian graphs are clearly subhamiltonian, and thus, two-page embeddable. We have large classes of two-page embeddable graphs due to the following results of Whitney [16] and Tutte [15].

Theorem 5. (Whitney) A maximal planar graph without separating triangles has a Hamiltonian cycle.

Theorem 6. (Tutte) A 4-connected planar graph with at least two edges has a Hamiltonian cycle.

Maximal planar graphs without separating triangles are embeddable in two-page books by Whitney's Theorem. To find examples of planar graphs that are not two-page embeddable, we seek maximal planar graphs with separating triangles that are not subhamiltonian. The *stellation* $St(G)$ of a planar graph G is formed as the result of placing a new vertex in every face (including the outer face) of G and connecting it to each vertex around the face. We can repeat this process by letting $St^n(G) = St(St^{n-1}(G))$. The smallest such maximal non-subhamiltonian planar graph $St^2(K_3)$ is shown in Figure 5. It is formed by starting with a triangle (K_3) and stellating twice.

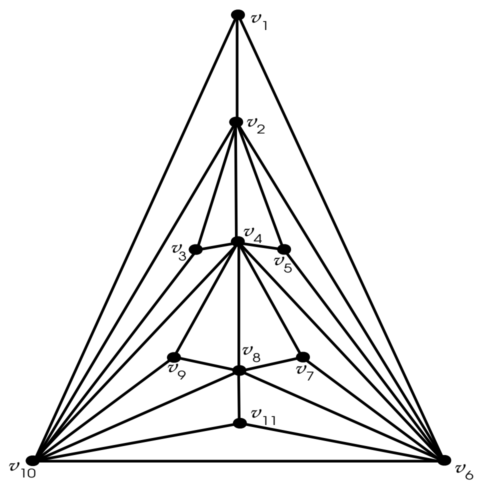


Figure 5.
The second stellation of the triangle $St^2(K_3)$.

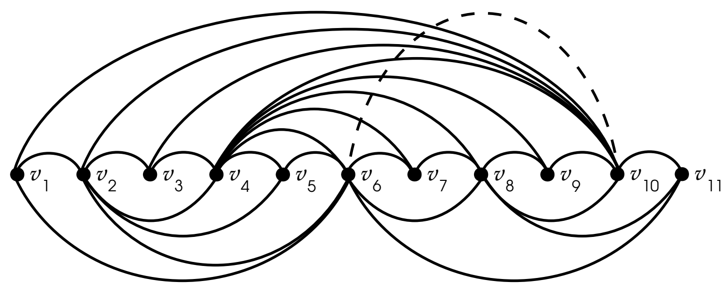


Figure 6.
Three-page book embedding of $St^2(K_3)$.

In their 1979 paper, Bernhart and Kainen conjectured that the book thickness of planar graphs is unbounded. Specifically, they suggest that $bt(St^n(G))$ can be made arbitrarily large if G is any maximal planar graph. Heath [8] disproves the specific claim by showing that $St^n(K_3)$ are all embeddable on three pages. Figure 6 depicts a three-page book embedding of $St^2(K_3)$. Several authors have disproved the larger conjecture for planar graphs [2, 8, 9, 11, 17, 18] by giving various finite bounds for the book thickness of a planar graph. Yannakakis settles the issue for planar graphs by offering a best bound of four pages [17, 18].

Theorem 7. (Yannakakis) If G is a planar graph, then $bt(G) \leq 4$.

In his paper, Yannakakis also outlines the construction of a planar graph which needs four pages. Hence, four pages are also necessary to accommodate all planar graphs. Since the example of Yannakakis is extremely large and complex and it is the only published example needing four pages, it appears that three pages are sufficient for most small planar graphs. If the original triangulation has sparse separating triangles, Kainen [10] suggests that only three pages are needed for the book embedding. In fact, it has been shown that if G is planar and bipartite (has no cycles of odd length), then two pages are enough [4, 13].

Theorem 8. If G is a planar bipartite graph, then $bt(G) \leq 2$.

In fact we can even allow odd cycles, other than triangles, and we get the following result [13].

Theorem 9. If G is a triangle-free planar graph, then $bt(G) \leq 2$.

Theorem 8 and Theorem 9 give optimal book-thickness for several common graphs. The $n \times n$ *square grid* is the planar graph formed by taking the Cartesian product of two paths of length n (see Figure 7). This set of graphs, previously shown to be 2-page embeddable in [3], contains no odd cycles. Thus, the two-page book thickness of square grids follows from these two theorems.

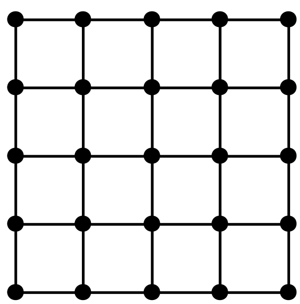


Figure 7.
The 4×4 square grid.

3. Beyond Planar Graphs. We observed that every two-page embeddable graph is planar. Thus, embedding a non-planar graph requires at least a three-page book. We also noted that the book thickness of any planar graph is at most four. When we move beyond the plane, is there a limit to how large a book we will need? This question can be answered by examining the complete graph K_n. It is the graph with n vertices and all possible $\binom{n}{2}$ edges joining pairs of distinct vertices.

First, we will examine the standard k-book. A k-page embeddable graph with n vertices can have at most $e = n + k(n-3)$ distinct edges. We can have n edges for the outer cycle and up to $n-3$ non-cycle edges on each of the k pages. Solving this for k, we get $k \geq \frac{e-n}{n-3}$. This gives us a lower bound for the book thickness in terms of the number of vertices and edges. We can now use this bound to obtain the optimal book thickness of K_n [1, 12].

<u>Theorem 10</u>. If $n \geq 4$, then $bt(K_n) = \lceil n/2 \rceil$.

<u>Proof</u>. Let $n \geq 4$. First, we show that $bt(K_n) \geq \lceil n/2 \rceil$. The graph K_n has n vertices and $\binom{n}{2}$ edges. By the previous observation, we have that

$$bt(K_n) \geq \frac{\binom{n}{2} - n}{n-3} = \frac{n(n-1)/2 - n}{n-3} = n/2.$$

Since the book thickness of a graph must be an integer, it follows that $bt(K_n) \geq \lceil n/2 \rceil$.

To obtain the other inequality, we will assume that n is even. Suppose that $n = 2m$. We will show that $bt(K_{2m}) \leq 2m/2 = m$. The result for odd n will follow from the fact that K_{2m-1} is a subgraph of K_{2m}. The m pages of the book are formed by rotating the triangulated $2m$-gon of Figure 8 through m successive positions.

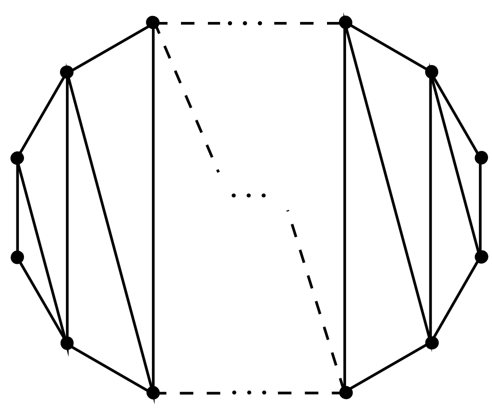


Figure 8.
Triangulation of the $2m$-gon.

A triangulation of the $2m$-cycle has $2m-3$ edges. It is easy to see that each inner diagonal of this $2m$-cycle cannot appear in more than one of the m rotations. We get $2m$ edges for the outer cycle and $m(2m-3)$ edges for the m triangulations for a total of $2m + m(2m-3) = 2m^2 - m = \binom{2m}{2}$ distinct edges. Hence, all $\binom{2m}{2}$ edges of K_{2m} are accounted for and we have the desired result.

As n increases, it is clear that the book thickness of K_n can be made arbitrarily large. Beyond K_n, the book thickness question becomes difficult since the ordering of the vertices along the spine must be considered as well as the assignment of edges to pages. Despite this difficulty, there are several classes of graphs for which the book thickness, or at least good bounds for book thickness, is known [5].

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