SOLUTIONS

No problem is ever permanently closed. Any comments, new solutions, or new insights on old problems are always welcomed by the problem editor.

173. *Proposed by Mohammad K. Azarian, University of Evansville, Evansville, Indiana.*

Show that

$$\sum_{n=1}^{\infty} \frac{x_n}{x_{n-1}} = \frac{7}{2},$$

provided

$$x_{n-1}(x_{n-2}^2 + x_{n-1}x_{n-3}) - 6x_{n-3}(x_{n-1}^2 - x_n x_{n-2}) = 0, \quad n \geq 3,$$

and $x_0 = x_1 = x_2 = 1$.

Solution by Panagiotis T. Krasopoulos, Athens, Greece. First, let us observe that from the statement of the problem it is assumed implicitly that $x_k \neq 0$ for any $k \geq 0$. This fact will be proved in the process of the following proof.

Let us assume that $x_k \neq 0$ for any $0 \leq k \leq n-1$. We divide the given equation by the product $x_{n-1}x_{n-2}x_{n-3}$ and we define $a_n = x_n/x_{n-1}$, so we obtain

$$a_{n-2} + a_{n-1} - 6a_{n-1} + 6a_n = 0 \text{ if and only if } 6a_n - 5a_{n-1} + a_{n-2} = 0,$$

where $n \geq 3$ and $a_1 = a_2 = 1$. This is a linear homogeneous difference equation with constant coefficients and can be solved directly by using its characteristic equation. After some algebraic calculations we have

$$a_n = 8\left(\frac{1}{2}\right)^n - 9\left(\frac{1}{3}\right)^n \text{ for } n \geq 1.$$

It can easily be seen that $\frac{8}{9} > \left(\frac{2}{3}\right)^n$ for $n \geq 1$ and so $a_n > 0$. Since $a_n > 0$ and $x_0 = x_1 = x_2 = 1 > 0$, by induction we obtain that $x_k > 0$ for any

$k \geq 0$ and so the division by x_k is allowed. Now the result follows directly since

$$\sum_{n=1}^{\infty} \frac{x_n}{x_{n-1}} = \sum_{n=1}^{\infty} a_n = 8 \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n - 9 \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n = 8 - 9 \cdot \frac{1}{2} = \frac{7}{2}.$$

We have used the infinite geometric series

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = 1 \text{ and } \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n = \frac{1}{2}.$$

The proof is complete.

Also solved by Shang Nina, Shandong University of Technology, Zibo, China; Mihai Cipu, Institute of Mathematics of the Romanian Academy, Bucharest, Romania; G. C. Greubel, Newport News, Virginia; Kandasamy Muthuvel, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin; Kenneth B. Davenport, Dallas, Pennsylvania; Dr. Louis Scheinman, Toronto, Canada; and the proposer.

174. *Proposed by Ovidiu Furdui, Cluj, Romania.*

Let $k \geq 1$ and $p \geq 0$ be two nonnegative integers. Find the sum

$$S(p) = \sum_{m_1, \ldots, m_k = 1}^{\infty} \frac{1}{m_1 m_2 \cdots m_k (m_1 + m_2 + \cdots + m_k + p)}.$$

Solution by Paolo Perfetti, Dipartimento di Matematica, Università degli studi "Tor Vergata" Roma, Italy. We write

$$\frac{1}{m_1 + \cdots + m_k + p} = \int_0^1 x^{m_1 + \cdots + m_k + p - 1}\, dx$$

and then

$$\begin{aligned}
S(p) &= \sum_{m_1,\ldots,m_k=1}^{\infty} \frac{1}{m_1 m_2 \cdots m_k} \int_0^1 x^{m_1+\cdots+m_k+p-1}\,dx \\
&= \int_0^1 x^{p-1}\,dx \sum_{m_1,\ldots,m_k=1}^{\infty} \frac{x^{m_1+\cdots+m_k}}{m_1 m_2 \cdots m_k} \\
&= \int_0^1 x^{p-1}(-1)^k(\ln(1-x))^k\,dx \\
&= (-1)^k \int_0^1 (1-x)^{p-1}(\ln x)^k\,dx.
\end{aligned}$$

Set $p = 0$. The integral is

$$\begin{aligned}
&(-1)^k \int_0^1 (1-x)^{-1}(\ln x)^k\,dx = \sum_{n=0}^{\infty}(-1)^k \int_0^1 x^n(\ln x)^k\,dx \\
&= \sum_{n=0}^{\infty}(-1)^k \left(\frac{x^{n+1}}{n+1}(\ln x)^k \Big|_0^1 - \frac{k}{n+1}\int_0^1 x^n(\ln x)^{k-1}\,dx \right) \\
&= \sum_{n=0}^{\infty}(-1)^k \left(-\frac{k}{n+1}\int_0^1 x^n(\ln x)^{k-1}\,dx \right).
\end{aligned}$$

This means that if we define

$$I_{n,k} = \int_0^1 x^n(\ln x)^k\,dx,$$

we have $I_{n,k} = \frac{-k}{n+1} I_{n,k-1}$ which implies

$$I_{n,k} = \frac{(-1)^k k!}{(n+1)^k} I_{n,0} = \frac{(-1)^k k!}{(n+1)^{k+1}}$$

and yields

$$S(0) = \sum_{n=0}^{\infty} \frac{k!}{(n+1)^{k+1}} = k!\zeta(k+1).$$

Let $p \geq 1$. The integral is

$$\begin{aligned}
(-1)^k \sum_{n=0}^{p-1} \binom{p-1}{n} \int_0^1 (-x)^n (\ln x)^k \, dx \\
= (-1)^k \sum_{n=0}^{p-1} \binom{p-1}{n} (-1)^n \frac{(-1)^k k!}{(n+1)^{k+1}} \\
= \sum_{n=0}^{p-1} \binom{p-1}{n} (-1)^n \frac{k!}{(n+1)^{k+1}}.
\end{aligned}$$

The proof is complete.

Also solved by G. C. Greubel, Newport News, Virginia and the proposer.

175. *Proposed by N. J. Kuenzi, Oshkosh, Wisconsin.*

The positive integer 45 can be written as a sum of five consecutive positive integers (SCPI): $45 = 7 + 8 + 9 + 10 + 11$; furthermore, 45 can be written as a SCPI in *exactly* five ways, namely, $45 = 22 + 23 = 14 + 15 + 16 = 7+8+9+10+11 = 5+6+7+8+9+10 = 1+2+3+4+5+6+7+8+9+10$. Is there a positive integer that can be written as a sum of 2009 consecutive positive integers and which can be written as a SCPI in *exactly* 2009 ways?

Solution by Kandasamy Muthuvel, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin. We shall generalize the given problem as follows.

Prove that 3^s, where $s > 1$, can be written as a sum of s consecutive positive integers and which can be written as a sum of consecutive positive integers in exactly s ways. In particular, 3^{2009} can be written as a sum of s consecutive positive integers and which can be written as a sum of consecutive positive integers in exactly 2009 ways.

Proof. We say that $a + (a+1) + \cdots + (a+n) = 3^s$ is solvable if a and n are positive integers. We shall show that $a + (a+1) + \cdots + (a+n) = 3^s$ is solvable if and only if $n = 3^t - 1$ for some positive integer t with $1 \leq t \leq s/2$, or $n = 2 \cdot 3^t - 1$ for some integer t with $0 \leq t < s/2$.

Suppose that $n = 3^t - 1$ for some positive integer t with $1 \leq t \leq s/2$. Let

$$a = \frac{3^s}{n+1} - \frac{n}{2}.$$

Clearly a is an integer and

$$a = \frac{3^s}{3^t} - \frac{(3^t - 1)}{2} = \frac{2 \cdot 3^s - 3^{2t} + 3^t}{2 \cdot 3^t} \geq \frac{2 \cdot 3^s - 3^s + 3^t}{2 \cdot 3^t} > 0.$$

Suppose that $n = 2 \cdot 3^t - 1$, for some integer t with $0 \leq t < s/2$. Let

$$a = \frac{3^s}{n+1} - \frac{n}{2}.$$

Clearly

$$a = \frac{3^{s-t} - 2 \cdot 3^t + 1}{2}$$

is an integer. Since $2t < s$, $2t \leq s - 1$ and

$$\begin{aligned} a &= \frac{3^s}{2 \cdot 3^t} - \frac{2 \cdot 3^t - 1}{2} = \frac{3^s - 2 \cdot 3^{2t} + 3^t}{2 \cdot 3^t} \\ &\geq \frac{3^s - 2 \cdot 3^{s-1} + 3^t}{2 \cdot 3^t} = \frac{3^{s-1} + 3^t}{2 \cdot 3^t} > 0. \end{aligned}$$

Hence $a + (a+1) + \cdots + (a+n) = 3^s$ is solvable if $n = 3^t - 1$ for some positive integer t with $1 \leq t \leq s/2$ or $n = 2 \cdot 3^t - 1$ for some integer t with $0 \leq t < s/2$.

Conversely, suppose that $a + (a+1) + \cdots + (a+n) = 3^s$ is solvable. Since

$$3^s = a + (a+1) + \cdots + (a+n) = \frac{(n+1)(2a+n)}{2},$$

$$a = \frac{3^s}{n+1} - \frac{n}{2}.$$

Consider the case that n is even. Since a and $n/2$ are positive integers, $n+1$ divides 3^s and hence $n = 3^t - 1$ for some $1 \le t$. Thus,

$$a = \frac{3^s}{3^t} - \frac{3^t - 1}{2} = \frac{2 \cdot 3^{s-t} - 3^t + 1}{2} > 0 \text{ is an integer}$$

implies $2 \cdot 3^{s-t} - 3^t + 1 > 1$ which implies $2 \cdot 3^{s-t} - 3^t > 0$ which implies $2 > 3^{2t-s}$ which implies $2t - s \le 0$ which implies $t \le s/2$.

Consider the case that n is odd. Since

$$a = \frac{3^s}{n+1} - \frac{n}{2} = \frac{2 \cdot 3^s - n(n+1)}{2(n+1)},$$

$n+1$ divides $2 \cdot 3^s$. Consequently, $\frac{n+1}{2} = 3^t$ for some $0 \le t$. Thus,

$$a = \frac{3^s}{2 \cdot 3^t} - \frac{2 \cdot 3^t - 1}{2} = \frac{3^{s-t} - 2 \cdot 3^t + 1}{2} > 0 \text{ is an integer}$$

implies $3^{s-t} - 2 \cdot 3^t + 1 > 1$ which implies $3^{s-t} - 2 \cdot 3^t > 0$ which implies $2 < 3^{s-2t}$ which implies $s - 2t > 0$ which implies $t < s/2$. Hence if $a + (a+1) + \cdots + (a+n) = 3^s$ is solvable, then $n = 3^t - 1$, for some positive integer t with $1 \le t \le s/2$ or $n = 2 \cdot 3^t - 1$, for some integer t with $0 \le t < s/2$.

It is easy to see that the cardinality of the set

$$\{s : n = 3^t - 1, \text{ for some positive integer } t \text{ with } 1 \le t \le s/2 \text{ or}$$

$$n = 2 \cdot 3^t - 1, \text{ for some integer } t \text{ with } 0 \le t < s/2\}$$

is s. This completes the solution of the generalized problem. $\square$

Also solved by Calvin A. Curtindolph, Black River Falls, Wisconsin and the proposer.

176. *Proposed by José Luis Díaz-Barrero, Universidad Politécnica de Cataluña, Barcelona, Spain.*

Let a, b, c be the lengths of the sides of a triangle ABC with altitudes h_a, h_b, and h_c, respectively. Prove that

$$\frac{1}{3}\sum_{cyclic}\frac{a^2}{bc(b+c-a)} \geq \frac{h_a+h_b+h_c}{ah_a+bh_b+ch_c}.$$

Solution by Panagiotis T. Krasopoulos, Athens, Greece. Let E be the area of the triangle. Then $ah_a = bh_b = ch_c = 2E$. The inequality then becomes

$$\sum_{cyclic}\frac{a^3}{abc(b+c-a)} \geq \frac{3}{6E}(h_a+h_b+h_c)$$

or

$$\sum_{cyclic}\frac{a^3}{(b+c-a)} \geq \frac{3}{6E}(2Ebc+2Eac+2Eab) = bc+ac+ab.$$

Now, since the triangle is not degenerate, $b+c-a>0$, $a+c-b>0$, and $a+b-c>0$ holds. We multiply both sides by $(b+c-a)(a+b-c)(a+c-b) > 0$. After some algebraic calculations we obtain

$$(a^3+b^3+c^3+3abc-a^2b-a^2c-b^2a-b^2c-c^2a-c^2b)(a+b+c)^2 \geq 0.$$

It is enough to prove that

$$a^3+b^3+c^3+3abc-a^2b-a^2c-b^2a-b^2c-c^2a-c^2b \geq 0,$$

or equivalently

$$a(a-b)(a-c)+b(b-a)(b-c)+c(c-a)(c-b) \geq 0.$$

The last inequality holds directly from Schur's inequality, i.e.,

$$a^t(a-b)(a-c)+b^t(b-a)(b-c)+c^t(c-a)(c-b)\geq 0,$$

for non-negative real numbers a, b, c and for $t=1$. This completes the proof.

Also solved by Kee-Wai Lau, Hong Kong, China; Mihai Cipu, Institute of Mathematics of the Romanian Academy, Bucharest, Romania (2 solutions); Oleh Faynshteyn, Leipzig, Germany; and the proposer.