REAL COMPACT OPERATORS IN FACTORS OF TYPE I, II, AND III$_\lambda$, $0 < \lambda < 1$

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Abstract. In the present paper the real ideals of relatively compact operators of W^*-algebras are considered. Similar to the complex case, a description (up to isomorphism) of the real two-sided ideal of relatively compact operators of the complex W^*-factors is given.

1. Introduction. Let $B(H)$ be the algebra of all bounded linear operators on a complex Hilbert space H. A weakly closed *-subalgebra M with identity element $\mathbf{1}$ in $B(H)$ is called a W^*-algebra. Let $P(M)$ be the set of all projections of M, I be the ideal of all operators with the finite range projection relative to M, $J = \overline{I}$ be the ideal of compact operators relatively to M. It is known [2], that I and J are proper if and only if M is infinite; and that J is the maximal two-sided ideal of M without infinite projections. The compact operators relative to M were defined by Sonis [6] (in the case of the algebras with Segal measure, i.e. for finite W^*-algebras) as the operators which are mapping bounded sets into relatively compact sets. In [4], an analogous notion of finiteness and compactness in purely infinite W^*-algebras was introduced and considered.

In the present paper we introduce and consider the ideal of compact operators relative to a real W^*-algebra. In a manner similar to the complex case, a description and classification (up to isomorphism) of the real two-sided ideal of the relatively compact operators is given.

2. Preliminary Information. A real *-subalgebra R with $\mathbf{1}$ in $B(H)$ is called a *real* W^**-algebra* if it is closed in the weak operator topology and $R \bigcap iR = \{\mathbf{0}\}$. A real W^*-algebra R is called a *real factor* if its center $Z(R)$ contains only elements of the form $\{\lambda \mathbf{1}\}$, $\lambda \in \mathbb{R}$. We say that a real W^*-algebra R is of the type I_{fin}, I_∞, II_1, II_∞, or III_λ $(0 \leq \lambda \leq 1)$ if the enveloping W^*-algebra $U(R) = R \dot{+} iR$ has the corresponding type in the ordinary classification of W^*-algebras [1].

A linear mapping α with $\alpha(x^*) = \alpha(x)^*$ of the algebra R into itself is called

-an *-*automorphism* if $\alpha(xy) = \alpha(x)\alpha(y)$;
-an *-*antiautomorphism* if $\alpha(xy) = \alpha(y)\alpha(x)$,
-an *involutive* if $\alpha^2(x) = \alpha(\alpha(x)) = x$, for all $x, y \in R$.

A *trace* on a (complex or real) W^*-algebra N is a linear function τ on the set N^+ of positive elements of N with values in $[0, +\infty]$, satisfying the following condition:

$$\tau(uxu^*) = \tau(x), \text{ for an arbitrary unitary } u \text{ and } x \text{ in } N.$$

The trace τ is said to be *finite* if $\tau(\mathbf{1}) < +\infty$; *semifinite* if given any $x \in N^+$ there is a nonzero $y \in N^+$, $y \leq x$ with $\tau(y) < +\infty$.

Let $R \subset B(H)$ be a real W^*-algebra, $M = R + iR$ be the *enveloping* W^*-*algebra* for R. Let τ be a semifinite trace on R. Subspace K of H with $K \eta R$, i.e. $P_K \in R$, is called

-*finite* relative to τ if $\tau(P_K) < \infty$, where P_K projection of H on K;

-*compact* relative to τ if K is an approximate of the bounded sets from relatively finite subspaces.

Real operator x of H (i.e. $x \in R$) is called *real compact* relative to τ if it is the operator mapping bounded sets into relatively compact sets.

3. Compact Operators in Semifinite Real Factor. Let $I(R)$ be the set of all relatively compact operators of R.

Theorem 1. Let R be a semifinite real factor. Then $I(R)$ is a unique (nonzero) uniformly closed two-sided ideal of R.

Proof. See [5] for details.

Theorem 2. Let R be a semifinite real factor, $U = R \dot{+} iR$ is its enveloping factor. Let $I(U)$ be a unique (nonzero) uniformly closed two-sided ideal of U. Then

$$I(U) = I(R) \dot{+} iI(R).$$

Proof. Since $I(R)$ is a uniformly closed two-sided ideal, then $I(R) \dot{+} iI(R)$ is also a uniformly closed two-sided ideal. In fact, let $\{c_n = a_n + ib_n\}$ be a Cauchy sequence in $I(R) \dot{+} iI(R)$, i.e. $\|c_n - c_m\| \to 0$ as $n, m \to \infty$. Then $\|(a_n - a_m) + i(b_n - b_m)\| \to 0$ as $n, m \to \infty$. Using Lemma 1.1.3 (iii) from [1] we have

$$\max\{\|a_n - a_m\|, \|b_n - b_m\|\} \leq \|(a_n - a_m) + i(b_n - b_m)\|.$$

Therefore, $\|a_n - a_m\| \to 0$ and $\|b_n - b_m\| \to 0$ as $n, m \to \infty$. Thus, $\{a_n\}$ and $\{b_n\}$ are Cauchy sequences in $I(R)$. Hence, they converge to a and b in $I(R)$, respectively. Thus, $c_n = a_n + ib_n \to a + ib$ in $I(R) \dot{+} iI(R)$, which is thus uniformly closed. Now, if $x = a + ib \in U$, $y = c + id \in I(R) + iI(R)$, then $xy = (ac - bd) + i(ad + bc) \in I(R) \dot{+} iI(R)$. Similarly, $yx \in I(R) \dot{+} iI(R)$. Therefore, $I(R) \dot{+} iI(R)$ is a uniformly closed two-sided ideal of U. Thus, we have proved that $I(R) \dot{+} iI(R) \subset I(U)$. Now, since for $x \in I(U)$ we have $x = a + ib, a, b \in R$, let $I(U) = A \dot{+} iB$, for some $A, B \subset R$. But, for $a \in A$, $b \in B$ we have $ab, ba \in I(U)$. Therefore, $ab, ba \in A$, hence, $A = B$, i.e.,

$I(U) = A\dot{+}iA$. Then $I(R) \subset A$ as $A, I(R) \subset R$. Let $\{a_n\}$ be a Cauchy sequence in $A \subset I(U)$. Since $I(U)$ is uniformly closed, $\{a_n\}$ converges to $a \in I(U)$. But, R is also uniformly closed. Therefore, $a \in R$. Then $a \in A$. Now, let $x \in A$, $y \in R$. Since $I(U)$ is a two-sided ideal of U, $xy, yx \in I(U)$, i.e. $xy, yx \in A$ as $xy, yx \in R$. Therefore, A is a uniformly closed two-sided ideal of R with $I(R) \subset A$. Then by Theorem 1 we have $A = I(R)$. This completes the proof of the theorem.

4. Real Ideals of Compact Operators of Factors of Type III$_\lambda$, ($\lambda \neq 1$). Let us recall [3] the notion of the *crossed product* of a W^*-algebra by a locally compact topological group by its *-automorphism. Let N be a (complex or real) W^*-algebra in $B(H)$, $\gamma\colon G \to Aut(M)$ be a group homomorphism such that each map $g \to \gamma_g$ is strongly continuous. Let $L_2(G, H)$ be the Hilbert space of all H-valued square integrable functions on G. We consider a *-algebra $U \subset B(L_2(G, H))$, generated by operators of the form: $\pi_\gamma(a)(a \in M)$ and $u(g)(g \in G)$, where

$$(\pi_\gamma(a)\xi)(h) = \gamma_h^{-1}(a)\xi(h), \quad (u(g)\xi)(h) = \xi(g^{-1}h),$$
$$\xi = \xi(h) \in L_2(G, H),\ g, h \in G.$$

The algebra U is called *crossed product* of M by G, and denoted by $W^*(M, G)$ (or $M \times_\gamma G$). Moreover, there exists a canonical embedding $\pi\gamma\colon M \to \pi_\gamma(M) \subset U$. Each element $x \in U$ has the form: $x = \sum_{g\in G} \pi_\gamma(x(g))u(g)$, where $x(\cdot)$ is an M-valued function on G.

Let θ be a *-automorphism of N. For the action $\{\theta^n\}$ of the group Z on N we denoted by $W^*(\theta, N)$, (or $N \times_\theta Z$) the crossed product of N by θ.

Now, let R be a factor of type III$_\lambda$, ($\lambda \neq 1$). Then by [7], either

-there exist a real factor F of type II$_\infty$ and an automorphism θ of F such that R is isomorphic to the crossed product $F \times_\theta Z$ or

-there exist a complex factor N of a type II$_\infty$ and an antiautomorphism σ of N such that R is isomorphic to $((N \oplus N^{op}) \times_\sigma Z, \beta)$, where N^{op} is the opposite W^*-algebra for N, $\beta(x, y) = (y, x)$, for all $x, y \in N$.

In the first case, let $I(R)$ be the norm closure of $span\{x \in R^+ \mid E(x) \in I(F)\}$, where $E\colon R \to F$ is a unique faithful normal conditional expectation.

In the second case, let $I(R)$ be the norm closure of $span\{x \in R^+ \mid E(x) \in I(N \oplus N^{op})\}$, where E is a unique faithful normal conditional expectation from M to $N \oplus N^{op}$.

If we now apply Theorem 1 and use the scheme of proof of Theorem 6.2 from [4], then we prove a real analogue of the theorem of Halpern-Kaftal.

<u>Theorem 3</u>. In each case $I(R)$ is a unique (nonzero) uniformly closed two-sided ideal of R.

Similar to Theorem 2, we can prove the following theorem.

Theorem 4. Let M be an injective factor of type III_λ, $0<\lambda<1$, R and Q are non-isomorphic real factors with the enveloping factor M, i.e. $R \dot{+} iR = Q \dot{+} iQ = M$. If $I(M)$ is a (nonzero) uniformly closed two-sided ideal of M, then

$$I(M) = I(R)\dot{+} iI(R) \text{ and } I(M) = I(Q)\dot{+} iI(Q),$$

where $I(R)$ and $I(Q)$ are non-isomorphic unique uniformly closed two-sided ideals of R and Q, respectively.

5. Main Result. Let M be a factor, α - an involutive *-antiautomorphism of M. Then from [1], the set $R = \{x \in M : \alpha(x) = x^*\}$ is a real factor and the enveloping W^*-algebra $U(R)$ of R coincides with M, and conversely, given an arbitrary real factor R there exists a unique involutive *-antiautomorphism α_R of the W^*-algebra $U(R)$ such that $R = \{x \in U(R) : \alpha(x) = x^*\}$. Moreover, R_1 and R_2 are two real *-isomorphic factors if and only if the enveloping factors $U(R_1)$ and $U(R_1)$ are *-isomorphic and the involutive *-antiautomorphism α_{R_1} and α_{R_2} are conjugate, i.e. $\alpha_{R_1} = \theta \cdot \alpha_{R_2} \cdot \theta^{-1}$, for some *-automorphism θ.

It is known [1] that

-in factor of type I_n, n even, there exists unique conjugacy class on involutive *-antiautomorphism;
-in factor of type I_n, n odd or $n=\infty$, there exist exactly two conjugacy classes on involutive *-antiautomorphism;
-in injective factor of type II_1 there exists unique conjugacy class on involutive *-antiautomorphism;
-in injective factor of type II_∞ there exists unique conjugacy class on involutive *-antiautomorphism;
-in injective factor of type III_λ, $0<\lambda<1$, there exist exactly two conjugacy classes on involutive *-antiautomorphism;
-in injective factor of type III_1 there exists unique conjugacy class on involutive *-antiautomorphism.

Hence, from Theorems 1 and 3 we obtain the following theorem.

Theorem 5. Let M be a factor.

1) If M has type I_n, n even, then in M there exist two (nonzero) uniformly closed two-sided real ideals up to isomorphisms;
2) If M has type I_n, n odd or $n=\infty$, then in M there exist three (nonzero) uniformly closed two-sided real ideals up to isomorphisms;
3) If M is an injective factor of type II_1 or type II_∞, then in M there exist two (nonzero) uniformly closed two-sided real ideals up to isomorphisms;
4) If M is an injective factor of type III_λ $(0<\lambda<1)$, then in M there exist three (nonzero) uniformly closed two-sided real ideals up to isomorphisms.

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