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The torsion generating set of the mapping class groups and the Dehn twist subgroups of non-orientable surfaces of odd genus

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Abstract. Let N_g be the non-orientable surface of genus g, $\mathrm{MCG}(N_g)$ the mapping class group of N_g, $\mathscr{T}(N_g)$ the index 2 subgroup generated by all Dehn twists of $\mathrm{MCG}(N_g)$. We prove that for odd genus, (1) if $g = 4k + 3$ $(k \geq 1)$, $\mathrm{MCG}(N_g)$ can be generated by three elements of finite order; (2) if $g = 4k + 1$ $(k \geq 2)$, $\mathscr{T}(N_g)$ can be generated by three elements of finite order.

1. Introduction

Let N_g be the closed non-orientable surface of genus g. We denote by $\mathrm{Homeo}(N_g)$ the group consisting of all self-homeomorphisms of N_g, and by $\mathrm{Homeo}_0(N_g)$ the normal subgroup consisting of homeomorphisms which are isotopic to the identity. Then the quotient group $\mathrm{Homeo}(N_g)/\mathrm{Homeo}_0(N_g)$ is called the mapping class group of N_g and is denoted by $\mathrm{MCG}(N_g)$. The subgroup of $\mathrm{MCG}(N_g)$ generated by all Dehn twists is denoted by $\mathscr{T}(N_g)$.

Lickorish is the first one who discovered that $\mathscr{T}(N_g)$ is an index 2 subgroup of $\mathrm{MCG}(N_g)$ ([6, 7]). Outside $\mathscr{T}(N_g)$, there is a mapping class called a "Y-homeomorphism" or a "crosscap slide". Chillingworth in [2] gave a finite set of generators for $\mathscr{T}(N_g)$ and hence also a finite set of generators for $\mathrm{MCG}(N_g)$. When the genus g is low, for example, $g = 2$, Lickorish found $\mathrm{MCG}(N_2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and Chillingworth found $\mathscr{T}(N_2)$ can be generated by one Dehn twist ([6, 2]). When $g = 3$, Birman and Chillingworth gave a concrete presentation for $\mathrm{MCG}(N_3)$ and then proved that $\mathrm{MCG}(N_3)$ can be generated by three elements ([1]). Chillingworth found $\mathscr{T}(N_3)$ can be generated by two Dehn twists ([2]), and Szepietowski simplified Birman and Chillingworth's generating set into a set consisting of three involutions ([10]).

It is a natural question to what extent we can simplify the generating sets for $\mathrm{MCG}(N_g)$ and $\mathscr{T}(N_g)$ when g is large. We would like to reduce both

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the number and the orders of the generators. When $g \geq 4$, a generating set for $\mathrm{MCG}(N_g)$ consisting of four involutions was constructed by Szepietowski. Szepietowski also proved when $g \geq 4$, $\mathrm{MCG}(N_g)$ can be generated by three elements (see [10]). The first homology of $\mathrm{MCG}(N_g)$ has been calculated by Korkmaz [5]. As a consequence, Korkmaz proved that when $g = 4$, the minimal number of the generators for $\mathrm{MCG}(N_4)$ is 3. About $\mathscr{T}(N_g)$, Stukow gave a finite presentation of $\mathscr{T}(N_g)$ in [9]. Omori reduced the number of Dehn twist generators for $\mathscr{T}(N_g)$ to $g+1$ when $g \geq 4$ ([8]).

In [3], the author proved the following: when the genus $g' \geq 5$ and $S_{g'}$ is an orientable closed surface of genus g', the extended mapping class group $\mathrm{MCG}^{\pm}(S_{g'})$ can be generated by two elements of finite order. One is of order 2 and the other is of order $4g'+2$. In the preprint [4], the author proved that the above result is also true for $g' = 3, 4$. We found that the method in [3, 4] can be used in some of the cases of $\mathrm{MCG}(N_g)$'s and $\mathscr{T}(N_g)$'s. We have the following result:

Theorem 1. *Let N_g, $\mathrm{MCG}(N_g)$, $\mathscr{T}(N_g)$ be as above.*

(1) *If $g = 4k+3$ $(k \geq 1)$ (i.e. $g = 7, 11, 15\dots$), $\mathrm{MCG}(N_g)$ can be generated by three elements of finite order. In the generating set, one of the generators is of order $2g$, and the other two are of order 2.*

(2) *If $g = 4k+1$ $(k \geq 2)$ (i.e. $g = 9, 13, 17\dots$), $\mathscr{T}(N_g)$ can be generated by three elements of finite order. In the generating set, one of the generators is of order $2g$, and the other two are of order 2.*

2. Preliminary

Crosscap slide.

In [6, 7], Lickorish proved that $[\mathrm{MCG}(N_g) : \mathscr{T}(N_g)] = 2$. As an example of the mapping classes which do not lie in $\mathscr{T}(N_g)$, he described a mapping class so-called a "Y-homeomorphism" or a "crosscap slide" as shown in Figure 1.

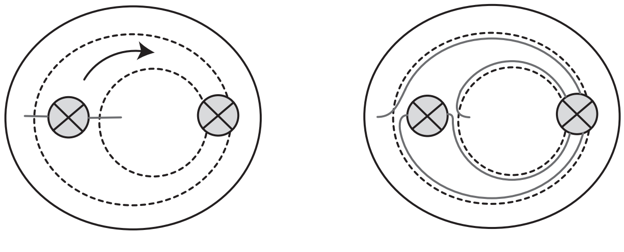

Fig. 1

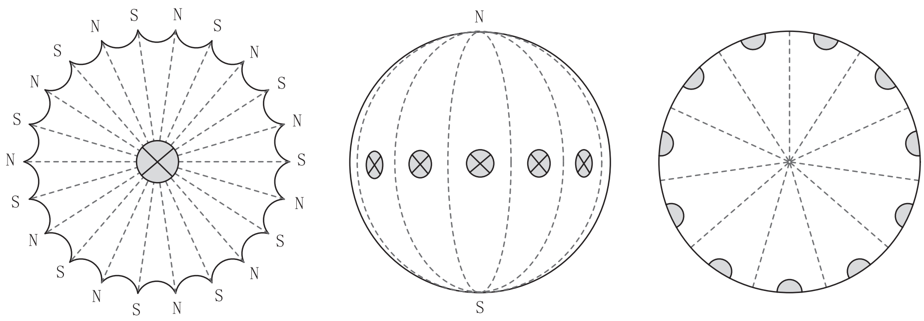

Fig. 2

Two points of view for the Möbius band partition of a non-orientable surface of odd genus.

If g is odd, we can decompose the non-orientable surface N_g into g Möbius bands. Figure 2 shows two points of view to do this.

(1) The left picture of Figure 2 is a $2g$-gon with a crosscap in the middle, and the opposite sides glued together pairwise. Under this gluing, the vertices of this $2g$-gon are divided into two equivalence classes. After the gluing, they form two points on N_g. We denote them by N and S. There are g arcs in dotted lines connecting pairs of antipodal vertices and passing through the crosscap in the middle of the $2g$-gon. They cut the $2g$-gon into g strips. After the gluing of the opposite sides of the $2g$-gon, they form g Möbius bands. We call it the *$2g$-gon presentation* of N_g.

(2) The middle and the right pictures of Figure 2 show a 2-sphere with g crosscaps. This is also N_g. Suppose the g crosscaps sit on the equator. Denote the north pole and the south pole by N and S, respectively. There are g arcs in dotted lines connecting N and S. They cut N_g into g Möbius bands. We call it the *g-crosscap presentation* of N_g.

We can check the above two presentations of N_g are equivalent. In fact, in both presentations, we cut N_g into g Möbius bands. The points N and S are on the boundaries of these Möbius bands. We can build a homeomorphism on each Möbius band and then glue them together to make a global homeomorphism between the $2g$-gon presentation of the surface and the g-crosscap presentation of the surface. In the following, we will go back and forth between the two presentations.

Notations.

(a) We use the convention of functional notation, namely, elements of the mapping class group are applied right to left, i.e. the composition FG means that G is applied first.

(b) On an orientable surface, for each explicit two-sided simple closed curve, a Dehn twist means a right-handed Dehn twist along such a curve, and a left-handed Dehn twist is the inverse of a right-handed Dehn twist. On a non-orientable surface of odd genus, such as the left picture of Figure 2, we can cut off the crosscap in the middle of the $2g$-gon presentation to get an orientable subsurface. So for each simple closed curve which is disjoint from the crosscap in the middle of the $2g$-gon presentation, we can still define the right-handed Dehn twist in the oriented subsurface.

(c) We denote the curves by lower-case letters a, b, c, d (possibly with subscripts) and the Dehn twists about them by the corresponding capital letters A, B, C, D. Notationally we do not distinguish a diffeomorphism/curve and its isotopy class.

The curves needed for generating $\mathscr{T}(N_g)$.

Omori constructed a generating set which consists of $g+1$ Dehn twists for $\mathscr{T}(N_g)$ ([8]). When we use the g-crosscap presentation of N_g, the curves for those Dehn twists are $a_1, a_2, \dots, a_{g-1}, b_0, e$ shown in Figure 3. We can check that a Dehn twist along a_1 maps e to the curve c in Figure 3. Hence the Dehn twists along $a_1, a_2, \dots, a_{g-1}, b_0, c$ can also generate $\mathscr{T}(N_g)$.

We can also use the $2g$-gon presentation to see what these curves are. See Figure 4.

We illustrate the verification of the correspondence of such curves as follows. The curves $a_1, a_2, \dots, a_{g-1}$ form a chain of curves on N_g. Here a chain of curves means a set of curves $a_1, a_2, \dots, a_{g-1}$ satisfying the following geometric intersection number conditions: (1) $i(a_j, a_{j+1}) = 1$ $(j = 1, 2, \dots, g-1)$; (2) $i(a_j, a_k) = 0$ $(|j-k| > 1)$. If we cut N_g along $a_1, a_2, \dots, a_{g-1}$, we can check that $N_g - \bigcup_{j=1}^{g-1} a_j$ is a Möbius band or a disk with a crosscap in the middle. The boundary of $N_g - \bigcup_{j=1}^{g-1} a_j$ consists of subarcs of a_j's. Each

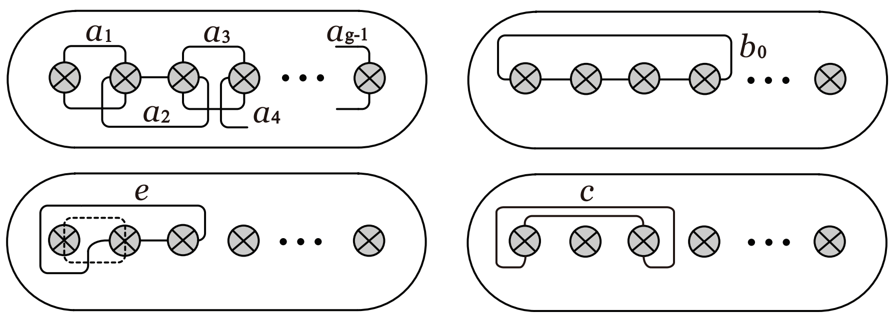

Fig. 3

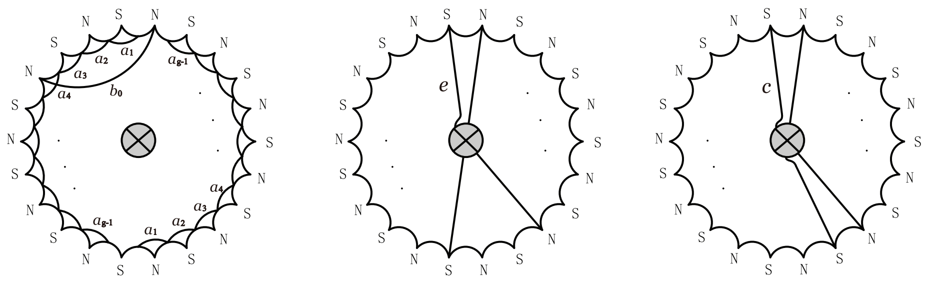

Fig. 4

two-sided curve γ on N_g will be cut into a union of arcs on $N_g - \bigcup_{j=1}^{g-1} a_j$. The end points of these arcs lie on the boundary of $N_g - \bigcup_{j=1}^{g-1} a_j$. These end points correspond to the intersection points of γ with a_j's. Each arc on $N_g - \bigcup_{j=1}^{g-1} a_j$ is determined by its end points on the boundary and its relative position with the crosscap in the middle of the disk. Hence we can detect γ by its intersection points with a_j's and the resulting arcs on $N_g - \bigcup_{j=1}^{g-1} a_j$. This gives the correspondence of the curves in both presentations of the non-orientable surface.

3. The proof of the main theorem

We now give the proof of Theorem 1.1.

Proof (Proof of Theorem 1.1). We first give the torsion generators. Suppose g is odd. See Figure 5. Let σ be the rotation of the $2g$-gon presentation, τ_1 the reflection of the $2g$-gon presentation that preserves the curve b_0, and τ_2 the reflection of the g-crosscap presentation that preserves c. We can easily see that $(\tau_1 \circ B_0)^2 = 1$, $(\tau_2 \circ C)^2 = 1$, $\sigma^{2g} = 1$.

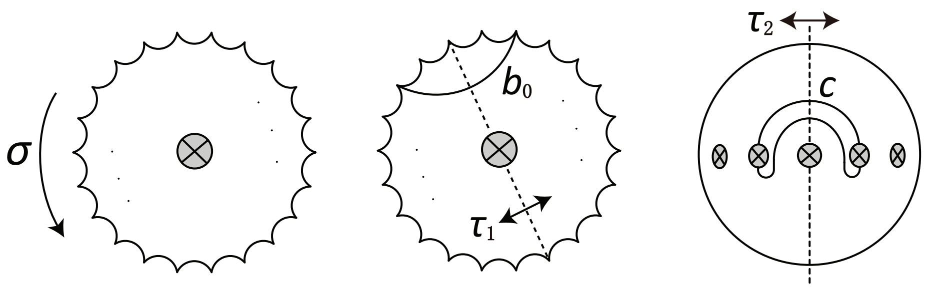

Fig. 5

Let $G = \langle \sigma, \tau_1 \circ B_0, \tau_2 \circ C \rangle$ be the subgroup of $\mathrm{MCG}(N_g)$ generated by these three elements of finite orders. We will prove that: (1) if $g = 7, 11, 15\ldots$, then $G = \mathrm{MCG}(N_g)$; (2) if $g = 9, 13, 17\ldots$, then $G = \mathscr{T}(N_g)$.

The proof is by the following steps:

Step 1. Under the given conditions, we prove G includes $A_1, \ldots, A_{g-1}, B_0, \tau_1$, and σ. Here $A_1, \ldots, A_{g-1}$, and B_0 are the Dehn twists along the curves $a_1, \ldots, a_{g-1}$, and b_0, respectively. They are shown in Figure 3 and 4.

Step 2. We check τ_2 is conjugate to τ_1 by some power of σ and then τ_2 is in G. Hence C is also in G. Here C is the Dehn twist along the curve c shown in Figure 3 and 4.

Step 3. By Omori's result [8], the fact that $A_1, \ldots, A_{g-1}, B_0, C$ are in G implies G includes $\mathscr{T}(N_g)$. Recall that $[\mathrm{MCG}(N_g) : \mathscr{T}(N_g)] = 2$. Hence G is either $\mathscr{T}(N_g)$ or $\mathrm{MCG}(N_g)$.

Step 4. We check whether τ_1 lies in $\mathscr{T}(N_g)$. If τ_1 lies in $\mathscr{T}(N_g)$, then all the generators of G is in $\mathscr{T}(N_g)$. Hence $G = \mathscr{T}(N_g)$. If τ_1 does not lie in $\mathscr{T}(N_g)$, then $G = \mathrm{MCG}(N_g)$.

The proof of Step 1:

Take the $2g$-gon presentation of N_g (g is odd). If we remove the crosscap in the middle, then we get an orientable surface with genus $\frac{g-1}{2}$. In [3] and [4], for orientable surfaces, using the $2g$-gon presentation, we generate $\mathrm{MCG}^{\pm}(S_{(g-1)/2})$ by σ and $\tau_1 \circ B_0$ when $\frac{g-1}{2} \geq 3$. Here for the non-orientable surfaces, the method is similar. All the curves in the proof will not pass through the crosscap in the middle of the $2g$-gon. In the following, we illustrate the main idea. For details, see [3] and [4]. We use the lantern relation $ABCD = XYZ$, where a, b, c, d, x, y, z are the curves on a 4-holed sphere. The lantern relation can also be written as $D = (XA^{-1})(YB^{-1})(ZC^{-1})$. So one Dehn twist can be decomposed into a product of three pairs. Each pair consists of a left-handed Dehn twist and a right-handed Dehn twist. If we denote $b_k = \sigma^k(b_0)$, then we can see $\sigma^k(\tau_1 \circ B_0)\sigma^k(\tau_1 \circ B_0) = B_k^{-1}B_0$. Hence from σ and $\tau_1 \circ B_0$, we can get a pair, which consists of a left-handed Dehn twist and a right-handed Dehn twist. Conjugate such a pair by elements in G, we get many similar pairs, which include the three pairs XA^{-1}, YB^{-1}, and ZC^{-1} in the lantern relation. So there is at least one Dehn twist in G. We can also check such a Dehn twist can be chosen to be some A_j or B_k. All a_j's are in the same σ-orbit. So every A_j is in G. Similar for B_k's. The elements $\tau_1 \circ B_0$ and B_0 are in G, so τ_1 is in G. The neighbourhood of $\bigcup_{j=1}^{g-1} a_j$ is a one-holed orientable surface of genus $\frac{g-1}{2}$. By the chain relation, $(A_{g-1}A_{g-2}\ldots A_1)^{2g}$ is a Dehn twist along the boundary curve of such a one-holed orientable surface. Such a curve bounds the crosscap in the middle of the $2g$-gon presentation of N_g.

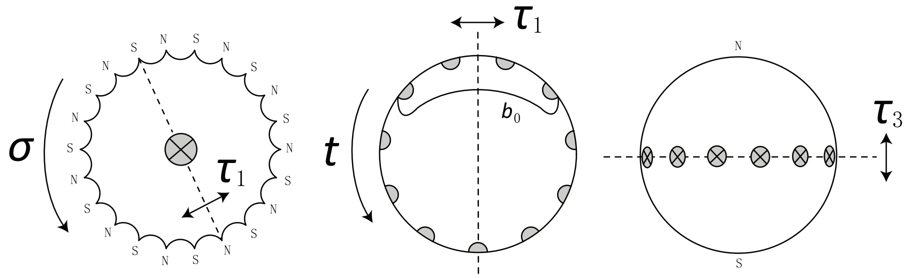

Fig. 6

The Dehn twist along such a curve is trivial. Hence $A_{g-1}A_{g-2}\dots A_1$ equals the rotation σ^{-1}, and so σ is in G.

The proof of Step 2:

We can interpret some of the torsion elements in more geometric ways. See Figure 6. We can check that τ_1 is not only a reflection in the $2g$-gon presentation but also a reflection in the g-crosscap presentation. Let τ_3 be the north-south reflection of the g-crosscap presentation of N_g, t be the order g rotation. Since σ gives a permutation of the g Möbius bands and interchanges N and S, we can see $\sigma = t \circ \tau_3$ and $\tau_3 = \sigma^g$. Hence τ_3 and t are also in G. Now τ_2 is conjugated to τ_1 by some power of t. So τ_2 also lies in G. Hence C lies in G.

The proof of Step 3 is trivial.

The proof of Step 4:

In [7], Lickorish gave the following result: for a mapping class f in $\mathrm{MCG}(N_g)$ and its induced automorphism f_* on the $\mathbb{R}$-coefficient homology group $H_1(N_g;\mathbb{R})$, the element f lies in $\mathscr{T}(N_g)$ (resp. does not lies in $\mathscr{T}(N_g)$) if and only if f_* has determinant $+1$ (resp. -1). In the g-crosscap presentation of N_g, take g one-sided simple closed curves which are the core curves of the g crosscaps. Since τ_1 is a reflection of the g-crosscap presentation, it exchanges $g-1$ core curves pairwise and reverse their orientations. These $g-1$ core curves form a basis for $H_1(N_g;\mathbb{R})$. The induced automorphism $(\tau_1)_*$ of $H_1(N_g;\mathbb{R})$ with respect to such a basis gives a $(g-1)\times(g-1)$-matrix

$$\begin{pmatrix} 0 & 0 & \dots & 0 & -1 \\ 0 & 0 & \dots & -1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & -1 & \dots & 0 & 0 \\ -1 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

When $g = 4k + 3$, the determinant is -1, τ_1 does not lie in $\mathscr{T}(N_g)$. When $g = 4k + 1$, the determinant is $+1$, τ_1 lies in $\mathscr{T}(N_g)$. This completes the proof. $\square$

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