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# Degeneration of Fermat hypersurfaces in positive characteristic

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**Abstract.** We work over an algebraically closed field $k$ of positive characteristic $p$. Let $q$ be a power of $p$. Let $A$ be an $(n+1) \times (n+1)$ matrix with coefficients $a_{ij}$ in $k$, and let $X_A$ be a hypersurface of degree $q+1$ in the projective space $\mathbf{P}^n$ defined by $\sum a_{ij} x_i x_j^q = 0$. It is well-known that if the rank of $A$ is $n+1$, the hypersurface $X_A$ is projectively isomorphic to the Fermat hypersuface of degree $q+1$. We investigate the hypersurfaces $X_A$ when the rank of $A$ is $n$, and determine their projective isomorphism classes.

## 1. Introduction

We work over an algebraically closed field $k$ of positive characteristic $p$. Let $q$ be a power of $p$. Let $n$ be a positive integer. We denote by $M_{n+1}(k)$ the set of square matrices of size $n+1$ with coefficients in $k$. For a nonzero matrix $A = (a_{ij})_{0 \le i,j \le n} \in M_{n+1}(k)$, we denote by $X_A$ the hypersurface of degree $q+1$ defined by the equation

$$\sum a_{ij} x_i x_j^q = 0$$

in the projective space $\mathbf{P}^n$ with homogeneous coordinates $(x_0, x_1, \dots, x_n)$. The following is well-known ([2], [10], [14], see also §4 of this paper).

Proposition 1. *Let $A = (a_{ij})_{0 \le i,j \le n} \in M_{n+1}(k)$ and $X_A \subset \mathbf{P}^n$ be as above. Then the following conditions are equivalent:*

( i ) $\operatorname{rank}(A) = n+1$,
( ii ) *$X_A$ is smooth,*
(iii) *$X_A$ is isomorphic to the Fermat hypersurface of degree $q+1$, and*
(iv) *there exists a linear transformation of coordinates $T \in GL_{n+1}(k)$ such that ${}^tTAT^{(q)} = I_{n+1}$, where ${}^tT$ is the transpose of $T$, $T^{(q)}$ is the matrix obtained from $T$ by raising each coefficient to its $q$-th power, and $I_{n+1}$ is the identity matrix.*

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The Fermat hypersurface of degree $q+1$ defined over an algebraically closed field of positive characteristic $p$ has been a subject of numerous papers. It has many interesting properties, such as supersingularity ([15], [16], [17]), or unirationality ([13], [15], [16]). Moreover, the hypersurface $X_A$ associated with the matrix $A$ with coefficients $a_{ij}$ in the finite field $\mathbf{F}_{q^2}$, which is called a Hermitian variety, has also been studied for many applications, such as coding theory ([8]). (The general results on Hermitian varieties are due to Segre [11]; see also [6]). Therefore it is important to extend these studies to degenerate cases.

In the case where characteristic $p \neq 2$, the following is well-known and can be found in any standard textbook on quadratic forms: the hypersurface defined by the quadratic form $\sum a_{ij}x_i x_j = 0$ is projectively isomorphic to the hypersurface defined by

$$x_0^2 + \cdots + x_{r-1}^2 = 0,$$

where $r$ is the rank of $A = (a_{ij})$. This result has been extended the case of characteristic 2 (see [3]). Therefore we have a question what is the normal form of the hypersurfaces defined by a form $\sum a_{ij}x_i x_j^q = 0$. When $A$ satisfies ${}^tA = A^{(q)}$ and hence this form is the Hermitian form over $\mathbf{F}_q$, the hypersurface $X_A$ is projectively isomorphic over $\mathbf{F}_{q^2}$ to

$$x_0^{q+1} + \cdots + x_{r-1}^{q+1} = 0,$$

where $r$ is the rank of $A$ ([5]).

In this paper, we classify the hypersurfaces $X_A$ associated with the matrices $A$ of rank $n$ *over an algebraically closed field*. Note that two hypersurfaces $X_A$, $X_{A'}$ associated with the matrices $A$, $A'$ are projectively isomorphic if and only if there exists a linear transformation $T \in GL_{n+1}(k)$ such that $A' = {}^tTAT^{(q)}$. In this case, we write $A \sim A'$.

We define $I_s$ to be the $s \times s$ identity matrix, and $E_r$ to be the $r \times r$ matrix

$$\begin{pmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}.$$

In particular, $E_1 = (0)$ and $E_0$ is the $0 \times 0$ matrix. Throughout this paper, a blank in a block decomposition of a matrix means that all the components of the block are 0. Our main result is as follow.

Theorem 1. *Let* $A = (a_{ij})_{0 \le i,j \le n}$ *be a nonzero matrix in* $M_{n+1}(k)$*, and let* $X_A$ *be the hypersurface of degree* $q+1$ *defined by* $\sum a_{ij} x_i x_j^q = 0$ *in the projective space* $\mathbf{P}^n$ *with homogeneous coordinates* $(x_0, x_1, \dots, x_n)$*. Suppose that the rank of* $A$ *is* $n$*. Then the hypersurface* $X_A$ *is projectively isomorphic to one of the hypersurfaces* $X_s$ *associated with the matrices*

$$W_s = \left( \begin{array}{c|c} I_s & \\ \hline & E_{n-s+1} \end{array} \right),$$

*where* $0 \le s \le n$*. Moreover, if* $s \ne s'$*, then* $X_s$ *and* $X_{s'}$ *are not projectively isomorphic.*

Corollary 1. *If* $A$ *is a general point of* $\{A \in M_{n+1}(k) \mid rank(A) = n\}$*, then* $A \sim W_{n-1}$.

Corollary 2. *Suppose that* $n \ge 2$, $s < n$ *and* $(n,s) \ne (2,0)$*. Then* $X_s$ *is rational.*

We also determine the automorphism group

$$\mathrm{Aut}(X_s) = \{g \in PGL_{n+1}(k) \mid g(X_s) = X_s\},$$

of the hypersurface $X_s$ for each $s$. For $M \in GL_{n+1}(k)$, we denote by $[M] \in PGL_{n+1}(k)$ the image of $M$ by the natural projection.

Theorem 2. *Let* $X_s$ *be the hypersurface associated with the matrix* $W_s$ *in the projective space* $\mathbf{P}^n$*. The projective automorphism group* $\mathrm{Aut}(X_s)$ *with* $s \le n-2$ *is the group consisting of* $[M]$*, with*

$$M = \left( \begin{array}{c|c|c} T & {}^t\mathbf{a} & 0 \\ \hline 0 & d & 0 \\ \hline \mathbf{c} & e & 1 \end{array} \right),$$

*where* $T \in GL_{n-1}(k)$, $\mathbf{a}, \mathbf{c}$ *are row vectors of dimension* $n-1$, $d, e \in k$*, and they satisfy the following conditions:*

(i) $[T] \in \mathrm{Aut}(X_s^{n-2})$, ${}^tTW_s'T^{(q)} = \delta W_s'$, $\delta = \delta^q \ne 0$, *where* $X_s^{n-2}$ *is the hypersurface defined in* $\mathbf{P}^{n-2}$ *by the matrix*

$$W_s' = \left( \begin{array}{c|c} I_s & \\ \hline & E_{n-s-1} \end{array} \right)$$

(ii) $d = \delta$,
(iii) $[\mathbf{a}W_s' + d(0,\dots,0,1)] \cdot T^{(q)} = \delta(0,\dots,0,1)$,
(iv) ${}^tTW_s' \cdot {}^t\mathbf{a}^{(q)} + {}^t\mathbf{c}d^q = 0$, *and*
(v) $[\mathbf{a}W_s' + d(0,\dots,0,1)] \cdot {}^t\mathbf{a}^{(q)} + ed^q = 0$.

*Moreover, we have*

$$\mathrm{Aut}(X_n) = \left\{ \left[ \begin{array}{c|c} T_n & \\ \hline \mathbf{u} & 1 \end{array} \right] \;\middle|\; \begin{array}{l} {}^tT_nT_n^{(q)} = \lambda I_n,\ T_n \in GL_n(k),\ \lambda \neq 0, \\ \mathbf{u} \textit{ is a row vector of dimension } n \end{array} \right\},$$

*and*

$$\mathrm{Aut}(X_{n-1}) = \left\{ \left[ \begin{array}{c|c|c} T_{n-1} & & \\ \hline & \beta & \\ \hline & & 1 \end{array} \right] \;\middle|\; \begin{array}{l} {}^tT_{n-1}T_{n-1}^{(q)} = \beta^q I_{n-1}, \\ T_{n-1} \in GL_{n-1}(k),\ 0 \neq \beta \in k \end{array} \right\}$$

We give a brief outline of our paper. In §2, we prove Theorem 1 and its corollaries. In §3, we prove Theorem 2. In §4, we recall the proof of Proposition 1 because this proposition plays an important role in the proof of Theorem 1. In §5, we investigate the plane curve $X_A$ associated with the matrix $A$ of rank $\leq 2$ in the projective plane $\mathbf{P}^2$, and recovers Homma's unpublished work [9] (see Remark 5).

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## 2. Proofs of Theorem 1 and its corollaries

We present several preliminary lemmas. The following remark may be helpful in reading the proof of lemmas.

Remark 1. *Let*

$$T = \begin{pmatrix} t_{00} & \cdots & t_{0n} \\ \vdots & \ddots & \vdots \\ t_{n0} & \cdots & t_{nn} \end{pmatrix}$$

*be an invertible matrix. Suppose that* $\sum a_{ij}x_ix_j^q = 0$ *is the equation associated to a matrix* $A = (a_{ij})_{0 \leq i,j \leq n}$. *Then the operation*

$$A \mapsto {}^tTAT^{(q)}$$

*on the matrix is equivalent to the transformation of the coordinates*

$$x_i \mapsto \sum_{j=0}^{n} t_{ij} x_j,$$

*where* $0 \le i \le n$.

Lemma 1. *Put*

$$G_{s,r} = \left( \begin{array}{c|c|c} I_s & & \\ \hline & E_r & \\ \hline \mathbf{a} & 0 \cdots 0 \ 1 & \\ 0 & 0 & \\ \vdots & \vdots & E_{n-s-r+1} \\ 0 & 0 & \end{array} \right),$$

*and*

$$G_{s,r+2} = \left( \begin{array}{c|c|c} I_s & & \\ \hline & E_{r+2} & \\ \hline \mathbf{a}^{(q^2)} & 0 \cdots 0 \ 1 & \\ 0 & 0 & \\ \vdots & \vdots & E_{n-s-r-1} \\ 0 & 0 & \end{array} \right),$$

*where* $s \ge 1$, $r \ge 0$, $n - s - r - 1 \ge 0$, *and* $\mathbf{a}$ *is a nonzero row vector of dimension* $s$. *Then*

$$G_{s,r} \sim G_{s,r+2}.$$

Proof. By the transformation

$$T_G = \left( \begin{array}{c|c|c|c|c} I_s & & -{}^t\mathbf{a} & & \\ \hline & I_r & & & \\ \hline & & 1 & & \\ \hline \mathbf{a}^{(q)} & & & 1 & \\ \hline & & & & I_{n-s-r-1} \end{array} \right),$$

we have

$$ {}^{t}T_G G_{s,r} T_G^{(q)} = G_{s,r+2}. \qquad \square $$

Remark 2. *Lemma* 1 *holds when $r = 0$ or $n - s - r - 1 = 0$. In particular, when $n - s - r - 1 = 0$, we have $G_{s,r+2} = W_s$.*

Lemma 2. *Put*

$$ H_{s,r} = \left( \begin{array}{c|c|c|c} D_{s-1} & -{}^{t}\mathbf{a}''\ 0 \cdots 0 & & \\ \hline \begin{matrix} -\mathbf{a}' \\ 0 \\ \vdots \\ 0 \end{matrix} & E_r & & \\ \hline & 0 \cdots 0\ 1 & 1 & \\ \hline & & \begin{matrix} 1 \\ 0 \\ \vdots \\ 0 \end{matrix} & E_{n-s-r+1} \end{array} \right), $$

*where $s \geq 1$, $r \geq 2$, $n - s - r - 1 \geq 1$, $D_{s-1} \in M_{s-1}(k)$, $\mathbf{a}'$ and $\mathbf{a}''$ are row vectors of dimension $s - 1$. Then*

$$ H_{s,r} \sim H_{s,r+2}. $$

Proof. By the transformation

$$ T_H = \left( \begin{array}{c|c|c|c|c} I_{s+r-1} & & & & \\ \hline & 1 & & & \\ \hline & -1 & 1 & 1 & \\ \hline & & & 1 & \\ \hline & & & & I_{n-s-r-1} \end{array} \right), $$

we have

$$ {}^{t}T_H H_{s,r} T_H^{(q)} = H_{s,r+2}. \qquad \square $$

LEMMA 3. *Put*

$$H'_{s,r} = \left(\begin{array}{c|c|c|c|c|c}
D_{s-1} & & & & & \\ \hline
-\mathbf{a}' & 0 & & & & \\ \hline
 & \begin{matrix}1\\0\\\vdots\\0\end{matrix} & E_r & & & \\ \hline
 & & 0\cdots0\ 1 & 0 & 1 & \\ \hline
 & & & 1 & 0 & \\ \hline
 & & & & \begin{matrix}1\\0\\\vdots\\0\end{matrix} & E_{n-s-r-1}
\end{array}\right),$$

*where* $s \geq 1$, $r \geq 2$, $n-s-r-3 \geq 1$, $D_{s-1} \in M_{s-1}(k)$, *and* $\mathbf{a}'$ *is a row vector of dimension* $s-1$. *Then*

$$H'_{s,r} \sim H'_{s,r+2}.$$

PROOF. By the transformation

$$T_{H'} = \left(\begin{array}{c|c|c|c|c|c}
I_{s+r} & & & & & \\ \hline
 & 1 & & & & \\ \hline
 & & 1 & & 1 & \\ \hline
 & -1 & & 1 & & \\ \hline
 & & & & 1 & \\ \hline
 & & & & & I_{n-s-r-3}
\end{array}\right),$$

we have

$${}^tT_{H'}H'_{s,r}T_{H'}^{(q)} = H'_{s,r+2}. \qquad \square$$

REMARK 3. *Lemma* 2 *and* 3 *will be used only in the case where* $n-s+1$ *is odd. Hence, we do not need to prove the case* $n-s-1=0$ *in Lemma* 2 *nor the case* $n-s-3=0$ *in Lemma* 3.

LEMMA 4. *Put*

$$P_s = \left(\begin{array}{c|c} I_s & \\ \hline \mathbf{a} & \\ 0 & \\ \vdots & E_{n-s+1} \\ 0 & \end{array}\right),$$

*where* $s \geq 1$, $n - s + 1 \geq 1$, *and* $\mathbf{a}$ *is a nonzero row vector of dimension* $s$. *Then*

(1) *If* $n - s + 1$ *is even, then* $P_s \sim W_s$.

(2) *If* $n - s + 1$ *is odd, then*

$$P_s \sim B_{s-1} = \left(\begin{array}{c|c} D_{s-1} & \\ \hline \mathbf{b}_{s-1} & \\ 0 & \\ \vdots & E_{n-s+2} \\ 0 & \end{array}\right),$$

*where* $D_{s-1} \in M_{s-1}(k)$, $\mathbf{b}_{s-1}$ *is the row vector of dimension* $s-1$. *In particular, if* $s = 1$ *and* $n$ *is odd, then* $P_1 \sim W_0$.

PROOF. (1) Suppose that $n - s + 1$ is even. Using Lemma 1 and Remark 2, we have

$$P_s = G_{s,0} \sim G_{s,n-s+1} = W_s.$$

(2) Next, suppose that $n - s + 1$ is odd. By interchanging the coordinates $x_0, \dots, x_{s-1}$, and scalar multiplication of the coordinates $x_s, \dots, x_n$ if nessesary, we can show that

$$P_s \sim P_s' = \left(\begin{array}{c|c|c|c} I_{s-1} & & & \\ \hline & 1 & & \\ \hline \mathbf{a}' & 1 & 0 & \\ \hline & & 1 & \\ & & 0 & \\ & & \vdots & E_{n-s} \\ & & 0 & \end{array}\right),$$

with $\mathbf{a}'$ being a row vector of dimension $s-1$. By the transformation

$$T_1 = \left(\begin{array}{c|c|c|c} I_{s-1} & & & \\ \hline -\mathbf{a}'' & 1 & & \\ \hline & & 1 & \\ \hline & & & I_{n-s} \end{array}\right),$$

with $\mathbf{a}''^{(q)} = \mathbf{a}'$, we have

$$Q_s = {}^tT_1 P_s' T_1^{(q)} = \left(\begin{array}{c|c|c|c} D_{s-1} & -{}^t\mathbf{a}'' & & \\ \hline -\mathbf{a}' & 1 & & \\ \hline & 1 & 0 & \\ \hline & & \begin{matrix} 1 \\ 0 \\ \vdots \\ 0 \end{matrix} & E_{n-s} \end{array}\right),$$

where $D_{s-1} = I_{s-1} + {}^t\mathbf{a}'' \cdot \mathbf{a}'$. If $n-s+1 = 1$, by the transformation

$$T_2 = \left(\begin{array}{c|c|c} I_{n-1} & & \\ \hline & 1 & \\ \hline \mathbf{a}'' & -1 & 1 \end{array}\right),$$

we have

$${}^tT_2 Q_n T_2^{(q)} = B_{n-1}.$$

Suppose that $n-s+1 > 1$. Note that, since we are in the case where $n-s+1$ is odd, we have $n-s+1 \geq 3$. By the transformation

$$T_3 = \left(\begin{array}{c|c|c|c|c} I_{s-1} & & & & \\ \hline & 1 & & & \\ \hline & -1 & 1 & 1 & \\ \hline & & & 1 & \\ \hline & & & & I_{n-s-1} \end{array}\right),$$

we have

$$
Q_s' = {}^tT_3 Q_s T_3^{(q)} = \left(\begin{array}{c|c|c|c|c}
D_{s-1} & -{}^t\mathbf{a}'' & & & \\ \hline
-\mathbf{a}' & 0 & & & \\ \hline
 & 1 & 0 & & \\ \hline
 & & 1 & 1 & \\ \hline
 & & & \begin{matrix} 1 \\ 0 \\ \vdots \\ 0 \end{matrix} & E_{n-s-1}
\end{array}\right) = H_{s,2}.
$$

Using Lemma 2, we have

$$
Q_s' = H_{s,2} \sim H_{s,n-s} = Q_s'' = \left(\begin{array}{c|c|c|c}
D_{s-1} & -{}^t\mathbf{a}''\ 0 \cdots 0 & & \\ \hline
\begin{matrix} -\mathbf{a}' \\ 0 \\ \vdots \\ 0 \end{matrix} & E_{n-s} & & \\ \hline
 & 0 \cdots 0\ 1 & 1 & \\ \hline
 & & 1 & 0
\end{array}\right).
$$

Then by the transformation

$$
T_4 = \left(\begin{array}{c|c|c}
I_{n-1} & & \\ \hline
 & 1 & \\ \hline
 & -1 & 1
\end{array}\right),
$$

we have

$$
R_s = {}^tT_4 Q_s'' T_4^{(q)} = \left(\begin{array}{c|c}
D_{s-1} & -{}^t\mathbf{a}''\ 0 \cdots 0 \\ \hline
\begin{matrix} -\mathbf{a}' \\ 0 \\ \vdots \\ 0 \end{matrix} & E_{n-s+2}
\end{array}\right).
$$

If $s = 1$, $R_1 \sim W_0$. Suppose that $s > 1$. By the transformation

$$T_5 = \left(\begin{array}{c|c|c|c|c} I_{s-1} & & & & \\ \hline & 1 & & 1 & \\ \hline \mathbf{a}'' & & 1 & & \\ \hline & & & 1 & \\ \hline & & & & I_{n-s-1} \end{array}\right),$$

we obtain

$$R_s' = {}^tT_5 R_s T_5^{(q)} = \left(\begin{array}{c|c|c|c|c} D_{s-1} & & & & \\ \hline -\mathbf{a}' & 0 & & & \\ \hline & 1 & 0 & 1 & \\ \hline & & 1 & 0 & \\ \hline & & & \begin{matrix} 1 \\ 0 \\ \vdots \\ 0 \end{matrix} & E_{n-s-1} \end{array}\right).$$

If $n - s - 1 = 1$, by the tranformation

$$T_6 = \left(\begin{array}{c|c|c|c} I_{n-2} & & & \\ \hline & 1 & & \\ \hline & & 1 & \\ \hline & -1 & & 1 \end{array}\right),$$

we have

$${}^tT_6 R_{n-2}' T_6^{(q)} = B_{n-3}.$$

Suppose that $n - s - 1 > 1$. Then by the transformation

$$T_7 = \left(\begin{array}{c|c|c|c|c|c} I_s & & & & & \\ \hline & 1 & & & & \\ \hline & & 1 & & 1 & \\ \hline & -1 & & 1 & & \\ \hline & & & & 1 & \\ \hline & & & & & I_{n-s-3} \end{array}\right),$$

we have

$$R_s'' = {}^tT_7 R_s' T_7^{(q)} = \left(\begin{array}{c|c|c|c|c|c}
D_{s-1} & & & & & \\ \hline
-\mathbf{a}' & 0 & & & & \\ \hline
 & \begin{matrix} 1 \\ 0 \end{matrix} & E_2 & & & \\ \hline
 & & 0\ 1 & 0 & 1 & \\ \hline
 & & & 1 & 0 & \\ \hline
 & & & & \begin{matrix} 1 \\ 0 \\ \vdots \\ 0 \end{matrix} & E_{n-s-3}
\end{array}\right) = H_{s,2}'.$$

Using Lemma 3, we have

$$R_s'' = H_{s,2}' \sim H_{s,n-s-2}' = R_s''' = \left(\begin{array}{c|c|c|c|c|c}
D_{s-1} & & & & & \\ \hline
-\mathbf{a}' & 0 & & & & \\ \hline
 & \begin{matrix} 1 \\ 0 \\ \vdots \\ 0 \end{matrix} & E_{n-s-2} & & & \\ \hline
 & & 0 \cdots 0\ 1 & 0 & 1 & \\ \hline
 & & & 1 & 0 & \\ \hline
 & & & & 1 & 0
\end{array}\right).$$

It is easy to see that

$${}^tT_6 R_s''' T_6^{(q)} = B_{s-1}. \qquad \square$$

Lemma 5. *Put*

$$B_s = \left(\begin{array}{c|c}
D_s & \\ \hline
\begin{matrix} \mathbf{b}_s \\ 0 \\ \vdots \\ 0 \end{matrix} & E_{n-s+1}
\end{array}\right),$$

*where* $s \geq 1$, $n - s + 1 \geq 1$, $D_s \in M_s(k)$, *and* $\mathbf{b}_s$ *is a row vector of dimension* $s$. *Suppose that the rank of* $B_s$ *is* $n$. *Then*

$$B_s \sim W_s = \left(\begin{array}{c|c} I_s & \\ \hline & E_{n-s+1} \end{array}\right),$$

*or*

$$B_s \sim B_{s-1} = \left(\begin{array}{c|c} D_{s-1} & \\ \hline \mathbf{b}_{s-1} & \\ 0 & \\ \vdots & E_{n-s+2} \\ 0 & \end{array}\right),$$

*where* $D_{s-1} \in M_{s-1}(k)$, *and* $\mathbf{b}_{s-1}$ *is a row vector of dimension* $s - 1$.

Proof. Suppose that $\det D_s \neq 0$. By Proposition 1, there exists a linear transformation of coordinates $T_D \in GL_s(k)$ such that ${}^tT_D D_s T_D^{(q)} = I_s$. By the transformation

$$T = \left(\begin{array}{c|c} T_D & \\ \hline & I_{n-s+1} \end{array}\right),$$

we have

$${}^tTB_sT^{(q)} = \left(\begin{array}{c|c} I_s & \\ \hline \mathbf{b}_s' & \\ 0 & \\ \vdots & E_{n-s+1} \\ 0 & \end{array}\right),$$

where $\mathbf{b}_s' = \mathbf{b}_s T_D^{(q)}$. If $\mathbf{b}_s' = 0$, then $B_s \sim W_s$. Suppose that $\mathbf{b}_s' \neq 0$. By Lemma 4, we have $B_s \sim W_s$, or $B_s \sim B_{s-1}$.

Suppose that $\det D_s = 0$. Then one row of the matrix $D_s$ is a linear combination of the other rows. By interchanging coordinates $x_0, \dots, x_{s-1}$ if nessesary, we can assume that the $s$-th row is a linear combination of the other rows. We write the matrix $D_s$ as

$$D_s = \left(\begin{array}{c|c} P & {}^t\mathbf{g} \\ \hline \mathbf{h} & d \end{array}\right),$$

where $P \in M_{s-1}(k)$, $\mathbf{g}$, $\mathbf{h}$ are row vectors of dimension $s - 1$, $d \in k$, and that satisfy $\mathbf{h} = \mathbf{w}P$, $d = \mathbf{w}{}^t\mathbf{g}$ with $\mathbf{w}$ being a row vector of dimension $s - 1$. Then

$$B_s \sim B_s' = \left(\begin{array}{c|c|c} P & {}^t\mathbf{g} & \\ \hline \mathbf{h} & d & \\ \hline \mathbf{f} & e & \\ 0 & 0 & \\ \vdots & \vdots & E_{n-s+1} \\ 0 & 0 & \end{array}\right),$$

where $\mathbf{f}$ is a row vector of dimension $s-1$, and $e \in k$. By the tranformation

$$T' = \left(\begin{array}{c|c|c} I_{s-1} & -{}^t\mathbf{w} & \\ \hline & 1 & \\ \hline & & I_{n-s+1} \end{array}\right),$$

we obtain

$$B_s'' = {}^tT'B_s'T'^{(q)} = \left(\begin{array}{c|c|c} P & -P \cdot {}^t\mathbf{w}^{(q)} + {}^t\mathbf{g} & \\ \hline & & \\ \hline \mathbf{f} & -\mathbf{f} \cdot {}^t\mathbf{w}^{(q)} + e & \\ 0 & 0 & \\ \vdots & \vdots & E_{n-s+1} \\ 0 & 0 & \end{array}\right).$$

Put

$$Q = \left(\begin{array}{c|c} P & -P \cdot {}^t\mathbf{w}^{(q)} + {}^t\mathbf{g} \\ \hline \mathbf{f} & -\mathbf{f} \cdot {}^t\mathbf{w}^{(q)} + e \end{array}\right).$$

Because the rank of $B_s'$ is $n$, we have $\det Q \neq 0$. Let $Q' \in GL_s(k)$ such that $QQ'^{(q)} = I_s$,

$$Q' = \left(\begin{array}{c|c} P' & {}^t\mathbf{g}' \\ \hline \mathbf{f}' & e' \end{array}\right),$$

where $P' \in M_{s-1}(k)$, $\mathbf{g}'$, $\mathbf{f}'$ are row vectors of dimension $s-1$, $e' \in k$. By the transformation

$$T'' = \left(\begin{array}{c|c|c} P' & {}^t\mathbf{g}' & \\ \hline \mathbf{f}' & e' & \\ \hline & & I_{n-s+1} \end{array}\right),$$

we obtain

$$ {}^tT''B''_sT''^{(q)} = \left(\begin{array}{c|c|c} {}^tP' & & \\ \hline \mathbf{g}' & 0 & \\ \hline & 1 & \\ & 0 & \\ & \vdots & E_{n-s+1} \\ & 0 & \end{array}\right). $$

Putting $D_{s-1} = {}^tP'$ and $\mathbf{b}_{s-1} = \mathbf{g}'$, we have $B''_s \sim B_{s-1}$. □

Remark 4. *When $s = 1$, we have*

$$ B_{s-1} = B_0 = E_{n+1} = W_0. $$

Now we prove Theorem 1 and Corollary 1.

Proof. Because the rank of the matrix $A$ is $n$, Proposition 1 implies that the hypersurface $X_A$ is singular. By using a linear transformation of coordinates if nessesary, we can assume that $X_A$ has a singular point $(0, \dots, 0, 1)$. Then we have $a_{in} = 0$ for any $0 \le i \le n$. The matrix $A$ is now of the form

$$ A = \left(\begin{array}{c|c} D_n & \\ \hline \mathbf{b}_n & \end{array}\right) = B_n, $$

where $D_n \in M_n(k)$, and $\mathbf{b}_n$ is a row vector of dimension $n$. Using Lemma 5 repeatedly and Remark 4, we have that the hypersurface $X_A$ is isomorphic to one of the hypersurfaces defined by the matrixes $W_s$ with $0 \le s \le n$.

If $A$ is general, then $\det(D_n) \neq 0$, and hence by the first paragraph of the proof of Lemma 5 and Lemma 4, we have $A \sim W_{n-1}$.

Next we prove that $s \neq s'$ implies $W_s \not\sim W_{s'}$. For this, we introduce some notions. Let $X^n_s$ be the hypersurface defined by the matrix $W_s$ in the projective space $\mathbf{P}^n$. The defining equation of $X^n_s$ can be written as

$$ F_q x_n + F_{q+1} = 0, $$

where

$$ F_q = \begin{cases} 0 & \text{if } s = n \\ x_{n-1}^q & \text{if } s < n, \end{cases} $$

and

$$ F_{q+1} = \begin{cases} x_0^{q+1} + \cdots + x_{n-1}^{q+1} & \text{if } s = n \\ x_0^{q+1} + \cdots + x_{s-1}^{q+1} + x_s^q x_{s+1} + \cdots + x_{n-2}^q x_{n-1} & \text{if } s < n. \end{cases} $$

It is easy to see that $X_s^n$ has only one singular point $P_0 = (0, \dots, 0, 1)$. The variety of lines in $\mathbf{P}^n$ passing through $P_0$ can be naturally identified with the hypersurface $\mathscr{H}_\infty = \{x_n = 0\}$ in $\mathbf{P}^n$ by the correspondence $Q \in \mathscr{H}_\infty$ to the line $\overline{QP_0}$. Let $\varphi$ be the map defined by

$$\begin{aligned} \varphi : \mathbf{P}^n \backslash \{P_0\} &\to \mathbf{P}^{n-1} \\ P &\mapsto \overline{PP_0}. \end{aligned}$$

Let $\overline{X_s^n} = \varphi(X_s^n \backslash \{P_0\})$. For $Q = (y_0, \dots, y_{n-1}, 0) \in \mathscr{H}_\infty$, we consider the line

$$l = \overline{QP_0} = \{(\lambda y_0, \dots, \lambda y_{n-1}, \mu) \mid (\lambda, \mu) \in \mathbf{P}^1\}.$$

We have $l \in \overline{X_s^n}$ if and only if there exists $P = (p_0, \dots, p_{n-1}, p_n) \in X_s^n \backslash \{P_0\}$ satisfying $P \in l$, i.e. there exists an element $\mu \in k$ such that

$$(p_0, \dots, p_{n-1}, p_n) = (y_0, \dots, y_{n-1}, \mu),$$

for some $P \in X_s^n \backslash \{P_0\}$, or equivalently there exists an element $\mu \in k$ such that

$$F_q(y_0, \dots, y_{n-1})\mu + F_{q+1}(y_0, \dots, y_{n-1}) = 0.$$

Then

$$\varphi^{-1}(l) \cap (X_s^n \backslash \{P_0\}) = \begin{cases} \varnothing & \text{if } F_q(y_0, \dots, y_{n-1}) = 0 \text{ and} \\ & \quad F_{q+1}(y_0, \dots, y_{n-1}) \neq 0, \\ \{\text{a single point}\} & \text{if } F_q(y_0, \dots, y_{n-1}) \neq 0, \\ l \backslash \{P_0\} & \text{if } F_q(y_0, \dots, y_{n-1}) = 0 \text{ and} \\ & \quad F_{q+1}(y_0, \dots, y_{n-1}) = 0. \end{cases}$$

Putting $V_s = \{F_q = 0, F_{q+1} = 0\} \subset \mathbf{P}^{n-1}$, and $H_s = \{F_q = 0\} \subset \mathbf{P}^{n-1}$, we have

$$V_s = \begin{cases} X_s^{n-2} & \text{if } s \leq n-2, \\ \text{nonsingular Fermat hypersurface in } \mathbf{P}^{n-1} & \text{if } s = n, \\ \text{nonsingular Fermat hypersurface in } \mathbf{P}^{n-2} & \text{if } s = n-1, \end{cases}$$

where $X_s^{n-2}$ is the hypersurface in $\mathbf{P}^{n-2}$ associated with the matrix

$$\left(\begin{array}{c|c} I_s & \\ \hline & E_{n-s-1} \end{array}\right).$$

For any $s \neq s'$, suppose that $X_s^n$ and $X_{s'}^n$ are isomorphic and let $\psi : X_s^n \to X_{s'}^n$ be an isomorphism. Because each of $X_s^n$ and $X_{s'}^n$ has only one singular point

$P_0$, we have $\psi(P_0) = P_0$, and hence $\psi$ induces an isomorphism $\bar{\psi}$ from $\overline{X_s^n}$ to $\overline{X_{s'}^n}$. For any line $l \in \overline{X_s^n}$ and $l' \in \overline{X_{s'}^n}$ such that $\bar{\psi}(l) = l'$, we have

$$\#(\varphi^{-1}(l) \cap (X_s^n \backslash \{P_0\})) = \#(\varphi^{-1}(l') \cap (X_{s'}^n \backslash \{P_0\})).$$

Thus $V_s \cong V_{s'}$ and $H_s \cong H_{s'}$. Hence, for any $s \neq s'$, if $V_s \not\cong V_{s'}$ or $H_s \not\cong H_{s'}$ then $X_s^n \not\cong X_{s'}^n$.

In the case $n = 1$, we have that $X_0^1$ consists of two points, and $X_1^1$ consists of a single point. In the case $n = 2$, we have that $X_0^2$ consists of two irreducible components, $X_1^2$ is irreducible, and $X_2^2$ consists of $(q+1)$ lines. Hence, in the case $n = 1$ and $n = 2$, we see that $s \neq s'$ implies $W_s \not\sim W_{s'}$. By induction on $n$, we have the proof. $\square$

Next, we prove Corollary 2.

Proof. Under the condition $n \geq 2$, $s < n$ and $(n, s) \neq (2, 0)$, we have $x_{n-1}$ does not divide $F_{q+1}$, and hence $V_s$ is of codimension 2 in $\mathbf{P}^{n-1}$. By induction on $n$, $X_s^n$ is irreducible. The morphism

$$\varphi|_{X_s^n \backslash \{P_0\}} : X_s^n \backslash \{P_0\} \to \mathscr{H}_\infty \cong \mathbf{P}^{n-1}$$

is birational with the inverse rational map

$$Q = (y_0, \dots, y_{n-1}, 0) \mapsto \left( y_0, \dots, y_{n-1}, -\frac{F_{q+1}(y_0, \dots, y_{n-1})}{y_{n-1}^q} \right). \qquad \square$$

## 3. Proof of Theorem 2

For any $s \leq n-2$, the matrix $W_s$ can be written

$$W_s = \left( \begin{array}{c|c|c} W_s' & & \\ \hline 0 \cdots 0 \ 1 & 0 & \\ \hline & 1 & 0 \end{array} \right).$$

For any $g \in \mathrm{Aut}(X_s)$, we have $g(P_0) = P_0$ because $X_s$ has only one singular point $P_0 = (0, \dots, 0, 1)$. The automorphism $g$ is defined by a matrix of the form

$$M = \left( \begin{array}{c|c|c} T & {}^t\mathbf{a} & 0 \\ \hline \mathbf{b} & d & 0 \\ \hline \mathbf{c} & e & 1 \end{array} \right),$$

where $T \in M_{n-1}(k)$, $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are row vectors of dimension $n-1$, and $d, e \in k$. We have ${}^tMW_sM^{(q)} = \delta W_s$ for some $0 \neq \delta \in k$, which implies

$$\begin{cases} {}^tTW_s'T^{(q)} = \delta W_s' & (1) \\ [\mathbf{a}W_s' + d(0,\dots,0,1)]\cdot T^{(q)} = \delta(0,\dots,0,1) & (2) \\ {}^tTW_s'\cdot {}^t\mathbf{a}^{(q)} + {}^t\mathbf{c}d^q = 0 & (3) \\ [\mathbf{a}W_s' + d(0,\dots,0,1)]\cdot {}^t\mathbf{a}^{(a)} + ed^q = 0 & (4) \\ \mathbf{b} = 0 & (5) \\ d^q = \delta & (6) \end{cases}$$

By (1), we see that $T$ is a matrix defining an automorphism of $X_s^{n-2}$ in $\mathbf{P}^{n-2}$. Because $s \le n-2$, by (2) we have $d = \delta$. Hence, we can calculate $T$ by induction on $n$. The vectors $\mathbf{a}$, $\mathbf{c}$ and $d$, $e$ can be find by using the equations (2)–(6). Conversely, it is easy to show that if the matrix $M$ satifies the conditions (i)–(v) then it defines a projective automorphism of $X_s$. The projective automorphism groups of $X_n$ and $X_{n-1}$ are easy to calculate. □

## 4. Proof of Proposition 1

For reader's convenience, we give a proof of Proposition 1, which is based on arguments of [12], chapter VI. The implications (iv) ⇒ (iii) ⇒ (ii) ⇒ (i) are clear. We prove (i) ⇒ (iv). For $B \in GL_{n+1}(k)$, consider the map $f_B$ defined by

$$f_B : GL_{n+1}(k) \to GL_{n+1}(k)$$
$$T \mapsto {}^tTBT^{(q)}.$$

Because the differential of the Frobenius map $F : T \mapsto T^{(q)}$ is identically zero, we can deduce that

$$d(f_B) = d({}^tT)BT^{(q)}.$$

Therefore, the tangent map of $f_B$ is surjective for any $B \in GL_{n+1}(k)$. Hence, $f_B$ is generically surjective, and the image of $f_B$ contains a non-empty open subset $U_B$. Let $A$ be any matrix of $M_{n+1}(k)$ such that the hypersurface $X_A$ is nonsingular, i.e. $A \in GL_{n+1}(k)$. Because $GL_{n+1}(k)$ is irreducible, we have $U_A \cap U_I \ne \emptyset$, where $I$ is the identity matrix of size $n+1$. There exist $T_1, T_2 \in GL_m(k)$ such that $f_A(T_1) = f_I(T_2)$. Putting $T = T_1T_2^{-1}$, we have ${}^tTAT^{(q)} = I$. □

## 5. The case of plane curves

Next we will study the plane curves $X_A$ associated with matrices $A$ of rank $\le 2$ in the projective plane $\mathbf{P}^2$.

Theorem 3. *Let* $A = (a_{ij})_{0 \le i,j \le 2} \in M_3(k)$ *be a nonzero matrix and let* $X_A$ *be the curve defined by* $\sum a_{ij} x_i x_j^q = 0$ *in* $\mathbf{P}^2$. *Suppose that the rank of* $A$ *is smaller than* 3.

(i) *When the rank of* $A$ *is* 1, *the curve* $X_A$ *is projectively isomorphic to one of the following curves*

$$Z_0 : x_0^{q+1} = 0, \qquad or \qquad Z_1 : x_0^q x_1 = 0.$$

(ii) *When the rank of* $A$ *is* 2, *the curve* $X_A$ *is projectively isomorphic to one of the following curves*

$$X_0 : x_0^q x_1 + x_1^q x_2 = 0, \quad X_1 : x_0^{q+1} + x_1^q x_2 = 0, \quad or \quad X_2 : x_0^{q+1} + x_1^{q+1} = 0.$$

Proof. In the case the rank of $A$ is 2. By Theorem 1, the plane curve $X_A$ is projectively isomorphic to one of the plane curves $X_0$, or $X_1$, or $X_2$.

In the case rank of $A$ is 1. With the same argument of the proof of Theorem 1, we can assume that the matrix $A$ is as following form

$$A = \begin{pmatrix} a_{00} & a_{01} & 0 \\ a_{10} & a_{11} & 0 \\ a_{20} & a_{21} & 0 \end{pmatrix}.$$

By interchanging with $x_0$ and $x_1$ if nessesary, we can assume that $(a_{01}, a_{11}, a_{21}) \neq (0,0,0)$. Because rank of $A$ is 1, there exists $\lambda \in k$ such that $(a_{00}, a_{10}, a_{20}) = \lambda(a_{01}, a_{11}, a_{21})$. The curve $X_A$ is defined by the equation

$$(a_{00}x_0 + a_{10}x_1 + a_{20}x_2)(x_0^q + \lambda x_1^q) = 0.$$

It is easy to show that $X_A$ is projectively isomorphic to the curve $Z_0$ or $Z_1$. $\square$

Remark 5. *In fact, the case when the plane curve* $X_A$ *of degree* $p+1$ *has been proved by Homma in* [9].

Note that the plane curve $X_1$ has a special property such that the tangent line of $X_1$ at every smooth point passes through the point $(0,1,0)$. Therefore, the plane curve $X_1$ is strange. Moreover, this curve is irreducible and nonreflexive. In [1], Ballico and Hefez proved that a reduced irreducible nonreflexive plane curve of degree $q+1$ is isomorphic to one of the following curves:

(1) $X_I : x_0^{q+1} + x_1^{q+1} + x_2^{q+1} = 0$,
(2) a nodal curve whose defining equation is given in [4] and [7], or
(3) strange curves.

Let $\mathscr{L}$ be the space of all reduced irreducible projective plane curves of degree $q+1$, which is open in the space $\mathscr{P} \cong \mathbf{P}^{\binom{q+3}{2}}$ of all projective plane curves of degree $q+1$. Let $\mathscr{L}_*$ be the locus of $\mathscr{P}$ consisting of curves isomorphic to $X_I$, and let $\mathscr{L}_1$ be the locus of $\mathscr{P}$ consisting of strange curves. Let $(\xi_J)$ be the homogeneous coordinates of $\mathscr{P}$ where $J = (j_0, j_1, j_2)$ ranges over the set of all ordered triples on non-negative integer such that $j_0 + j_1 + j_2 = q+1$. The point $(\xi_J)$ corresponds to the curve $\sum \xi_J x^J = 0$ where $x^J = x_0^{j_0} x_1^{j_1} x_2^{j_2}$. Then the locus of all curves defined by the equation of the form $\sum a_{ij} x_i x_j^q = 0$ is the linear subspace of $\mathscr{P}$ defined by $\xi_J = 0$, unless $J \in \{(q+1,0,0), (0,q+1,0), (0,0,q+1), (q,1,0), (q,0,1), (1,q,0), (1,0,q), (0,q,1), (0,1,q)\}$. By Theorem 3, we have that because $Z_0$, $Z_1$, $X_0$, $X_2$ are reducible, the closure $\overline{\mathscr{L}_*}$ of $\mathscr{L}_*$ in $\mathscr{L}$ consists of curves isomorphic to $X_I$ or to $X_1$, and the intersection of $\overline{\mathscr{L}_*}$ and $\mathscr{L}_1$ consist of curves isomorphic to $X_1$.

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