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A note on the value distribution of $f^l(f^{(k)})^n$

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ABSTRACT. Let f be a transcendental meromorphic function in the complex plane $\mathbf{C}$, and a be a nonzero complex number. We give quantitative estimates for the characteristic function $T(r,f)$ in terms of $N(r,1/(f^l(f^{(k)})^n-a))$, for integers k, l, n greater than 1. We conclude that $f^l(f^{(k)})^n$ assumes every nonzero finite value infinitely often.

1. Introduction

Let f be a transcendental meromorphic function in the complex plane $\mathbf{C}$. In this article, we use the standard notations in the sense of Nevanlinna [7], such as $T(r,f)$, $N(r,f)$, $\overline{N}(r,f)$, $m(r,f)$, $S(r,f)$, $\delta(a,f)$. In particular, $T(r,f)$ is the characteristic function and $\overline{N}(r,f)$ is a counting function with respect to poles of f, ignoring multiplicities. We shall use the symbol $S(r,f)$ to denote an error term $v(r)$ satisfying $v(r)=o(T(r,f))$ as $r\to\infty$, possibly outside a set of finite linear measure. Throughout this paper a small function (with respect to f) means a function $\varphi(z)$ meromorphic in $\mathbf{C}$ satisfying $T(r,\varphi)=S(r,f)$. In addition, in this paper, we use another type of small function $S^*(r,f)$, which has the property $S^*(r,f)=o(T(r,f))$ as $r\to\infty$, $r\notin E$, where E is a set of logarithmic density 0.

A meromorphic function f is rational if and only if $T(r,f)=O(\log r)$ (see [6]). The quantity

$$\delta(a,f)=\liminf_{r\to\infty}\frac{m(r,1/(f-a))}{T(r,f)}=1-\limsup_{r\to\infty}\frac{N(r,1/(f-a))}{T(r,f)}$$

is called the deficiency of f at the point a. Another deficiency is defined by

$$\Theta(a,f)=1-\limsup_{r\to\infty}\frac{\overline{N}(r,1/(f-a))}{T(r,f)}.$$

Note that $0\le\delta(a,f)\le\Theta(a,f)\le 1$.

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The First Fundamental Theorem of the value distribution theory due to Nevanlinna is utilized frequently in this note. It is stated as the following property:

$$T\left(r, \frac{1}{f-a}\right) = T(r, f) + O(1)$$

for any constant $a \in \mathbf{C}$. The details can be found in [6] for example. A root of the equation $f(z) = a$ ($1/f(z) = 0$ for $a = \infty$) will be called an a-point of the function $f(z)$ for $a \in \mathbf{C} \cup \{\infty\}$. a is called a Picard exceptional value of a function $f(z)$ if the number of its a-points in $\mathbf{C}$ is finite.

The aim of this paper is to look for a lower estimate of $N(r, 1/(f^l(f^{(k)})^n - a))$. The following well-known estimate is due to Hayman [7, Theorem 3.5].

THEOREM A. *Let f be a transcendental meromorphic function in the plane, l be a positive integer, and a, b be constants with $b \neq 0$. Then*

$$T(r, f) \le \left(2 + \frac{1}{l}\right) N\left(r, \frac{1}{f-a}\right) + \left(2 + \frac{2}{l}\right) \overline{N}\left(r, \frac{1}{f^{(l)} - b}\right) + S(r, f). \tag{1}$$

Hayman also concluded a corollary from the previous inequality.

COROLLARY. *Under the same assumptions as in Theorem* A, *either f assumes every finite value infinitely often or $f^{(l)}$ assumes every finite value except possibly zero infinitely often.*

Moreover, Hayman conjectured that if f is a transcendental meromorphic function and $l \ge 1$, then $f^l f'$ takes every finite nonzero value infinitely often. This conjecture has been confirmed by himself in [7] for $l \ge 3$, by Mues [13] for $l = 2$ and by Bergweiler and Eremenko [3] for $l = 1$. During the past decades, a sequence of related research have been made. In 1982, Doeringer [4, Corollary 1] proved that for a transcendental meromorphic function f, the only possible Picard exceptional value is zero for a differential monomial $f^l(f^{(k)})^n$ when $l \ge 3$. In 1994, Tse and Yang [15] gave an estimate of $T(r, f)$ for $l = 1$ and $l = 2$ and confirmed the only possible Picard exceptional value is zero. In 1996, Yang and Hu [19, Theorem 2] proved that if $\delta(0, f) > 3/(3(l+n)+1)$ with positive integers k, l, n, then for a nonzero finite complex number a, $f^l(f^{(k)})^n - a$ has infinitely many zeros. In 2002, Li and Wu [12] obtained that for a nonzero finite complex number a and positive integers l, k with $l \ge 2$, there exists a constant $M > 0$ such that

$$T(r, f) < M\overline{N}\left(r, \frac{1}{f^l f^{(k)} - a}\right) + S(r, f).$$

In 2003, Wang [16] studied the zeros of $f^l f^{(k)} - \phi$ for a small meromorphic function $\phi(z) \not\equiv 0$, and verified that for $l \geq 2$, $f^l f^{(k)} - \phi$ had infinitely many zeros if the poles of f were multiple. In 2004, Alotaibi [2] gave an estimate and showed that the function $f(f^{(k)})^n - \phi$ has infinitely many zeros for a small function $\phi(z) \not\equiv 0$, when $n \geq 2$.

We introduce a result given by Lahiri and Dewan [9, Theorem 3.2].

Theorem B. *Let f be a transcendental meromorphic function and $a(z)$, $\alpha(z)$ be both small functions of f without being identically zero and infinity. If $\psi = \alpha f^l(f^{(k)})^n$, where $l(\geq 0)$, $n(\geq 1)$, $k(\geq 1)$ are integers, then*

$$(l+n)T(r,f) \leq \overline{N}(r,f) + \overline{N}\left(r,\frac{1}{f}\right) + nN_{(k)}\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{\psi - a}\right) + S(r,f), \tag{2}$$

where $N_{(k)}(r,1/f)$ is the counting function of zeros of f with multiplicity counted $\min\{q,k\}$ times.

Remark. Inequality (2) implies that for $l \geq 3$, $n \geq 1$, $k \geq 1$,

$$T(r,f) \leq \frac{1}{l-2} N\left(r,\frac{1}{f^l(f^{(k)})^n - a}\right) + S(r,f), \tag{3}$$

which implies

$$\delta(a,\psi) \leq \Theta(a,\psi) \leq 1 - \frac{l-2}{nk+n+l}. \tag{4}$$

However, this result is still worth refining. In the current paper, we obtained an estimate corresponding to the case k, l, n all greater than 1, and in our proof, we use a very important inequality of Yamanoi.

Theorem C ([18, Yamanoi]). *Let f be a meromorphic and transcendental function in the complex plane and let $k \geq 2$ be an integer, $A \subset \mathbf{C}$ be a finite set of complex numbers. Then we have*

$$(k-1)\overline{N}(r,f) + \sum_{a \in A} N_1\left(r,\frac{1}{f-a}\right) \leq N\left(r,\frac{1}{f^{(k)}}\right) + S^*(r,f), \tag{5}$$

where

$$N_1\left(r,\frac{1}{f-a}\right) = N\left(r,\frac{1}{f-a}\right) - \overline{N}\left(r,\frac{1}{f-a}\right).$$

Remark. Actually, when A is an empty set, the inequality (5) turns out to be the following one.

$$(k-1)\overline{N}(r,f) \leq N\left(r,\frac{1}{f^{(k)}}\right)+S^*(r,f), \tag{6}$$

which corresponds to the famous Gol'dberg Conjecture, which says that for a transcendental meromorphic function f and $k \geq 2$, then $\overline{N}(r,f) \leq N(r,1/f^{(k)})+S(r,f)$. The difference is that they apply small function with different exceptional set.

In this paper, we continue to consider the general form $f^l(f^{(k)})^n - a$ for a nonzero constant a. The following theorem improve the estimate (3) when $k,n \geq 2$.

Theorem 1. *Let f be a transcendental meromorphic function in* $\mathbf{C}$, *l, n, k be integers greater than* 1 *and a be a nonzero constant. Then*

$$T(r,f) \leq \frac{1}{l-1}N\left(r,\frac{1}{f^l(f^{(k)})^n-a}\right)+S^*(r,f). \tag{7}$$

Remark. When $l \geq 2$, $n \geq 2$, $k \geq 2$, (7) is better than (3) except the appearance of larger exceptional set of logarithmic density 0. If the case $k=1$, $l \geq 3$ or $n=1$, $l \geq 3$ occurs, (3) might be the best choice so far.

Another important remark should be made here. As we realized that for general form $f^l(f^{(k)})^n$, there are two cases that are excluded in Theorem B and Theorem 1: $l=1$, $n \geq 1$, $k \geq 1$ and $l=2$, $n \geq 1$, $k \geq 1$. We summarize the known estimates of these two cases. For the case $l=2$, $n=k=1$, Zhang [20] obtained a quantitative result, proving that the inequality $T(r,f) < 6N(r,1/(f^2f'-1))+S(r,f)$ holds. For the case $l=2$, $n=1$, $k>1$ the inequality is due to Huang and Gu [8]. For the case $l=1$, $n \geq 2$, $k \geq 1$, by Li and Yang [11] and Alotaibi [2] gave two different inequalities for the estimates independently. For the case $l=n=1$, $k \geq 1$, again Alotaibi [1] obtained an estimate provided that $N_{1)}\left(r,\frac{1}{f^{(k)}}\right)=S(r,f)$, where $N_{1)}\left(r,\frac{1}{f^{(k)}}\right)$ is the counting function of simple zeros of $f^{(k)}$, as well, Wang [16] gave an estimate but under the additional condition that multiplicities of all poles of f are at least 3 and $N_{1)}(r,1/f) \leq \lambda T(r,f)$, where $\lambda < 1/3$ is a constant.

Though these cases are excluded in Theorem 1, our estimate is considered to be stronger compared to the known results so far. Furthermore, it is natural to estimate the deficiency of $f^l(f^{(k)})^n$ by making use of Theorem 1. This leads us to the following.

THEOREM 2. *Let f be a transcendental meromorphic function in* $\mathbf{C}$, *k, l, n be positive integers all greater than* 1 *and a be a nonzero constant. Then*

$$\delta(a, f^l(f^{(k)})^n) \le 1 - \frac{l-1}{nk+n+l}.$$

REMARK. Since for a nonzero constant a, $\delta(a, f^l(f^{(k)})^n) < 1$, Theorem 2 also implies that the possible Picard exceptional value of $f^l(f^{(k)})^n$ is zero for $k \ge 2$, $l \ge 2$, $n \ge 2$. We would like to state these results as a corollary here.

COROLLARY 1. *Under the same conditions as Theorem* 1, *$f^l(f^{(k)})^n$ assumes every finite value except possibly zero infinitely often.*

REMARK. In fact, this kind of result is not brand new. There are already a couple of known results implying that for any positive integers k, l, n, the function $f^l(f^{(k)})^n$ assumes every finite value except possibly zero infinitely often. The readers should see Lahiri and Dewan [9, 10], Steinmetz [14], Wang [17], Alotaibi [2, 1] and Li and Wu [12] for further details.

The following section contains couples of lemmas used for the proofs of Theorem 1 and Theorem 2. The proofs are placed in Section 3 and 4 respectively, as well an application to the sum of deficiencies is followed.

2. Lemmas

Before we proceed to the proofs of the theorems, we need the following lemmas.

LEMMA 1 ([7, Theorem 3.1]). *Let f be a non-constant meromorphic function in the complex plane, l be a positive integer, $a_0(z), a_1(z), \dots, a_l(z)$ be meromorphic functions in the plane satisfying $T(r, a_v(z)) = S(r, f)$ for $v = 0, 1, \dots, l$ (as $r \to +\infty$) and*

$$\psi(z) = \sum_{v=0}^{l} a_v(z) f^{(v)}(z).$$

Then

$$m\left(r, \frac{\psi}{f}\right) = S(r, f).$$

In particular, this lemma implies $m(r, f^{(l)}/f) = S(r, f)$ and $m(r, f^{(l+1)}/f^{(l)}) = S(r, f^{(l)})$.

LEMMA 2 ([6, p. 99]). *Let f be a non-constant meromorphic function in the complex plane, k be a positive integer. Then*

$$T(r, f^{(k)}) \leq (k+1)T(r, f) + S(r, f). \tag{8}$$

In particular, $S(r, f^{(k)}) \leq S(r, f)$. Inequality (8) will be used often in this note without reference.

LEMMA 3. *Let f be a transcendental meromorphic function in the complex plane. Then the differential monomial*

$$\psi = f^l (f^{(k)})^n$$

is transcendental, where l, n and k are positive integers.

PROOF. We have

$$\frac{1}{f^{l+n}} = \left(\frac{f^{(k)}}{f}\right)^n \frac{1}{\psi}.$$

We obtain from Lemma 1 and the First Fundamental Theorem that

$$\begin{aligned}
(l+n)T(r, f) &\leq nT\left(r, \frac{f^{(k)}}{f}\right) + T\left(r, \frac{1}{\psi}\right) \\
&\leq nN\left(r, \frac{f^{(k)}}{f}\right) + T\left(r, \frac{1}{\psi}\right) + S(r, f) \\
&\leq nk\left[\overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f}\right)\right] + T\left(r, \frac{1}{\psi}\right) + S(r, f). \qquad (9)
\end{aligned}$$

Since $\overline{N}(r, f) \leq \overline{N}(r, \psi) + S(r, f)$ and $\overline{N}\left(r, \frac{1}{f}\right) \leq \overline{N}\left(r, \frac{1}{\psi}\right) + S(r, f)$, we can simplify inequality (9) to

$$(l+n)T(r, f) \leq (2nk+1)T\left(r, \frac{1}{\psi}\right) + S(r, f).$$

Because f is transcendental, we conclude that ψ is transcendental as well. □

LEMMA 4. *Let f be a transcendental meromorphic function in $\mathbf{C}$, let k, l, n be positive integers, and set*

$$g = f^l (f^{(k)})^n - 1.$$

Then,

$$T(r, g) \leq O(T(r, f)),$$

as $r \to \infty$, possibly outside a set of finite linear measure.

Proof. Note that $N(r, f^l(f^{(k)})^n) = O(N(r, f))$ and $m(r, f^{(k)}/f) = S(r, f)$ by Lemma 1. Applying the First Fundamental Theorem, we get

$$\begin{aligned} T(r, g) &= T(r, f^l(f^{(k)})^n - 1) \\ &= N(r, f^l(f^{(k)})^n) + m(r, f^l(f^{(k)})^n) + O(1) \\ &\le O(N(r, f)) + lm(r, f) + nm(r, f^{(k)}) + O(1) \\ &\le O(N(r, f)) + lm(r, f) + nm(r, f) + nm\left(r, \frac{f^{(k)}}{f}\right) + O(1) \\ &= O(T(r, f)) + S(r, f). \end{aligned}$$

We can see that

$$T(r, g') \le N(r, g') + m(r, g) + S(r, g) \le T(r, g) + S(r, g).$$

Hence

$$T(r, g) \le O(T(r, f)). \quad \square$$

3. Proof of Theorem 1

Without loss of generality, we assume $a = 1$, $g = f^l(f^{(k)})^n - 1$. By Lemma 3, we know that g is not constant. Since

$$\frac{1}{f^{l+n}} = \left(\frac{f^{(k)}}{f}\right)^n - \frac{g'}{f^{l+n}}\left(\frac{g}{g'}\right),$$

it follows that

$$m\left(r, \frac{1}{f^{l+n}}\right) \le m\left(r, \frac{g}{g'}\right) + m\left(r, \frac{g'}{f^{l+n}}\right) + S(r, f).$$

Note that

$$\frac{g'}{f^{l+n}} = l\frac{f'}{f}\left(\frac{f^{(k)}}{f}\right)^n + n\frac{f^{(k+1)}}{f}\left(\frac{f^{(k)}}{f}\right)^{n-1},$$

which implies

$$m\left(r, \frac{g'}{f^{l+n}}\right) = S(r, f).$$

Therefore, we have

$$m\left(r, \frac{1}{f^{l+n}}\right) \le m\left(r, \frac{g}{g'}\right) + S(r, f).$$

We know that the poles of g'/g come from the zeros and poles of g, and all are simple. The poles of g/g' come from zeros of g' which are not zeros of g, preserving multiplicity. Hence, we get

$$N\left(r, \frac{g'}{g}\right) = \overline{N}\left(r, \frac{1}{g}\right) + \overline{N}(r, g), \tag{10}$$

and

$$N\left(r, \frac{g}{g'}\right) = N\left(r, \frac{1}{g'}\right) - \left(N\left(r, \frac{1}{g}\right) - \overline{N}\left(r, \frac{1}{g}\right)\right). \tag{11}$$

By combining (10) with (11),

$$N\left(r, \frac{g'}{g}\right) - N\left(r, \frac{g}{g'}\right) = \overline{N}(r, g) + N\left(r, \frac{1}{g}\right) - N\left(r, \frac{1}{g'}\right). \tag{12}$$

By Lemma 4, we know that

$$m(r, g'/g) = S(r, g) \leq S(r, f), \qquad \overline{N}(r, g) = \overline{N}(r, f).$$

Applying the First Fundamental Theorem and (12),

$$\begin{aligned} m\left(r, \frac{1}{f^{l+n}}\right) &\leq m\left(r, \frac{g}{g'}\right) + S(r, f) \\ &\leq N\left(r, \frac{g'}{g}\right) - N\left(r, \frac{g}{g'}\right) + m\left(r, \frac{g'}{g}\right) + S(r, f) \\ &\leq N\left(r, \frac{g'}{g}\right) - N\left(r, \frac{g}{g'}\right) + S(r, g) + S(r, f) \\ &= \overline{N}(r, f) + N\left(r, \frac{1}{g}\right) - N\left(r, \frac{1}{g'}\right) + S(r, f). \end{aligned} \tag{13}$$

Here we add $N(r, 1/f^{l+n})$ to both sides of inequality (13), then

$$(l+n)T\left(r, \frac{1}{f}\right) \leq \overline{N}(r, f) + N\left(r, \frac{1}{g}\right) - N\left(r, \frac{1}{g'}\right) + N\left(r, \frac{1}{f^{l+n}}\right) + S(r, f). \tag{14}$$

Note that $g' = f^{l-1}(f^{(k)})^{n-1}(lf^{(k)}f' + nff^{(k+1)})$, which implies

$$(l-1)N\left(r, \frac{1}{f}\right) + (n-1)N\left(r, \frac{1}{f^{(k)}}\right) \leq N\left(r, \frac{1}{g'}\right). \tag{15}$$

Substituting (15) into (14), we get

$$T\left(r,\frac{1}{f^{l+n}}\right) \le \overline{N}(r,f) + N\left(r,\frac{1}{g}\right) + N\left(r,\frac{1}{f^{l+n}}\right) - (l-1)N\left(r,\frac{1}{f}\right) - (n-1)N\left(r,\frac{1}{f^{(k)}}\right) + S(r,f).$$

Hence,

$$(l+n)T(r,f) \le N\left(r,\frac{1}{g}\right) + \overline{N}(r,f) + (n+1)N\left(r,\frac{1}{f}\right) - (n-1)N\left(r,\frac{1}{f^{(k)}}\right) + S(r,f). \tag{16}$$

Inequality (6) implies that for $k \ge 2$,

$$\overline{N}(r,f) \le N\left(r,\frac{1}{f^{(k)}}\right) + S^*(r,f). \tag{17}$$

Now by combining inequality (16) and (17), we have for $n \ge 2$

$$\begin{aligned}(l+n)T(r,f) &\le N\left(r,\frac{1}{g}\right) + (n+1)N\left(r,\frac{1}{f}\right) - (n-2)N\left(r,\frac{1}{f^{(k)}}\right) \\ &\quad + S(r,f) + S^*(r,f) \\ &\le N\left(r,\frac{1}{g}\right) + (n+1)N\left(r,\frac{1}{f}\right) + S^*(r,f).\end{aligned}$$

Since $l-1>0$ for $(n+1)N(r,1/f) \le (n+1)T(r,f)$, then

$$T(r,f) \le \frac{1}{l-1}N\left(r,\frac{1}{g}\right) + (n+1)N\left(r,\frac{1}{f}\right) + S^*(r,f). \tag{18}$$

For any nonzero constant a, we replace $f^l(f^{(k)})^n - 1$ in above inequality by $f^l(f^{(k)})^n - a$, the inequality (7) is obtained. The proof is completed. □

4. Proof of Theorem 2

Set $\psi = f^l(f^{(k)})^n$. Inequality (18) is stated that

$$T(r,f) \le \frac{1}{l-1}N\left(r,\frac{1}{\psi - a}\right) + S^*(r,f)$$

for $l \ge 2$, $n \ge 2$, $k \ge 2$. By the definition of $\delta(a,f)$ and the First Fundamental Theorem, we obtain

$$\begin{aligned} T(r,\psi) &\le (nk+n+l)T(r,f) + S(r,f) \\ &\le \frac{nk+n+l}{l-1} N\left(r, \frac{1}{\psi - a}\right) + S^*(r,f). \end{aligned} \tag{19}$$

By inequality (19), we have

$$N\left(r, \frac{1}{\psi - a}\right) \ge \frac{l-1}{nk+n+l} T(r,\psi) - S^*(r,f).$$

Since

$$\begin{aligned} T(r,f) = \frac{1}{l} T(r,f^l) &\le \frac{1}{l}(T(r,(f^{(k)})^n) + T(r,\psi)) \\ &\le O(T(r,\psi)), \end{aligned}$$

then we deduce that

$$\liminf_{r\to\infty} \frac{S^*(r,f)}{T(r,\psi)} = \liminf_{r\to\infty} \frac{S^*(r,f)}{T(r,f)} \frac{T(r,f)}{T(r,\psi)} = 0.$$

Therefore, by the definition of deficiency,

$$\begin{aligned} \delta(a,\psi) &= 1 - \limsup_{r\to\infty} \frac{N\left(r, \frac{1}{\psi - a}\right)}{T(r,\psi)} \\ &\le 1 - \limsup_{r\to\infty} \frac{\frac{l-1}{nk+n+l} T(r,\psi) - S^*(r,f)}{T(r,\psi)} \\ &\le 1 - \frac{l-1}{nk+n+l} + \liminf_{r\to\infty} \frac{S^*(r,f)}{T(r,\psi)} \\ &= 1 - \frac{l-1}{nk+n+l}. \quad \square \end{aligned}$$

5. An application

After Yamanoi's result was published in 2013, there are some results about deficieny relations came out by using his important theorem. We take a result from Fang and Wang [5] as a good example here, and we analogize their steps to get an estimate of the sum of deficiencies of $f^l(f^{(k)})^n$.

We need the following lemma for our calculation. This lemma is as well used in paper [5].

Lemma 5 ([7, p. 33]). *Let $a_1, a_2, \dots, a_q$, where $q > 2$, be distinct finite complex numbers. Then*

$$\sum_{i=1}^{q} m\left(r, \frac{1}{f - a_i}\right) \leq m\left(r, \sum_{i=1}^{q} \frac{1}{f - a_i}\right) + O(1).$$

Theorem 3. *Let f be a transcendental meromorphic function in $\mathbf{C}$, k, l, n be positive integers all at least 2 and $a_i \in \mathbf{C}$ be constants, $i = 1, 2, \dots, q$. Then*

$$\sum_{i=1}^{q} \delta(a_i, f^l(f^{(k)})^n) \leq 1 + \frac{1}{nk + n + l}.$$

Proof. By the Nevanlinna theory, for constants $a_i \in \mathbf{C}$, the sum of deficiencies of function f is defined by

$$\sum_{i=1}^{q} \delta(a_i, f) = \liminf_{r \to \infty} \sum_{i=1}^{q} \frac{m\left(r, \frac{1}{f - a_i}\right)}{T(r, f)}. \tag{20}$$

Let $\psi = f^l(f^{(k)})^n$. By Lemma 5, we have

$$\begin{aligned}
\sum_{i=1}^{q} m\left(r, \frac{1}{\psi - a_i}\right) &\leq m\left(r, \sum_{i=1}^{q} \frac{1}{\psi - a_i}\right) + O(1) \\
&\leq m\left(r, \sum_{i=1}^{q} \frac{\psi''}{\psi - a_i}\right) + m\left(r, \frac{1}{\psi''}\right) + S(r, f) \\
&\leq T(r, \psi'') - N\left(r, \frac{1}{\psi''}\right) + S(r, f) \\
&\leq N(r, \psi'') + m(r, \psi'') - N\left(r, \frac{1}{\psi''}\right) + S(r, f) \\
&\leq N(r, \psi) + 2\overline{N}(r, \psi) + m(r, \psi) - N\left(r, \frac{1}{\psi''}\right) + S(r, f). \qquad (21)
\end{aligned}$$

By Yamanoi's result (6), it follows from inequality (21) that

$$\begin{aligned}
\sum_{i=1}^{q} m\left(r, \frac{1}{\psi - a_i}\right) &\leq T(r, \psi) + 2\overline{N}(r, \psi) - \overline{N}(r, \psi) + S(r, f) \\
&\leq T(r, \psi) + \overline{N}(r, f) + S(r, f) \\
&\leq T(r, \psi) + T(r, f) + S(r, f). \qquad (22)
\end{aligned}$$

By Theorem 1 and Theorem 2, it follows from inequality (22) that,

$$\begin{aligned}
\sum_{i=1}^{q} m\left(r, \frac{1}{\psi - a_i}\right) &= \liminf_{r\to\infty} \sum_{i=1}^{q} \frac{m\left(r, \frac{1}{\psi - a_i}\right)}{T(r,\psi)} \\
&\le 1 + \liminf_{r\to\infty} \frac{T(r,f)}{T(r,\psi)} + S(r,f) \\
&\le 1 - \frac{1}{l-1}\left(1 - \limsup_{r\to\infty} \frac{N\left(r, \frac{1}{\psi - a}\right)}{T(r,\psi)} - 1\right) \\
&= 1 - \frac{1}{l-1}(\delta(a,\psi) - 1) \\
&\le 1 - \frac{1}{l-1}\left(1 - \frac{l-1}{nk+n+l} - 1\right) \\
&= 1 + \frac{1}{nk+n+l}
\end{aligned}$$

in which a is a nonzero constant. □

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