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Properties of minimal charts and their applications II

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ABSTRACT. Let Γ be a minimal chart with exactly seven white vertices. In this paper, we show that Γ is a chart of type (7), $(5,2)$, $(4,3)$, $(3,2,2)$ or $(2,3,2)$ if necessary we change the labels. We investigate minimal charts with loops or lenses.

1. Introduction

Kamada introduced a method to describe surface braids as oriented labeled graphs in a disk, called charts ([2], [3], [4]) (see Section 2 for the precise definition of charts). In a chart there are three kinds of vertices; white vertices, crossings and black vertices. In this paper, we investigate properties of minimal charts which we need to prove that there is no minimal chart with exactly seven white vertices (see Section 2 for the definition of minimal charts).

Let Γ be a chart. For each label m, we denote by Γ_m the 'subgraph' of Γ consisting of edges of label m and their vertices. In this paper,

crossings are vertices of Γ but we do not consider crossings as vertices of Γ_m. The vertices of Γ_m are white vertices and black vertices.

An *edge* of Γ_m is the closure of a connected component of the set obtained by taking out all white vertices from Γ_m.

A chart Γ is of *type* $(m; n_1, n_2, \ldots, n_k)$ or of *type* $(n_1, n_2, \ldots, n_k)$ briefly if it satisfies the following three conditions:

(1) For each $i = 1, 2, \ldots, k$, the chart Γ contains exactly n_i white vertices in $\Gamma_{m+i-1} \cap \Gamma_{m+i}$.
(2) If $i < 0$ or $i > k$, then Γ_{m+i} does not contain any white vertices.
(3) Both of the two subgraphs Γ_m and Γ_{m+k} contain at least one white vertex.

Note that $n_1 \geq 1$ and $n_k \geq 1$ by the condition (3).

The following is the main result in this paper:

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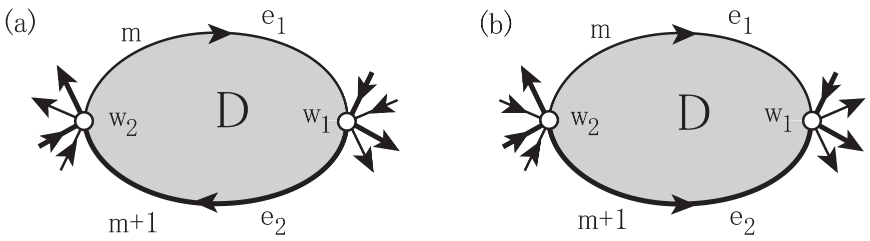

Fig. 1. (a) is of type 1 and (b) is of type 2.

THEOREM 1.1. *Let Γ be a minimal chart of type $(m; n_1, n_2, \ldots, n_k)$. Suppose that Γ contains exactly seven white vertices. If necessary we change the label $m+i$ by $m+k-i$ for all label i, then Γ is a chart of type* (7), $(5,2)$, $(4,3)$, $(3,2,2)$ *or* $(2,3,2)$.

Among six short arcs in a small neighborhood of a white vertex, a center arc of each three consecutive arcs oriented inward or outward is called *a middle arc* at the white vertex (see Figure 3). The other arcs are called *non-middle arcs*. There are two middle arcs in a small neighborhood of each white vertex.

Let Γ be a chart. Let D be a disk such that ∂D consists of an edge e_1 of Γ_m and an edge e_2 of Γ_{m+1} and that any edge containing a white vertex in e_1 does not intersect the open disk $Int(D)$. Let w_1 and w_2 be the white vertices in e_1. If the disk D satisfies one of the following conditions, then D is called *a lens of type* $(m, m+1)$ (see Figure 1):

(1) Neither e_1 nor e_2 contains a middle arc.

(2) One of the two edges e_1 and e_2 contains middle arcs at both white vertices w_1 and w_2.

If D satisfies the above condition (1) (resp. (2)), then the lens D is called *a lens of type* 1 (*resp. type* 2). We also say that D is a *lens of Γ*.

In [5] we showed that in a minimal chart, there exist at least three white vertices in the interior of any lens. In this paper we shall show the following theorem:

THEOREM 1.2. *Let Γ be a minimal chart. The complement of any lens contains at least three white vertices.*

Hence we have the following corollary:

COROLLARY 1.3. *Let Γ be a minimal chart with at most seven white vertices. Then there is no lens of Γ.*

Let Γ be a chart. A *loop* is a closed edge of Γ_m which contains only one white vertex but may contain crossings. Finally we shall investigate minimal charts with loops.

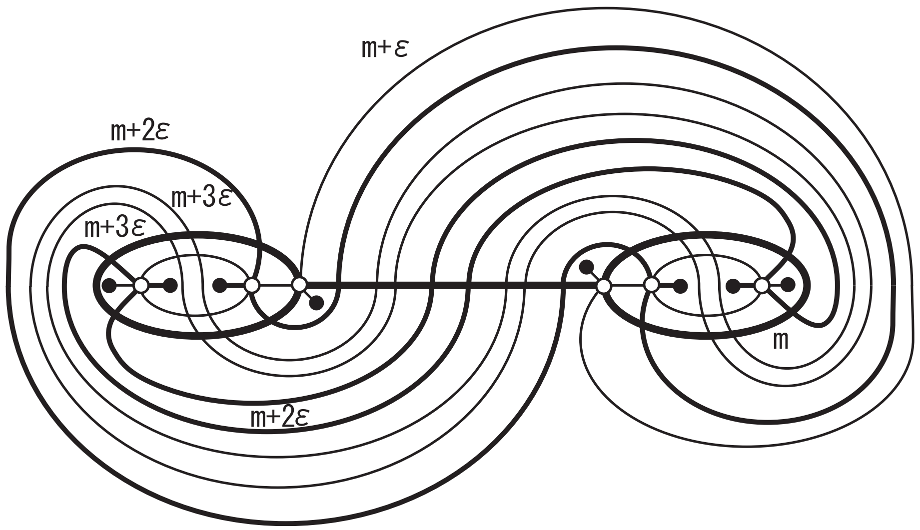

Fig. 2

Theorem 1.4. *Let Γ be a minimal chart with exactly six white vertices. Suppose that Γ contains a loop of label m. If necessary we take the reflection of the chart Γ, then Γ is C-move equivalent to a minimal chart which contains the chart as shown in Figure 2 where $\varepsilon \in \{+1, -1\}$.*

This paper is organized as follows. In Section 2, we give notations and definitions. In Section 3, we review useful lemmata proved in [5]. In Section 4 and 5, we investigate 2-angled disks and loops. In Section 6, we investigate the subgraph Γ_m containing at most three white vertices, and we prove Theorem 1.1. In Section 7, we prove Theorem 1.2. In Section 8, we prove Theorem 1.4.

In this paper for a set X we denote the interior of X, the boundary of X and the closure of X by $Int(X)$, ∂X and $Cl(X)$ respectively. If X is a polyhedron in S^2, we denote a regular neighborhood of X in S^2 by $N(X)$.

The following is the list of the new words in this paper:

(p. 1) edge, type $(m; n_1, n_2, \dots, n_k)$ (or type $(n_1, n_2, \dots, n_k)$),

(p. 2) middle arc, non-middle arc, lens (of type 1/of type 2), loop,

(p. 5) terminal edge, free edge, minimal chart,

(p. 6) ring, simple hoop,

(p. 7) pseudo chart,

(p. 8) admissible boundary arc, (D, α)-arc of label k, (D, α)-arc free, inward arc, outward arc,

(p. 10) bipartition of Γ with the partition point b wit respect to the label k, associated disk of the loop,
(p. 12) white vertex of type k with respect to Γ_m,
(p. 13) k-angled disk with s feelers,
(p. 22) θ-curve, pair of eyeglasses, oval, skew θ-curve, pair of skew eyeglasses (of type 1/of type 2),
(p. 26) bicolored 2-angled disk (of type (s_1, s_2)),
(p. 33) solar eclipse.

2. Preliminaries

In this section, we define charts and notations.

Let n be a positive integer. An *n-chart* is an oriented labeled graph in a disk, which may be empty or have closed edges without vertices, called *hoops*, satisfying the following four conditions:

(1) Every vertex has degree 1, 4, or 6.
(2) The labels of edges are in $\{1, 2, \ldots, n-1\}$.
(3) In a small neighborhood of each vertex of degree 6, there are six short arcs, three consecutive arcs are oriented inward and the other three are outward, and these six are labeled i and $i+1$ alternately for some i, where the orientation and the label of each arc are inherited from the edge containing the arc.
(4) For each vertex of degree 4, diagonal edges have the same label and are oriented coherently, and the labels i and j of the diagonals satisfy $|i-j| > 1$.

A vertex of degree 1, 4, and 6 is called a *black vertex*, a *crossing*, and a *white vertex* respectively (see Figure 3).

C-moves are local modification of charts in a disk as shown in Figure 4 (see [1], [4] for the precise definition). Kamada originally defined CI-moves as follows (C-I-moves are special cases of CI-moves): A chart Γ is obtained from a chart Γ' by a *CI-move*, if there exists a disk D such that

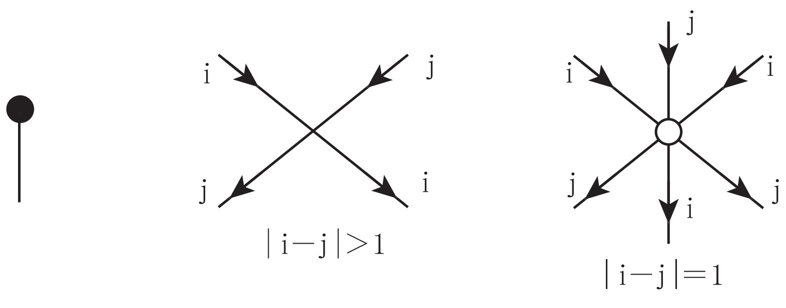

Fig. 3

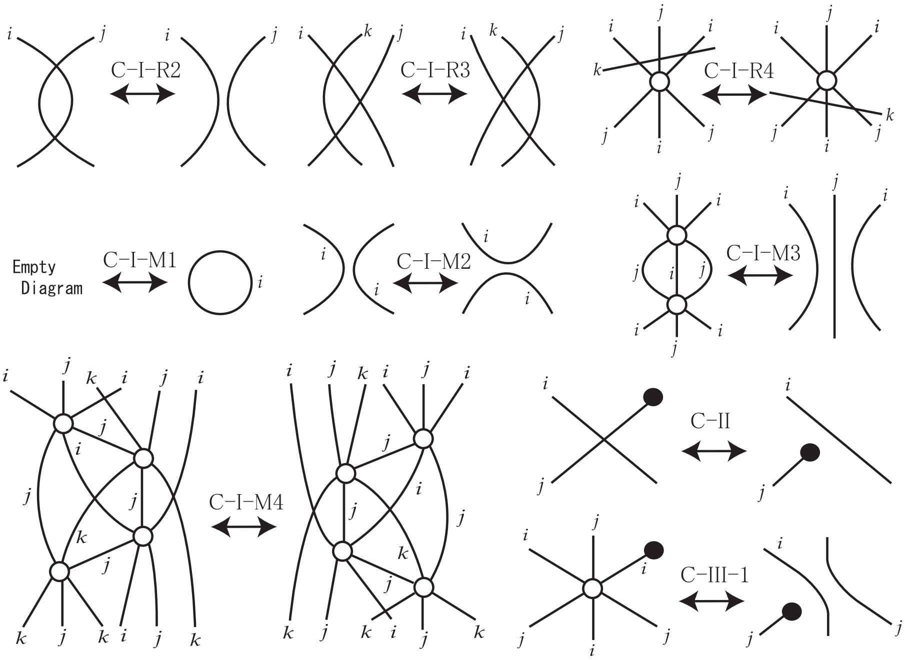

Fig. 4. For the C-III-1 move, the edge containing the black vertex does not contain a middle arc in the left figure.

(1) the two charts Γ and Γ' intersect the boundary of D transversely or do not intersect the boundary of D,

(2) $\Gamma \cap D^c = \Gamma' \cap D^c$, and

(3) neither $\Gamma \cap D$ nor $\Gamma' \cap D$ contains a black vertex,

where $(\cdots)^c$ is the complement of $(\cdots)$.

Two charts are said to be *C-move equivalent* if there exists a finite sequence of C-moves which modify one of the two charts to the other.

Let Γ be a chart. An edge of Γ or Γ_m is called a *free edge* if it has two black vertices. An edge of Γ or Γ_m is called a *terminal edge* if it has a white vertex and a black vertex. Note that free edges of Γ_m, terminal edges of Γ_m, and loops may contain crossings of Γ.

For each chart Γ, let $w(\Gamma)$ and $f(\Gamma)$ be the number of white vertices, and the number of free edges respectively. The pair $(w(\Gamma), -f(\Gamma))$ is called the *complexity* of the chart. A chart is called a *minimal chart* if its complexity is minimal among the charts C-move equivalent to the chart with respect to the lexicographic order of pairs of integers.

In the following lemma, we investigate the difference of a chart in a disk and in a 2-sphere. This lemma follows from that there exists a natural one-to-one correspondence between {charts in S^2}/C-moves and {charts in D^2}/C-moves, conjugations ([3, Chapter 23 and Chapter 25]).

LEMMA 2.1 ([5, Lemma 2.1]). *Let Γ and Γ' be charts in a disk D. Suppose that Γ is ambient isotopic to Γ' in the one point compactification of the open disk $Int(D)$, i.e. the 2-sphere S^2. Then there exist hoops $C_1, C_2, \ldots, C_k$ in $Int(D)$ such that*

(1) *the chart Γ is obtained from $\Gamma' \cup \left(\bigcup_{i=1}^{k} C_i\right)$ by C-moves in the disk D,*

(2) *the chart Γ' and hoops $C_1, C_2, \ldots, C_k$ are mutually disjoint, and*

(3) *each hoop C_i bounds a disk containing the chart Γ' in the disk D.*

Moreover the chart Γ is minimal if and only if Γ' is minimal.

Lemma 2.1 says that we can move the point at infinity in S^2 to any complementary domain of the chart. To make the argument simple, we assume that the charts lie on the 2-sphere instead of the disk. In this paper,

all charts are contained in the 2-sphere S^2.

We have the special point in the 2-sphere S^2, called the point at infinity, denoted by ∞. In this paper, all charts are contained in a disk such that the disk does not contain the point at infinity ∞.

A *hoop* is a closed edge of a chart Γ without vertices (hence without crossings, neither). A *ring* is a closed edge of Γ_m containing crossings but not containing a white vertex. A hoop is said to be *simple* if one of the complementary domain of the hoop does not contain any white vertices.

It was shown in [5] the following: if a minimal chart Γ does not satisfy one of the following six conditions, then there exists another minimal chart Γ' such that Γ' satisfies all of the six conditions and Γ' is C-move equivalent to Γ, or we have a contradiction to the minimality of Γ. We can assume that all minimal charts Γ satisfy the following six conditions (see [5]):

ASSUMPTION 1. *No terminal edge of Γ_m contains a crossing. Hence any terminal edge of Γ_m is a terminal edge of Γ and any terminal edge of Γ_m contains a middle arc.*

ASSUMPTION 2. *No free edge of Γ_m contains a crossing. Hence any free edge of Γ_m is a free edge of Γ.*

ASSUMPTION 3. *All free edges and simple hoops in Γ are moved into a small neighborhood U_∞ of the point at infinity ∞.*

ASSUMPTION 4. *Each complementary domain of any ring must contain at least one white vertex.*

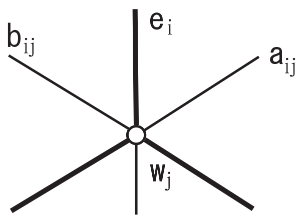

Fig. 5

Assumption 5. *Hence we assume that the subgraph obtained from Γ by omitting free edges and simple hoops does not meet the set U_∞. Also we assume that Γ does not contain free edges nor simple hoops, otherwise mentioned. Therefore we can assume that if an edge of Γ_m contains a black vertex, then it is not a free edge but a terminal edge and that each complementary domain of any hoops and rings of Γ contains a white vertex, otherwise mentioned.*

Assumption 6. *The point at infinity ∞ is moved in any complementary domain of Γ.*

We use the following notation:

In our argument, we often need a name for an unnamed edge by using a given edge and a given white vertex. For the convenience, we use the following naming: Let e', e_i, e'' be three consecutive edges containing a white vertex w_j. Here, the two edges e' and e'' are unnamed edges. There are six arcs in a neighborhood U of the white vertex w_j. If the three arcs $e' \cap U$, $e_i \cap U$, $e'' \cap U$ lie anticlockwisely around the white vertex w_j in this order, then e' and e'' are denoted by a_{ij} and b_{ij} respectively (see Figure 5). There is a possibility $a_{ij} = b_{ij}$ if they are contained in a loop.

3. Lemmata

Lemma 3.1 ([5, Theorem 1.1 and Corollary 6.3]). *Let Γ be a minimal chart. Then the following hold:*

(1) *There exist at least three white vertices in the interior of any lens.*

(2) *If Γ is a minimal chart of type $(m; n_1, n_2, \dots, n_k)$, then there does not exist any lens of type $(m, m+1)$ nor $(m+k-1, m+k)$.*

Let Γ be a chart. If an object consists of some edges of Γ, arcs in edges of Γ and arcs around white vertices, then the object is called *a pseudo chart*.

Let Γ and Γ' be C-move equivalent charts. Suppose that a pseudo chart X of Γ is also a pseudo chart of Γ'. Then we say that Γ is modified to Γ' by

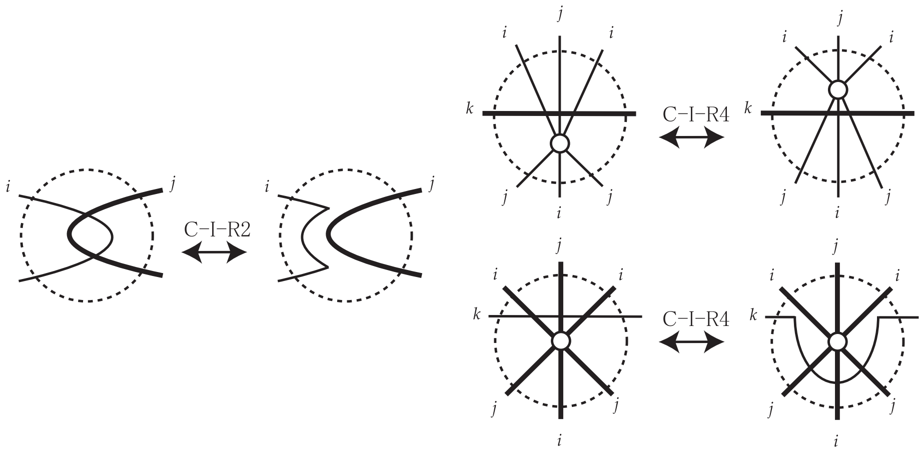

Fig. 6. C-moves keeping thicken figures fixed.

C-moves keeping X fixed. In Figure 6, we give examples of C-moves keeping pseudo charts fixed.

Let D be the closure of an open disk U. A simple arc α in $\partial U = D - U$ is called an *admissible boundary arc* of D provided that $\alpha \cap Cl(\partial U - \alpha) = \partial\alpha$.

Let Γ be a chart, and D the closure of an open disk U. Let α be a simple arc in $\partial U = D - U$. We call a simple arc γ in an edge of Γ_k a (D, α)*-arc* of label k provided that $\partial\gamma \subset Int(\alpha)$ and $Int(\gamma) \subset U$. If there is no (D, α)-arc in Γ, then the chart Γ is said to be (D, α)*-arc free*.

Let Γ be a chart and D the closure of an open disk U. Let α be a simple arc in ∂U. For each $k = 1, 2, \ldots,$ let Σ_k be the pseudo chart which consists of all arcs in $D \cap \Gamma_k$ intersecting the set $Cl(\partial U - \alpha)$. Let $\Sigma_\alpha = \bigcup_k \Sigma_k$.

LEMMA 3.2 ([5, Lemma 3.3]) (*Disk Lemma*). *Let Γ be a minimal chart and D the closure of an open disk U. Let α be an admissible boundary arc of D. Suppose that the interior of α contains neither white vertices, isolated points of $Cl(U) \cap \Gamma$, nor arcs of $Cl(U) \cap \Gamma$. If U does not contain white vertices of Γ, then for any neighborhood V of α, there exists a (D, α)-arc free minimal chart Γ' obtained from the chart Γ by C-moves in $V \cup D$ keeping Σ_α fixed (see Figure 7 and 8).*

Let Γ be a chart, and v a vertex. Let α be a short arc of Γ in a small neighborhood of v with $v \in \partial\alpha$. If the arc α is oriented to v, then α is called *an inward arc*, and the otherwise α is called *an outward arc*.

The following lemma will be used in the proof of Lemma 5.2 and 5.5.

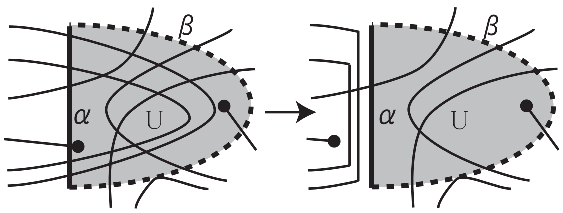

Fig. 7. The open disk U is a shaded area and $Cl(U)$ is a disk.

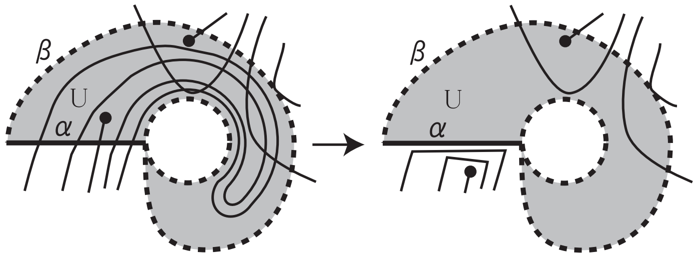

Fig. 8. The open disk U is a shaded area and $Cl(U)$ is not a disk.

LEMMA 3.3 ([5, Lemma 5.2]). *Let Γ be a minimal chart. Let e_1 be an edge of Γ_m with $\partial e_1 \subset \Gamma_{m+\varepsilon}$ $(\varepsilon \in \{+1, -1\})$. Let w_1 and w_2 be the white vertices of the edge e_1. Suppose that*

(1) *one of the two edges a_{11} and b_{12} contains an inward arc and the other contains an outward arc, and*

(2) *one of the two edges a_{12} and b_{11} contains an inward arc and the other contains an outward arc (see Figure 9).*

Then the edge e_1 contains at least one crossing in $\Gamma_m \cap \Gamma_{m+2\varepsilon}$. In particular if both edges a_{11} and b_{12} are terminal edges, or if both edges a_{12} and b_{11} are terminal edges, then e_1 contains at least two crossings in $\Gamma_m \cap \Gamma_{m+2\varepsilon}$.

Let α be an arc, and p, q points in α. We denote by $\alpha[p,q]$ the subarc of α whose end points are p and q.

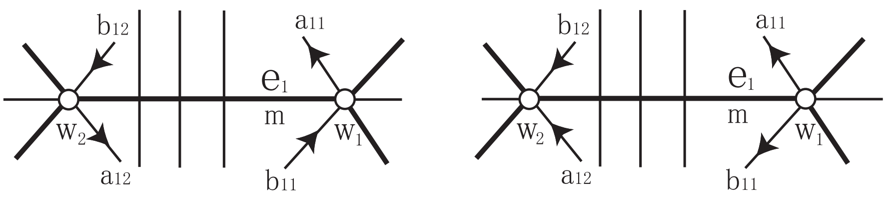

Fig. 9

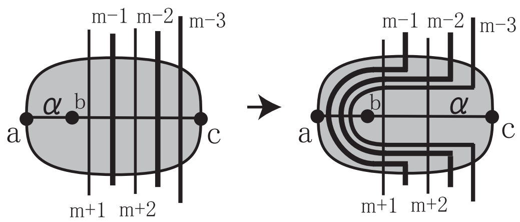

Fig. 10

Let Γ be a chart and a, b, c mutually different three points of an arc α with $b \in \alpha[a,c]$. The arc $\alpha[a,c]$ is said to be *a bipartition arc of Γ with the partition point b with respect to the label k* provided that

(1) $\alpha[a,c] \cap Cl(\Gamma_k - \alpha[a,c]) \subset \{a,c\}$,

(2) $\alpha[a,b] \cap \Gamma_j = \emptyset$ for all j $(j > k)$, and

(3) $\alpha[b,c] \cap \Gamma_i = \emptyset$ for all i $(i < k)$.

The following lemma will be used in the proof of Lemma 7.1.

LEMMA 3.4 ([5, Lemma 4.1]) (*Bipartition Lemma*). *Let Γ be a chart, and D a disk without any white vertices of Γ. Let α be a proper arc of the disk D. Let a, c be the end points of α, and b an interior point of α. Suppose that there exists an integer m with $Cl(\Gamma_m - \alpha) \cap Int(D) = \emptyset$ such that $\Gamma_i \cap \alpha$ is at most finitely many interior points of α for each i $(i \neq m)$. Then there exists a chart Γ^* obtained from Γ by C-I-R2 moves and C-I-R3 moves in D keeping Γ_m fixed such that (see Figure 10)*

(1) *the number of points in $\Gamma_i \cap \alpha$ is equal to the number of points in $\Gamma_i^* \cap \alpha$ for each i $(i \neq m)$, and*

(2) *the arc $\alpha[a,c]$ is a bipartition arc of Γ^* with the partition point b with respect to the label m.*

4. Loops

Let Γ be a chart. Let ℓ be a loop of label m, and w the white vertex in ℓ. Let e be the edge of Γ_m with $w \in e$ and $e \neq \ell$. Then the loop ℓ bounds two disks on the 2-sphere. One of the two disks does not contain the edge e. The disk is called *the associated disk of the loop ℓ* (see Figure 11).

Let D be a disk. We denote the number of white vertices in $Int(D)$ by $w(D)$.

LEMMA 4.1. *Let Γ be a minimal chart with a loop ℓ of label m, and let $\varepsilon \in \{+1,-1\}$ be the integer such that the white vertex in ℓ is contained in $\Gamma_{m+\varepsilon}$. Let D be the associated disk of ℓ. Then $Int(D)$ (resp. $S^2 - D$) contains at least one white vertex of $\Gamma_{m+\varepsilon}$ (resp. Γ_m). Hence $w(D) \geq 1$ and $w(Cl(S^2 - D)) \geq 1$.*

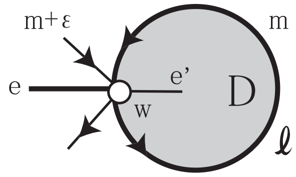

Fig. 11

Proof. Let e and e' be the edges of Γ_m and $\Gamma_{m+\varepsilon}$ respectively such that $w \in e$, $w \in e'$, $e \neq \ell$, and $e' \subset D$ (see Figure 11).

Since the edge e does not contain a middle arc at the white vertex w, it is not a terminal edge by Assumption 1. Hence there exists a white vertex of Γ_m in $S^2 - D$.

Since the edge e' of $\Gamma_{m+\varepsilon}$ does not contain a middle arc at w in D, we have $w(D) \geq 1$ in a similar way as above.

Lemma 4.2. *Let Γ be a minimal chart with a loop ℓ of label m. Let D be the associated disk of the loop ℓ. Then $w(D) \geq 2$ and $w(Cl(S^2 - D)) \geq 2$.*

Proof. By Lemma 4.1, there exists a white vertex of $\Gamma_{m+\varepsilon}$ in $Int(D)$. If $Int(D)$ contains only one white vertex of $\Gamma_{m+\varepsilon}$, then there exists a loop ℓ' of $\Gamma_{m+\varepsilon}$ in $Int(D)$. By Lemma 4.1, the associated disk of the loop ℓ' contains another white vertex in its interior. Hence we have $w(D) \geq 2$.

Similarly we have $w(Cl(S^2 - D)) \geq 2$.

We note that the statement "$w(Cl(S^2 - D)) \geq 2$" in Lemma 4.2 will be extended to "$w(Cl(S^2 - D)) \geq 3$" in Lemma 8.2.

5. 2-angled disks

Let Γ be an n-chart. Let F be a closed domain with $\partial F \subset \Gamma_{m-1} \cup \Gamma_m \cup \Gamma_{m+1}$ for some integer m, where $\Gamma_0 = \emptyset$ and $\Gamma_n = \emptyset$. By the condition (3) for charts, in a small neighborhood of each white vertex, there are three inward arcs and three outward arcs. Also in a small neighborhood of each black vertex, there exists only one inward arc or one outward arc. We often use the following fact, when we fix (inward or outward) arcs near white vertices and black vertices:

The number of inward arcs contained in $F \cap \Gamma_m$ is equal to the number of outward arcs in $F \cap \Gamma_m$.

When we use this fact, we say that we use *IO-Calculation with respect to Γ_m in F*. For example in a chart Γ, consider the pseudo chart as shown in Figure

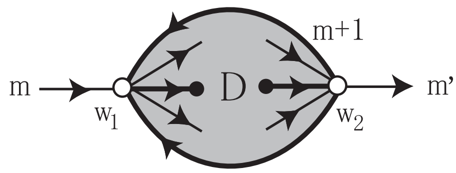

Fig. 12

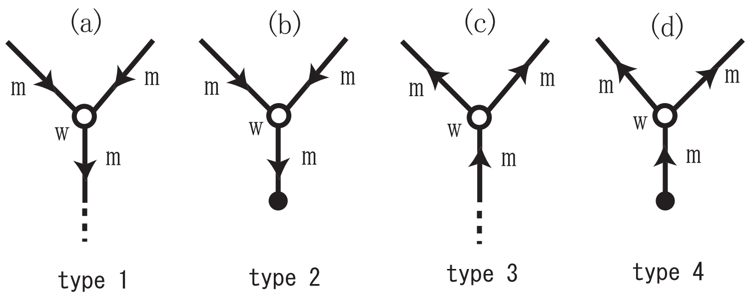

Fig. 13

12. Let D be the disk whose boundary is contained in Γ_{m+1} as shown in Figure 12. Suppose that $Int(D)$ contains neither white vertices nor other black vertices. Then we have $m' = m$. For, if $m' \neq m$, then the number of inward arcs in $D \cap \Gamma_m$ is zero, but the number of outward arcs in $D \cap \Gamma_m$ is two. This is a contradiction. Instead of the above argument, we say that

we have $m' = m$ by IO-Calculation with respect to Γ_m in D.

For each pseudo chart G,

$$\begin{aligned} IO(G; m) = {} & \text{the number of inward arcs of label } m \text{ in } G \\ & - \text{the number of outward arcs of label } m \text{ in } G. \end{aligned}$$

We often use the pseudo chart around a white vertex w as shown in Figure 13. The pseudo charts (a), (b), (c) and (d) are said to be of *type* 1, 2, 3 *and* 4 *with respect to* Γ_m respectively. When we want to emphasize white vertices, for the pseudo chart around a white vertex w of type k with respect to Γ_m, we often say that w is the *white vertex of type k with respect to Γ_m*. In IO-Calculation we often use the table of pseudo charts as shown in Figure 14. We call the table *IO-table*.

Let Γ be a chart. Let D be a disk. If ∂D consists of k edges of the subgraph Γ_m, then D is called *a k-angled disk of Γ_m*. Let N be a regular

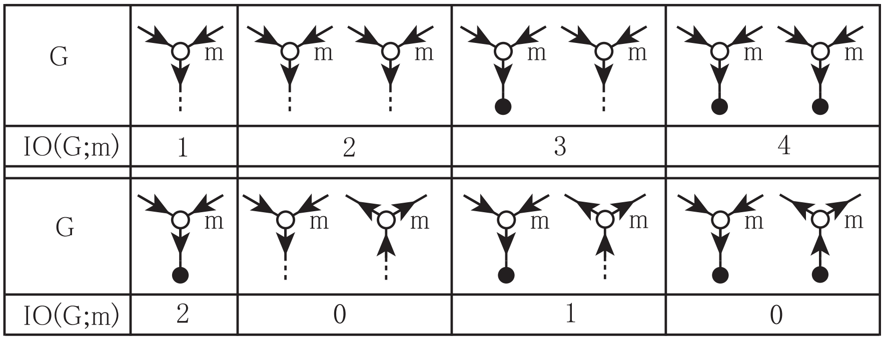

Fig. 14. IO-table.

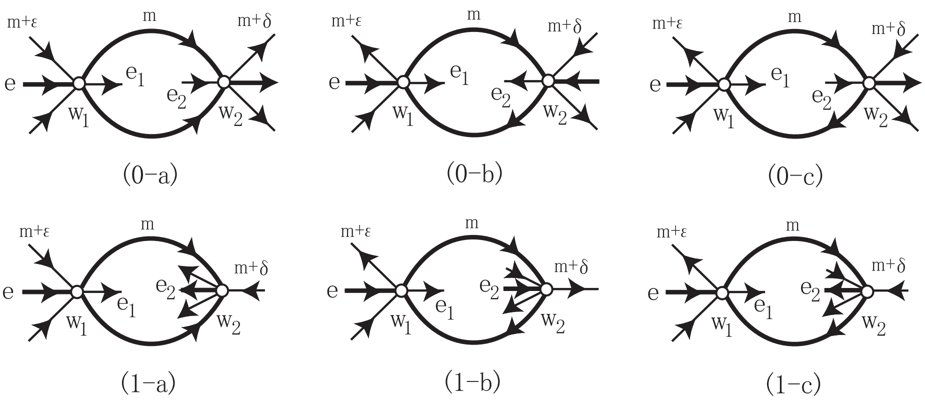

Fig. 15. The white vertex w_1 is in $\Gamma_m \cap \Gamma_{m+\varepsilon}$ and the white vertex w_2 is in $\Gamma_m \cap \Gamma_{m+\delta}$ where $\varepsilon, \delta \in \{+1, -1\}$.

neighborhood of ∂D in D. If $(N - \partial D) \cap \Gamma_m$ consists of s arcs, then D is called a *k-angled disk with s feelers*.

Let D be a 2-angled disk of Γ_m with at most one feeler, and e an edge of Γ_m containing a white vertex w_1 in ∂D but not contained in D. If necessary we take the reflection of the chart Γ or change the orientations of all of the edges, we have the above six 2-angled disks as shown in Figure 15. The three ones on the upper side are 2-angled disks without feelers and the others are 2-angled disks with one feeler.

By IO-Calculation with respect to $\Gamma_{m\pm1}$ or Γ_m in 2-angled disks and by Assumption 1, we have the following lemma:

Lemma 5.1. *Let Γ be a minimal chart. Let D be a 2-angled disk of Γ_m with at most one feeler. If D is not of type* (0–a) *nor* (0–c)*, then* $w(D) > 0$.

5.1. 2-angled disks without feelers

Lemma 5.2. *Let Γ be a minimal chart. Let D be a 2-angled disk of Γ_m of type* (0–a) *as shown in Figure* 15(0–a). *If $w(D) = 0$, then a regular neighborhood $N(D)$ contains one of the two pseudo charts as shown in Figure* 16.

Proof. We use the notations as shown in Figure 15(0–a). By Assumption 3 and Assumption 4, the disk D contains neither free edges, rings nor hoops.

If the edge e_1 is not a terminal edge of $\Gamma_{m+\varepsilon}$, then $e_1 = e_2$. Thus we have two lenses in D not containing white vertices in their interiors. This contradicts Lemma 3.1 (1). Hence e_1 is a terminal edge. Similarly, e_2 is a terminal edge of $\Gamma_{m+\delta}$.

If $\delta \neq \varepsilon$, then we have the pseudo chart as shown in Figure 16a.

Now suppose $\delta = \varepsilon$. Suppose that there exists at most one proper arc separating w_1 and w_2 in D which is contained in an edge of $\Gamma_{m+2\varepsilon}$.

Let e_3 and e_4 be the edges of Γ_m in ∂D. For $i = 1, 2$ let α_i be an arc almost parallel to the edge e_{i+2} such that $D \cap \alpha_i = \partial \alpha_i = \{w_1, w_2\}$. Let p_i and q_i be points in α_i near w_1 and w_2 respectively. Let $\alpha_i' = \alpha_i[p_i, q_i]$ for $i = 1, 2$ and D' the disk with $\partial D' = \alpha_1 \cup \alpha_2$ and $D' \supset D$ (see Figure 17).

Applying Disk Lemma (Lemma 3.2) for the disk D' and the boundary arc α_i', we have that Γ is (D', α_1')-arc free and (D', α_2')-arc free. Hence we can assume Γ is (D, e_3)-arc free and (D, e_4)-arc free. Hence for $i = 3, 4$ the edge

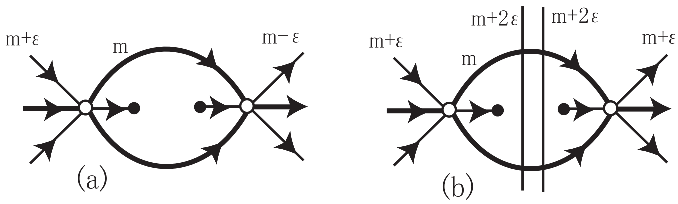

Fig. 16

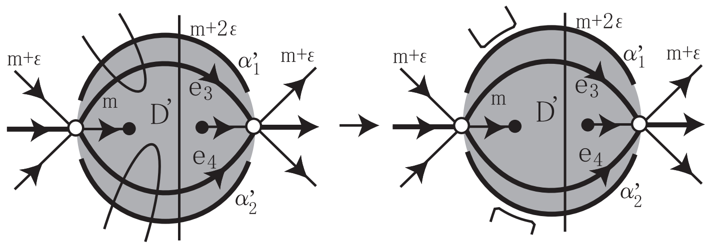

Fig. 17. The gray disk is the disk D'.

e_i contains at most one crossing in $\Gamma_m \cap \Gamma_{m+2\varepsilon}$. This contradicts Lemma 3.3. Therefore there exist at least two proper arcs separating w_1 and w_2 in D each of which is contained in an edge of $\Gamma_{m+2\varepsilon}$. Hence we have the pseudo chart as shown in Figure 16b.

LEMMA 5.3. *Let Γ be a minimal chart. Let D be a 2-angled disk of Γ_m of type* (0–b) *as shown in Figure* 15(0–b). *Then $w(D) \geq 1$. If $w(D) = 1$, then a regular neighborhood $N(D)$ contains one of the two pseudo charts as shown in Figure* 18.

PROOF. We use the notations as shown in Figure 15(0–b). By Lemma 5.1, we have $w(D) \geq 1$.

Suppose $w(D) = 1$. By IO-Calculation with respect to $\Gamma_{m+\varepsilon}$ in D, there exists a white vertex w_3 of type 2 with respect to $\Gamma_{m+\varepsilon}$ in $Int(D)$. Since $w_3 \in \Gamma_{m+\varepsilon}$, we have $w_3 \in \Gamma_m$ or $w_3 \in \Gamma_{m+2\varepsilon}$.

We show that $w_3 \in \Gamma_{m+2\varepsilon}$. If $w_3 \in \Gamma_m$, then there exists a terminal edge of Γ_m not containing a middle arc at w_3. This contradicts Assumption 1. Hence $w_3 \in \Gamma_{m+2\varepsilon}$. Therefore we have the pseudo charts as shown in Figure 18.

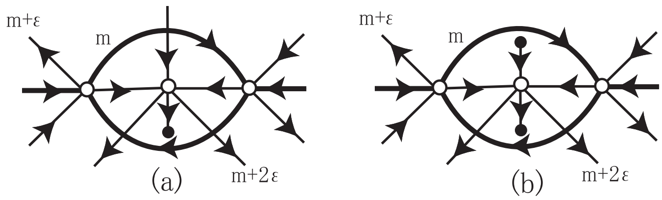

Fig. 18

LEMMA 5.4. *Let Γ be a minimal chart. Let D be a 2-angled disk of Γ_m of type* (0–c) *as shown in Figure* 15(0–c). *Then $w(D) \geq 2$.*

PROOF. We use the notations as shown in Figure 15(0–c).

If $e_1 = e_2$, then the edge e_1 separates D into two lenses. By Lemma 3.1 (1), $w(D) \geq 6$. Suppose that $e_1 \neq e_2$.

Neither e_1 nor e_2 contains a middle arc at w_1 or w_2, neither e_1 nor e_2 is a terminal edge by Assumption 1. If $\varepsilon = \delta$, then there exist at least two white vertices in D by IO-Calculation with respect to $\Gamma_{m+\varepsilon}$. Thus $w(D) \geq 2$. If $\varepsilon \neq \delta$, then each of e_1 and e_2 possesses a white vertex in $Int(D)$. Thus $w(D) \geq 2$.

LEMMA 5.5. *Let Γ be a minimal chart. Suppose that D is a 2-angled disk of Γ_m of type* (0–c) *as shown in Figure* 15(0–c) *and $w(D) = 2$. If necessary we change the orientations of all edges and if necessary we take the reflection of the*

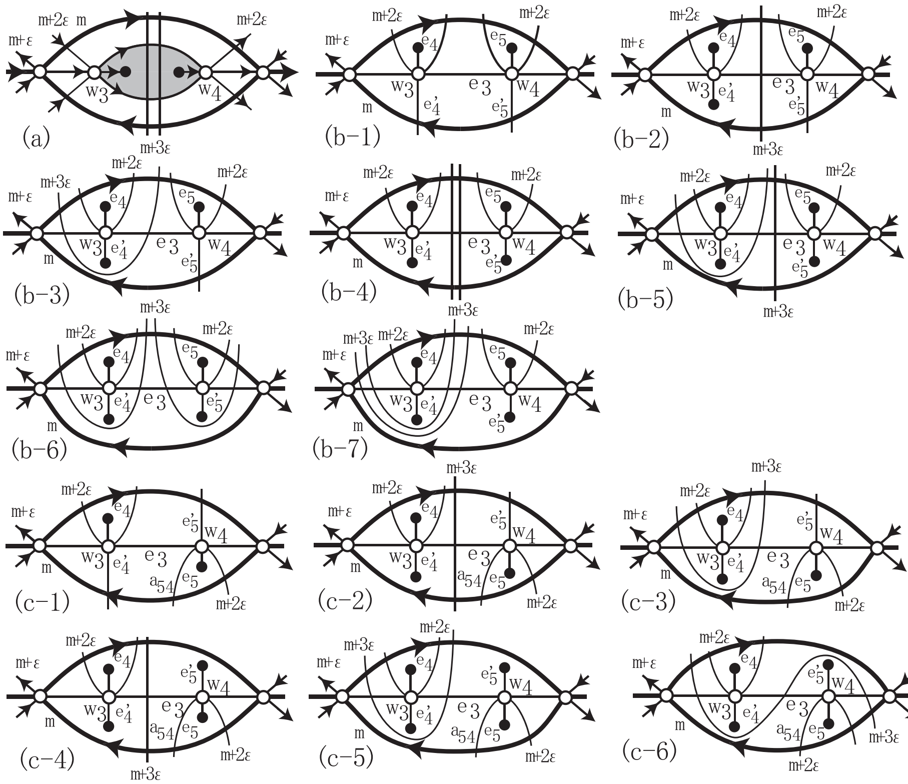

Fig. 19

chart Γ, then a regular neighborhood $N(D)$ contains one of the 14 *pseudo charts as shown in Figure* 19 *by C-moves in D keeping ∂D fixed.*

Proof. We use the notations as shown in Figure 15(0–c).

Suppose $\varepsilon \neq \delta$. By the proof of Lemma 5.4, both of edges e_1 and e_2 contain white vertices different from w_1 and w_2. Hence there exist a white vertex of $\Gamma_{m+\varepsilon}$ and a white vertex of $\Gamma_{m+\delta}$ in $Int(D)$. Since there exists only one white vertex of $\Gamma_{m+\varepsilon}$ in $Int(D)$, there exists a loop of label $m+\varepsilon$ in $Int(D)$. The associated disk of the loop contains at most one white vertex in its interior. This contradicts Lemma 4.2. Hence $\varepsilon = \delta$.

By the proof of Lemma 5.4 and IO-Calculation with respect to $\Gamma_{m+\varepsilon}$ in D, there exist two white vertices w_3 and w_4 of $\Gamma_{m+\varepsilon}$ in $Int(D)$ such that (1) one of w_3 and w_4 is of type 1 with respect to $\Gamma_{m+\varepsilon}$, and the other is of type 3 with respect to $\Gamma_{m+\varepsilon}$, or (2) one of w_3 and w_4 is of type 2 with respect to $\Gamma_{m+\varepsilon}$, and the other is of type 4 with respect to $\Gamma_{m+\varepsilon}$.

If there exists a loop of label $m+\varepsilon$ containing w_3 or w_4, then we have a contradiction in a similar way as above. Hence there does not exist any loop of label $m+\varepsilon$ containing w_3 or w_4.

For the case (1), we can show that there exists a 2-angled disk without feelers containing w_3 and w_4, say D' (see Figure 19a). Since $w(D')=0$, by Lemma 5.2, 5.3 and 5.4 the disk D' is a 2-angled disk of type (0–a). Hence the both edges in $\partial D'$ are oriented from one of w_3 and w_4 to the other. By IO-Calculation with respect to Γ_m in $Cl(D-D')$, we have $\{w_3, w_4\} \subset \Gamma_m$ or $\{w_3, w_4\} \subset \Gamma_{m+2\varepsilon}$.

If $\{w_3, w_4\} \subset \Gamma_m$, then there exist two lenses of $(m, m+\varepsilon)$ in D whose interiors do not contain any white vertices. This contradicts Lemma 3.1 (1). Hence $\{w_3, w_4\} \subset \Gamma_{m+2\varepsilon}$.

Applying Disk Lemma (Lemma 3.2) several times, we can assume that if a connected component of $D \cap \Gamma_{m+3\varepsilon}$ intersects the 2-angled disk D', then the arc is a proper arc in D separating w_3 and w_4 (cf. Figure 17). By Lemma 5.2, a regular neighborhood $N(D')$ contains the pseudo chart as shown in Figure 16b. Hence $D \cap \Gamma_{m+3\varepsilon}$ contains at least two proper arcs each of which separates w_3 and w_4. Hence a regular neighborhood $N(D)$ contains the pseudo chart as shown in Figure 19a.

For the case (2), there exists an edge e_3 of $\Gamma_{m+\varepsilon}$ containing w_3 and w_4. Since w_3 and w_4 are white vertices of type 2 or 4 with respect to $\Gamma_{m+\varepsilon}$, there exist terminal edges e_4 and e_5 of $\Gamma_{m+\varepsilon}$ containing w_3 and w_4 respectively.

The arc $e_1 \cup e_2 \cup e_3$ separates the disk D into two disks, say D_1 and D_2. By IO-Calculation with respect to Γ_m in D, we have $\{w_3, w_4\} \subset \Gamma_m$ or $\{w_3, w_4\} \subset \Gamma_{m+2\varepsilon}$.

Suppose $\{w_3, w_4\} \subset \Gamma_m$. By IO-Calculation with respect to Γ_m in D_1 and D_2, both of e_4 and e_5 are contained in D_1 or D_2. There exists a lens of $(m, m+\varepsilon)$ in D whose interior does not contain any white vertices. This contradicts Lemma 3.1 (1). Hence $\{w_3, w_4\} \subset \Gamma_{m+2\varepsilon}$.

There are two possibilities: (i) Both of e_4 and e_5 are contained in D_1 or D_2 (see Figure 19b), or (ii) one of e_4 and e_5 is contained in D_1 and the other is contained in D_2 (see Figure 19c).

Let e_4', e_5' be the edges of $\Gamma_{m+2\varepsilon}$ containing w_3 and w_4 respectively and different from a_{43}, b_{43}, a_{54} and b_{54}. There are three possibilities: (a) Neither e_4' nor e_5' is a terminal edge (see Figure 19b-1 and Figure 19c-1), (b) only one of e_4' and e_5' is a terminal edge (see Figure 19b-2, b-3 and Figure 19c-2, c-3), or (c) both of e_4' and e_5' are terminal edges (see Figure 19b-4, b-5, b-6, b-7 and Figure 19c-4, c-5, c-6).

Applying Disk Lemma (Lemma 3.2) twice, $D \cap \Gamma_{m+2\varepsilon}$ is one of pseudo charts as shown in Figure 19b and c. Applying Disk Lemma (Lemma 3.2) several times, we can assume that if a connected component of $D \cap \Gamma_{m+3\varepsilon}$

intersects the edge e_3, then the arc is a proper arc in D separating w_3 and w_4.

Suppose that the edges e_4' and e_5' satisfy the conditions (i) and (b) (see Figure 19b-2, b-3). By Lemma 3.3, e_3 contains at least one crossing in $\Gamma_{m+\varepsilon} \cap \Gamma_{m+3\varepsilon}$. Hence $D \cap \Gamma_{m+3\varepsilon}$ contains at least one proper arc separating w_3 and w_4.

Suppose that the edges e_4' and e_5' satisfy the conditions (i) and (c) (see Figure 19b-4, b-5, b-6, b-7). By Lemma 3.3, e_3 contains at least two crossings in $\Gamma_{m+\varepsilon} \cap \Gamma_{m+3\varepsilon}$. Hence $D \cap \Gamma_{m+3\varepsilon}$ contains at least two proper arcs separating w_3 and w_4.

Suppose that the edges e_4' and e_5' satisfy the conditions (ii) and (b) or the conditions (ii) and (c) (see Figure 19c-2, c-3, c-4, c-5, c-6). Without loss of generality we can assume that e_4' is a terminal edge. Suppose $e_3 \cap \Gamma_{m+3\varepsilon} = \emptyset$. Let α be an arc connecting the black vertex in e_4' and a point in a_{54} such that $Int(\alpha) \cap (\Gamma_{m+\varepsilon} \cup \Gamma_{m+2\varepsilon} \cup \Gamma_{m+3\varepsilon}) = \emptyset$. By C-II moves, we can assume $Int(\alpha) \cap \Gamma = \emptyset$. Since we apply a C-I-M2 move between e_4' and a_{54}, we have a new terminal edge containing w_4 but not containing a middle arc at w_4. This contradicts Assumption 1. Hence e_3 contains at least one crossing in $\Gamma_{m+\varepsilon} \cap \Gamma_{m+3\varepsilon}$. Hence $D \cap \Gamma_{m+3\varepsilon}$ contains at least one proper arc separating w_3 and w_4.

5.2. 2-angled disks with one feeler

Lemma 5.6. *Let Γ be a minimal chart. Let D be a 2-angled disk of Γ_m of type* (1–a) *as shown in Figure* 15(1–a). *Then $w(D) \geq 1$. If $w(D) = 1$, then a regular neighborhood $N(D)$ contains the pseudo chart as shown in Figure* 20.

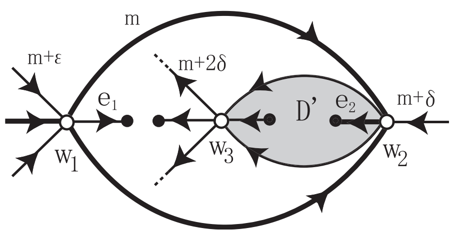

Fig. 20

Proof. We use the notations as shown in Figure 15(1–a). By Lemma 5.1, we have $w(D) \geq 1$.

Suppose $w(D) = 1$. By IO-Calculation with respect to $\Gamma_{m+\delta}$ in D, there exists a white vertex w_3 of type 2 with respect to $\Gamma_{m+\delta}$. There exists a 2-

angled disk D' of $\Gamma_{m+\delta}$ with $w(D')=0$ and $D' \subset D$. Since the both two edges in $\partial D'$ are oriented from w_2 to w_3, the 2-angled disk D' is of type (0–a) or (1–a). By Lemma 5.1 and 5.2, the 2-angled disk D' is of type (0–a). By IO-Calculation with respect to Γ_m in $Cl(D-D')$, we have $w_3 \in \Gamma_{m+2\delta}$. Hence e_1 is a terminal edge. By Lemma 5.2, we have the pseudo chart as shown in Figure 20.

Lemma 5.7. *Let Γ be a minimal chart. Let D be a* 2*-angled disk of Γ_m of type* (1–b) *or* (1–c) *as shown in Figure* 15(1–b) *or* (1–c). *Then $w(D) \geq 3$.*

Proof. We use the notations as shown in Figure 15(1–b) and (1–c).

Since the edge e_2 of Γ_m does not contain a middle arc at w_2, by Assumption 1 there exists a white vertex w_3 of Γ_m with $\partial e_2 = \{w_2, w_3\}$. Hence $w(D) \geq 1$.

If there exists only one white vertex of Γ_m in $Int(D)$, then there exists a loop of Γ_m in D. By Lemma 4.2, $w(D) \geq 3$. Hence we can assume that there exist at least two white vertices of Γ_m in $Int(D)$.

Suppose $w(D)=2$. Then there exists a 2-angled disk D' of Γ_m in D with $w(D')=0$ and $w_3 \in \partial D'$. Let w_4 be the white vertex in $\partial D'$ with $w_4 \neq w_3$. Let e_4 be the edge of Γ_m with $e_4 \ni w_4$ and $e_4 \not\subset \partial D'$. Then e_4 is a terminal edge. Thus the both edges in $\partial D'$ are oriented from one of w_3 and w_4 to the other one. Since $e_2 \not\subset D'$, the 2-angled disk D' is of type (0–a) or (1–a). Since $w(D')=0$, the 2-angled disk D' is of type (0–a) by Lemma 5.2 and 5.6.

If D is of type (1–b) (see Figure 21a), then none of the six edges e_1, b_{22}, a_{23}, b_{23}, a_{44}, b_{44} contain middle arcs at w_1, w_2, w_3, w_3, w_4, w_4 respectively. By Assumption 1 none of the six edges are terminal edges. The four edges e_1, b_{22}, a_{23}, b_{23} contain outward arcs at w_1, w_2, w_3, w_3 respectively. We have a contradiction by IO-Calculation with respect to $\Gamma_{m\pm\delta}$ in $Cl(D-D')$. Thus $w(D) \geq 3$.

If D is of type (1–c) (see Figure 21b), then none of the six edges e_1, a_{22}, a_{23}, b_{23}, a_{44}, b_{44} contain middle arcs at w_1, w_2, w_3, w_3, w_4, w_4 respectively. By

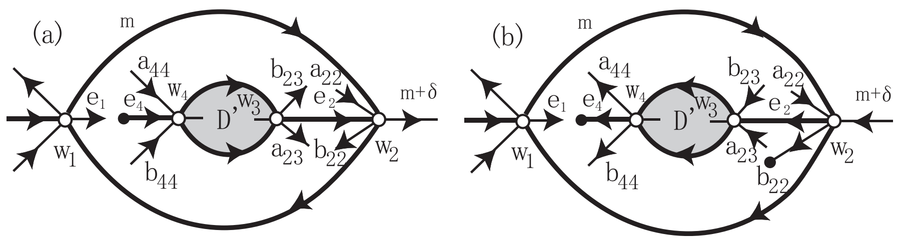

Fig. 21

Assumption 1, none of the six edges are terminal edges. The four edges e_1, b_{22}, a_{44}, b_{44} contain outward arcs at w_1, w_2, w_4, w_4 respectively and the three edges a_{22}, a_{23}, b_{23} contain inward arcs at w_2, w_3, w_3 respectively. By IO-Calculation with respect to $\Gamma_{m\pm\delta}$ in $Cl(D-D')$, the edge b_{22} of $\Gamma_{m+\delta}$ is a terminal edge. If $a_{44}=a_{22}$ or $a_{44}=a_{23}$, then we have a contradiction by IO-Calculation. Thus we have $a_{44}=b_{23}$. However there exists a lens of type $(m, m\pm\delta)$ in D whose interior does not contain any white vertices. This contradicts Lemma 3.1 (1). Therefore $w(D)\geq 3$.

By Lemma 5.2, 5.3, 5.4, 5.6 and 5.7, we have the following corollary:

COROLLARY 5.8. *Let Γ be a minimal chart. Let D be a* 2*-angled disk of Γ_m with at most one feeler. If $w(D)=0$, then D is of type* (0–a) *and a regular neighborhood $N(D)$ contains one of the two pseudo charts as shown in Figure* 16.

If D is a 2-angled disk of type (1–b) or (1–c), then so is $Cl(S^2-D)$. By Lemma 5.7, we have the following corollary. We shall use this corollary in [6].

COROLLARY 5.9. *Let Γ be a minimal chart with at most seven white vertices. Then there does not exist any* 2*-angled disk of Γ_m of type* (1–b) *nor* (1–c).

6. Types of charts

LEMMA 6.1. *Let Γ be a minimal chart of type $(m; n_1, n_2, \dots, n_k)$. For each label i, if a connected component of Γ_i contains a white vertex, then it contains at least two white vertices. Hence $n_1>1$ and $n_k>1$.*

PROOF. Let Σ be a connected component of Γ_i containing only one white vertex. In a neighborhood of the white vertex, among three arcs contained in edges of Γ_i there exists only one middle arc. Thus there exists a terminal edge of Σ which does not contain a middle arc. This contradicts Assumption 1.

In our argument we often construct a chart Γ. On the construction of a chart Γ, for a white vertex w, among the three edges of Γ_m containing w, if we have specified two edges and if the last edge of Γ_m containing w contains a black vertex (see Figure 22a and b), then we remove the edge containing the black vertex and put a black dot at the center of the white vertex as shown in Figure 22c, we call it a *BW-vertex*.

For example, the graph as shown in Figure 23a means one of the four graphs as shown in Figure 23b.

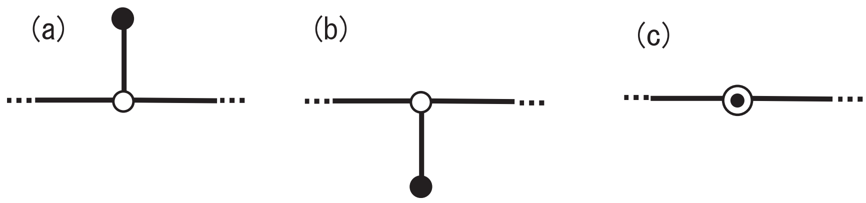

Fig. 22

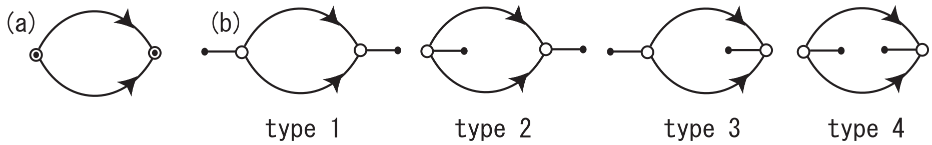

Fig. 23

LEMMA 6.2. *Let Γ be a minimal chart. Let Σ be a connected component of Γ_m containing a white vertex. If Σ contains at most three white vertices, then it is one of six subgraphs as shown in Figure* 24.

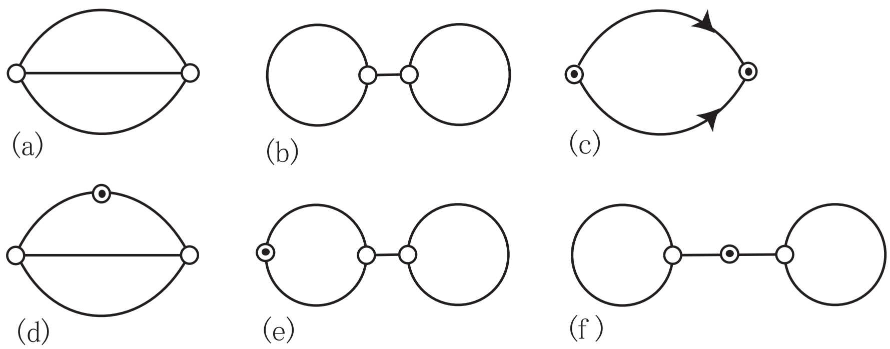

Fig. 24

PROOF. By Lemma 6.1, Σ contains at least two white vertices.

Suppose that Σ contains exactly two white vertices. If Σ does not contain any terminal edge, then Σ is one of the two subgraphs as shown in Figure 24a and b. If Σ contains at least one terminal edge, Σ is the subgraph as shown in Figure 24c.

Suppose that Σ contains exactly three white vertices. By IO-Calculation with respect to Γ_m, Σ contains exactly one terminal edge of Γ_m. Hence Σ is obtained by adding one BW-vertex on an edge of the two subgraphs as shown in Figure 24a and b. Now Σ is contained in the 2-sphere. Thus Σ is one of the three subgraphs as shown in Figure 24d, e and f.

We call the subgraphs a, b, c, d in Figure 24 a *θ-curve*, *a pair of eyeglasses*, an *oval* and a *skew θ-curve* respectively. We call the subgraphs as shown in Figure 24e and f *pairs of skew eyeglasses of type* 1 *and* 2 respectively.

LEMMA 6.3. *Let Γ be a minimal chart of type $(m; n_1, n_2, \dots, n_k)$. Suppose that $n_1 = 2$ (resp. $n_k = 2$), and Γ_m (resp. Γ_{m+k}) contains a θ-curve. Then $n_2 > 3$ (resp. $n_{k-1} > 3$).*

PROOF. Suppose $n_1 = 2$ and Γ_m contains a θ-curve. The θ-curve separates the 2-sphere into three 2-angled disks of Γ_m. Two of them are of type (0–c), say D_1 and D_2. We use the notations as shown in Figure 25a.

By Lemma 3.1 (2), $a_{11} \neq b_{12}$ and $b_{11} \neq a_{12}$. By Assumption 1, none of a_{11}, b_{11}, a_{12}, b_{12} are terminal edges. By IO-Calculation with respect to Γ_{m+1} in D_i for $i = 1, 2$, $Int(D_i)$ contains at least two white vertices of Γ_{m+1}. Therefore $n_2 > 3$.

Similarly we can show for the case $n_k = 2$.

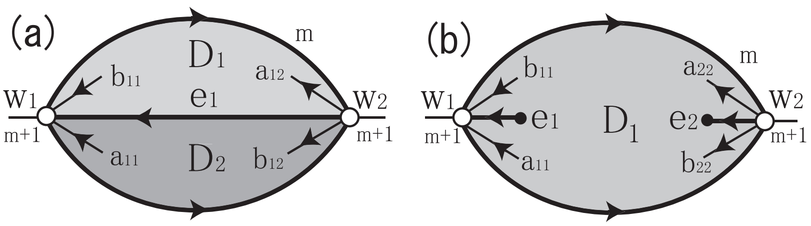

Fig. 25

LEMMA 6.4. *Let Γ be a minimal chart of type $(m; n_1, n_2, \dots, n_k)$. Suppose that $n_1 = 2$ (resp. $n_k = 2$), and Γ_m (resp. Γ_{m+k}) contains a pair of eyeglasses. Then $n_2 > 1$ (resp. $n_{k-1} > 1$).*

PROOF. Suppose that $n_1 = 2$ and Γ_m contains a pair of eyeglasses. Then Γ_m contains two loops. By Lemma 4.1, the associated disk of each loop contains at least one white vertex of Γ_{m+1}. Hence $n_2 > 1$.

Similarly we can show for the case $n_k = 2$.

LEMMA 6.5. *Let Γ be a minimal chart of type $(m; n_1, n_2, \dots, n_k)$. Suppose that $n_1 = 2$ (resp. $n_k = 2$), and Γ_m (resp. Γ_{m+k}) contains an oval. Then $n_2 > 1$ (resp. $n_{k-1} > 1$).*

Proof. Suppose that $n_1 = 2$ and that Γ_m contains an oval, say Σ. Then the oval Σ separates the 2-sphere into two 2-angled disks of Γ_m, say D_1 and D_2.

If D_1 is a 2-angled disk with one feeler, then so is D_2. By IO-Calculation with respect to Γ_{m+1} in D_i for $i = 1, 2$, there exists a white vertex of Γ_{m+1} in $Int(D_i)$. Hence $n_2 > 1$.

Hence we may assume that D_1 is a 2-angled disk with two feelers and D_2 is a 2-angled disk without feelers. We use the notations as shown in Figure 25b. Since none of the four edges a_{11}, b_{11}, a_{22} and b_{22} contain middle arcs at w_1, w_1, w_2 and w_2 respectively, none of the edges are terminal edges by Assumption 1.

By IO-Calculation with respect to Γ_{m+1} in D_1, we have $n_2 \neq 1$. If $n_2 = 0$, then $a_{11} = b_{22}$ and $b_{11} = a_{22}$. Hence there exist two lenses of type $(m, m+1)$. This contradicts Lemma 3.1 (2). Hence $n_2 > 1$.

Similarly we can show for the case $n_k = 2$.

By Lemma 6.2, 6.3, 6.4 and 6.5, we have the following proposition:

Proposition 6.6. *Let Γ be a minimal chart of type $(m; n_1, n_2, \ldots, n_k)$. If $n_1 = 2$ (resp. $n_k = 2$), then $n_2 > 1$ (resp. $n_{k-1} > 1$). Hence there does not exist a minimal chart of type $(m; 2, 0, \ldots, n_k)$, $(m; 2, 1, \ldots, n_k)$, $(m; n_1, n_2, \ldots, 0, 2)$ nor $(m; n_1, n_2, \ldots, 1, 2)$.*

In a similar way as the one of Lemma 6.5, we have the following lemma. We shall use this lemma in [6].

Lemma 6.7. *Let Γ be a minimal chart. Let D be a 2-angled disk of Γ_m with two feelers such that ∂D is contained in an oval of Γ_m. Then $w(D) \geq 2$.*

Lemma 6.8. *Let Γ be a minimal chart of type $(m; n_1, n_2, \ldots, n_k)$. Suppose that $n_1 = 3$ (resp. $n_k = 3$), and Γ_m (resp. Γ_{m+k}) contains a skew θ-curve. Then $n_2 > 1$ (resp. $n_{k-1} > 1$).*

Proof. Suppose $n_1 = 3$ and Γ_m contains a skew θ-curve, say Σ. Then Σ separates the 2-sphere into three disks. One is a 3-angled disk of Γ_m with one feeler, say A. One is a 2-angled disk of Γ_m without feelers, say B. We use the notations as shown in Figure 26. Without loss of generality we can assume that the terminal edge of Γ_m contains an outward arc at the white vertex w_1.

By Assumption 1, the terminal edge contains a middle arc. Hence the other edges of Γ_m containing w_1 are oriented to w_1. Without loss of generality we can assume the edge $\partial A \cap \partial B$ is oriented from w_2 to w_3. The other edge in ∂B is oriented from w_3 to w_2.

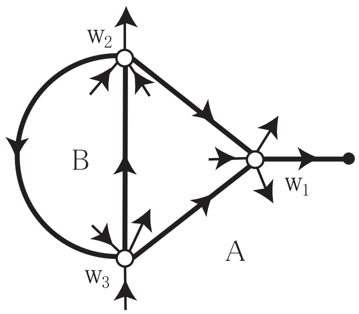

Fig. 26

By IO-Calculation with respect to Γ_{m+1} in the domains A and B, there exists a white vertex of Γ_{m+1} in both $Int(A)$ and $Int(B)$. Hence $n_2 > 1$.

Similarly we can show for the case $n_k = 3$.

Let Γ be a chart containing a pair of skew eyeglasses Σ of Γ_m of type 1 (see Figure 27a). Let D_1 be the associated disk of the loop ℓ in Σ and D_2 a 2-angled disk of Γ_m with $\partial D_2 \subset \Sigma$ and $D_1 \cap D_2 = \emptyset$. Let e_4 be the terminal edge in the pair of skew eyeglasses. Without loss of generality, we can assume that the two edges e_3 and a_{31} contain outward arcs at w_1. We use the notations as shown in Figure 27a. Suppose that Γ is a minimal chart. Then the loop ℓ and the other edges are oriented automatically as shown in Figure 27a. Note that the terminal edge e_4 is oriented from w_3 to the black vertex.

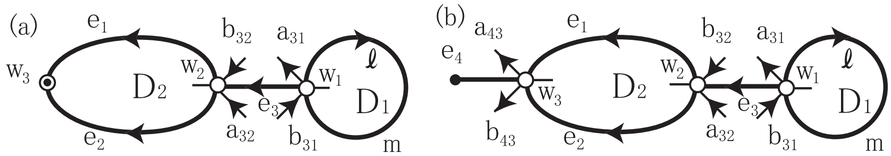

Fig. 27

LEMMA 6.9. *Let Γ be a minimal chart of type $(m; n_1, n_2, \dots, n_k)$. Suppose that $n_1 = 3$ (resp. $n_k = 3$), and Γ_m (resp. Γ_{m+k}) contains a pair of skew eyeglasses of type* 1. *Then $n_2 > 1$ (resp. $n_{k-1} > 1$).*

PROOF. Suppose $n_1 = 3$ and Γ_m contains a pair of skew eyeglasses of type 1. We use the notations as shown in Figure 27a and b. By Lemma 4.1, $Int(D_1)$ contains a white vertex of Γ_{m+1}.

If the terminal edge e_4 of Γ_m is contained in the disk D_2, then there exists a white vertex of Γ_{m+1} in $Int(D_2)$ by IO-Calculation with respect to Γ_{m+1} in D_2. Thus $n_2 > 1$.

Suppose that $e_4 \not\subset D_2$ (see Figure 27b). Suppose that there does not exist any white vertex of Γ_{m+1} in $S^2 - (D_1 \cup D_2)$. Since none of the five edges b_{31}, a_{32}, b_{32}, a_{43} and b_{43} contain middle arcs at w_1, w_2, w_2, w_3 and w_3 respectively, none of the edges are terminal edges. There are three possibilities: $a_{32} = a_{31}$, $a_{32} = a_{43}$ and $a_{32} = b_{43}$. By IO-Calculation with respect to Γ_{m+1}, we have $a_{32} = b_{43}$. However the curve $a_{32} \cup e_2$ bounds a lens of type $(m, m+1)$. This contradicts Lemma 3.1 (2). Hence there exists a white vertex of Γ_{m+1} in $S^2 - (D_1 \cup D_2)$. Therefore $n_2 > 1$.

Similarly we can show for the case $n_k = 3$.

Since a pair of skew eyeglasses of type 2 contain two loops, we can prove the following lemma by the similar way as the one of Lemma 6.4.

Lemma 6.10. *Let Γ be a minimal chart of type $(m; n_1, n_2, \dots, n_k)$. Suppose that $n_1 = 3$ (resp. $n_k = 3$), and Γ_m (resp. Γ_{m+k}) contains a pair of skew eyeglasses of type 2. Then $n_2 > 1$ (resp. $n_{k-1} > 1$).*

By Lemma 6.2, 6.8, 6.9 and 6.10, we have the following proposition:

Proposition 6.11. *Let Γ be a minimal chart of type $(m; n_1, n_2, \dots, n_k)$. If $n_1 = 3$ (resp. $n_k = 3$), then $n_2 > 1$ (resp. $n_{k-1} > 1$). Hence there does not exist a minimal chart of type $(m; 3, 0, \dots, n_k)$, $(m; 3, 1, \dots, n_k)$, $(m; n_1, n_2, \dots, 0, 3)$ nor $(m; n_1, n_2, \dots, 1, 3)$.*

We show the first main theorem as follows:

Proof of Theorem 1.1. Since Γ contains exactly seven white vertices, we have $n_1 + n_2 + \cdots + n_k = 7$. Moreover since $n_1 > 1$ and $n_k > 1$ by Lemma 6.1, we have $n_1 = 2, 3, 4, 5$ or 7. If necessary we change the label $m + i$ by $m + k - i$ for all label i, then we can assume $n_1 \geq n_k$.

If $n_1 = 7$, then the chart Γ is of type (7).

If $n_1 = 5$, then $n_k = 2$ $(k \geq 2)$. Since there is no chart of type $(\dots, 0, 2)$ by Proposition 6.6, the chart Γ is of type $(5, 2)$.

If $n_1 = 4$, then $n_k = 2$ or 3. Since there is no chart of type $(\dots, 0, 2)$ nor $(\dots, 1, 2)$ by Proposition 6.6, we have $n_k = 3$. Since there is no chart of type $(\dots, 0, 3)$ by Proposition 6.11, the chart Γ is of type $(4, 3)$.

If $n_1 = 3$, then $n_k = 2$ or 3. Since there is no chart of type $(3, 0, \dots)$ nor $(3, 1, \dots)$ by Proposition 6.11, we have $n_k = 2$. Since there is no chart of type $(\dots, 0, 2)$ nor $(\dots, 1, 2)$ by Proposition 6.6, the chart Γ is of type $(3, 2, 2)$.

If $n_1 = 2$, then $n_k = 2$. Since there is no chart of type $(2, 0, \dots)$, $(2, 1, \dots)$, $(\dots, 0, 2)$ nor $(\dots, 1, 2)$ by Proposition 6.6, the chart Γ is of type $(2, 3, 2)$.

7. Complements of lenses

Let Γ be a chart and D a disk. If ∂D consists of an edge of Γ_m and an edge of Γ_{m+1}, then D is called a *bicolored 2-angled disk*. Let w_1 and w_2 be the white vertices in ∂D. For $i = 1, 2$, let N_i be a regular neighborhood of w_i in D. If $(N_i - \partial D) \cap \Gamma$ consists of s_i arcs, then we say that D is a bicolored 2-angled disk of *type* (s_1, s_2). Note that a lens is a bicolored 2-angled disk of type $(0, 0)$.

LEMMA 7.1. *Let Γ be a chart. Let D be the bicolored 2-angled disk of type $(2, 2)$ as shown in Figure 28. Suppose that $w(D) = 2$ and the two white vertices w_3 and w_4 are in $\Gamma_{m+\varepsilon} \cap \Gamma_{m+2\varepsilon}$ where $\varepsilon \in \{+1, -1\}$. Then Γ is not minimal.*

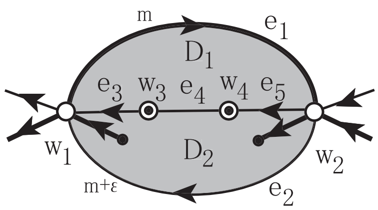

Fig. 28. The gray disk D is of type $(2, 2)$.

PROOF. We use the notations as shown in Figure 28. Suppose $\varepsilon = +1$.

Let b be a point in the interior of the edge e_5. By Bipartition Lemma (Lemma 3.4), the edge $e_5 = e_5[w_4, w_2]$ is a bipartition arc of Γ with the partition point b with respect to the label m. Using C-I-R2 moves, C-I-R3 moves and C-I-R4 moves, we can push the arcs in edges of Γ_i for all $i < m$ intersecting the edge e_5 to the other side of the white vertex w_4. Hence we can assume that $e_5 \cap \Gamma_i = \emptyset$ for all $i < m$. Hence $e_5 \cap \Gamma_{m-1} = \emptyset$.

Similarly we can assume that $e_4 \cap \Gamma_i = \emptyset$ for all $i < m$. Thus $e_4 \cap \Gamma_{m-1} = \emptyset$.

The arc $e_3 \cup e_4 \cup e_5$ separates D into two disks. For $i = 1, 2$ let D_i be the one of the two disks with $e_i \subset D_i$.

Since Γ_m does not contain any crossings of label $m - 1$ and since $(e_4 \cup e_5) \cap \Gamma_{m-1} = \emptyset$, we have $\partial D_1 \cap \Gamma_{m-1} = e_3 \cap \Gamma_{m-1}$. By Disk Lemma (Lemma 3.2), Γ is (D_1, e_3)-arc free (cf. Figure 17). Hence we can assume that $\partial D_1 \cap \Gamma_{m-1} = \emptyset$. By Disk Lemma (Lemma 3.2), Γ is (D_2, e_2)-arc free, too. Hence $\partial D_2 \cap \Gamma_{m-1} = \emptyset$ and $e_2 \cap \Gamma_{m-1} = \emptyset$.

Since $e_2 \cap \Gamma_{m-1} = \emptyset$, we can apply a C-I-M2 move between the two terminal edges of Γ_m along the edge e_2. Then we obtain a new free edge. Hence the number of free edges increases. Therefore Γ is not minimal.

Similarly we can show for the case $\varepsilon = -1$.

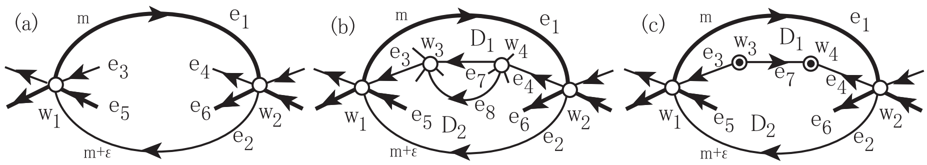

Fig. 29

Lemma 7.2. *Let Γ be a minimal chart. Let D be a bicolored* 2*-angled disk of type* $(2,2)$ *as shown in Figure* 29a. *Then* $w(D) \geq 3$.

Proof. We use the notations as shown in Figure 29a. Suppose $w(D) \leq 2$.

If $e_3 = e_4$, then the edge e_3 separates D into two disks. One of the two disks is a lens D' in D with $w(D') \leq 2$. This contradicts Lemma 3.1 (1). Hence $e_3 \neq e_4$. Similarly we have $e_5 \neq e_6$.

By IO-Calculation with respect to $\Gamma_{m+\varepsilon}$ in D, there exist two white vertices w_3 and w_4 of $\Gamma_{m+\varepsilon}$ in $Int(D)$. Without loss of generality we can assume that $\partial e_3 = \{w_1, w_3\}$.

Suppose that $\partial e_4 = \{w_2, w_3\}$. Let e be the edge of $\Gamma_{m+\varepsilon}$ adjacent to w_3 with $e \neq e_3$ nor $e \neq e_4$. Then e does not contain a middle arc at w_3. Hence e must contain the vertex w_4. Since there is no white vertex different from w_3 and w_4 in $Int(D)$, there exists a loop of label $m+\varepsilon$ containing the white vertex w_4 whose associated disk does not contain any white vertex in its interior. This contradicts Lemma 4.2. Therefore $\partial e_4 = \{w_2, w_4\}$.

Since there exist only two white vertices in $Int(D)$, there is no loop adjacent to the white vertices w_3 nor w_4. Hence there exists an edge e_7 of $\Gamma_{m+\varepsilon}$ with $\partial e_7 = \{w_3, w_4\}$ in D. Hence there are two possibilities of pseudo charts as shown in Figure 29b and c.

For the case (b), the set $e_3 \cup e_4 \cup e_7 \cup e_8$ separates D into three disks. For $i = 1, 2$ let D_i be the one of the three disks with $e_i \subset D_i$. Let D_3 be the last one.

Since e_3 contains an outward arc at w_3, one of e_7 and e_8 contains an inward arc at w_3. Since $\partial D_3 = e_7 \cup e_8$ and since the disk D_3 is a 2-angled disk without feelers such that $w(D_3) = 0$, we have that D_3 is of type (0–a) and both of e_7 and e_8 contain inward arcs at w_3 by Corollary 5.8.

By IO-Calculation with respect to Γ_m in D_1, both vertices w_3 and w_4 are in Γ_m or $\Gamma_{m+2\varepsilon}$ at the same time.

If w_3 and w_4 are in Γ_m, then in D_1 there exists a lens of type $(m, m+\varepsilon)$ whose interior does not contain any white vertices. If w_3 and w_4 are in $\Gamma_{m+2\varepsilon}$, then in D_2 there exists a lens of type $(m+\varepsilon, m+2\varepsilon)$ whose interior does not contain any white vertices. For each cases we have a contradiction to Lemma 3.1 (1).

For the case (c), the set $e_3 \cup e_4 \cup e_7$ separates D into two disks. For $i = 1, 2$ let D_i be the one of the two disks with $e_i \subset D_i$.

If $\{w_3, w_4\} \subset \Gamma_{m+2\varepsilon}$, then $e_5 \neq e_6$ implies that e_5 and e_6 are terminal edges. This contradicts Lemma 7.1. Hence $\{w_3, w_4\} \not\subset \Gamma_{m+2\varepsilon}$.

If one of the two vertices w_3 and w_4 is in $\Gamma_{m+2\varepsilon}$, then there exists a loop of label m in D whose associated disk contains at most one white vertex in its interior. This contradicts Lemma 4.2. Thus we have $\{w_3, w_4\} \subset \Gamma_m$.

Let e' and e'' be the terminal edges of $\Gamma_{m+\varepsilon}$ containing w_3 and w_4 respectively. By IO-Calculation with respect to Γ_m in D_1 and D_2 at the same time, both edges e' and e'' are in D_1 or D_2. Then there exists a lens D' of type $(m, m+\varepsilon)$ in D_1 or D_2 with $w(D') = 0$. This contradicts Lemma 3.1 (1). Therefore $w(D) \geq 3$.

If D is a bicolored 2-angled disk of type $(2, 2)$ as shown in Figure 29a, then so is $Cl(S^2 - D)$. By Lemma 7.2, we have the following corollary. We shall use this corollary in [6].

Corollary 7.3. *Let Γ be a minimal chart with at most seven white vertices. Then there does not exist any bicolored* 2*-angled disk of type* $(2, 2)$ *as shown in Figure* 29a.

Proposition 7.4. *Let Γ be a minimal chart. For any lens D of type* 1*, $S^2 - D$ contains at least three white vertices.*

Proof. Without loss of generality we can assume that D is of type $(m, m+1)$. Suppose that $S^2 - D$ contains at most two white vertices. Let $D_1 = Cl(S^2 - D)$. Then $w(D_1) \leq 2$ and D_1 is a bicolored 2-angled disk of type $(4, 4)$. We use the notations as shown in Figure 30a.

If $e_1 = e_2$ or $e_1' = e_2'$, then D_1 contains a bicolored 2-angled disk of type $(2, 2)$ as shown in Figure 29a. By Lemma 7.2, $w(D_1) \geq 3$. This is a contradiction. Hence $e_1 \neq e_2$ and $e_1' \neq e_2'$.

If e_1 is a loop, then by Lemma 4.2 the associated disk D' of the loop contains at least two white vertices in its interior. Since $w(D_1) \leq 2$, $Int(D_1) - D'$ does not contain any white vertices. This implies that e_2 is a loop whose associated disk does not contain any white vertices in its interior. This contradicts Lemma 4.2. Hence e_1 is not a loop. Similarly we can show that none of e_2, e_1' and e_2' are loops.

Since for $i = 1, 2$ neither e_i nor e_i' contains a middle arc at w_i, by Assumption 1 there exist white vertices w_{i+2} and w_{i+2}' different from w_1 and w_2 with $w_{i+2} \in e_i$ and $w_{i+2}' \in e_i'$ (see Figure 30b).

Suppose $w_3 = w_4 = w_3' = w_4'$. Then the four edges e_1, e_2, e_2', e_1' are situated around w_3 in this order. However e_1 and e_2' contain inward arcs

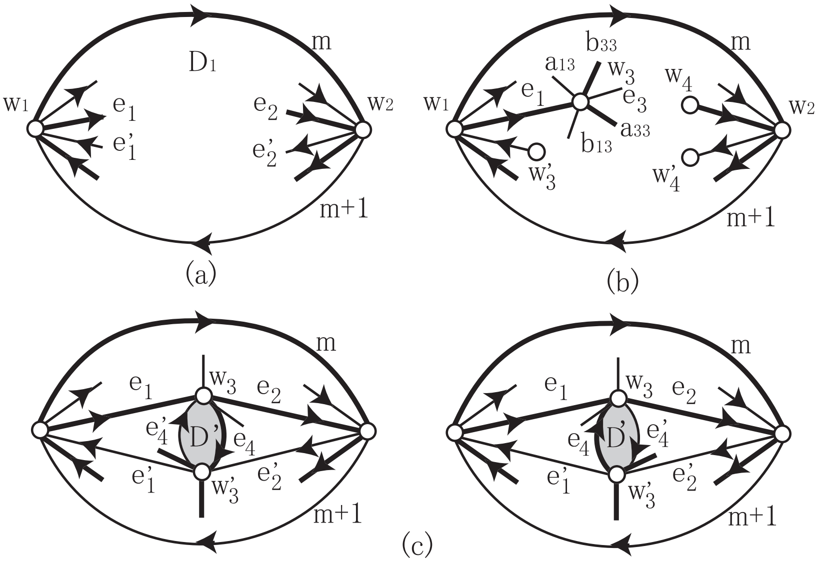

Fig. 30

at w_3, and e_2 and e_1' contain outward arcs at w_3. This contradicts the condition (3) for charts. Hence two of w_3, w_4, w_3', w_4' are different white vertices.

Suppose $w_3 = w_3'$. Then $w_4 \neq w_3$ or $w_4' \neq w_3$. Let e_3 be the edge of Γ_{m+1} containing w_3 but different from a_{13} and b_{13}. Let D_2 be the bicolored 2-angled disk bounded by $e_1 \cup e_1'$ in D_1. Since $Int(D_1) - D_2$ contains w_4 or w_4' and since $w(D_1) \leq 2$, we have $w(D_2) = 0$. There are three possibilities: $e_1' = b_{13}$, $e_1' = a_{13}$ and $e_1' = e_3$. If $e_1' = b_{13}$, then D_2 is a lens with $w(D_2) = 0$. This contradicts Lemma 3.1 (1). If $e_1' = a_{13}$, then $w(D_2) = 0$ implies that in D_2 there exists a loop of Γ_{m+1} containing w_3 whose associated disk does not contain any white vertices in its interior. This contradicts Lemma 4.2. Thus $e_1' = e_3$. Then b_{13} and a_{33} are contained in D_2, but one of them does not contain a middle arc at w_3. Since $w(D_2) = 0$, we have a contradiction by IO-Calculation with respect to Γ_m or Γ_{m+1} in D_2. Hence $w_3 \neq w_3'$.

Similarly we can show $w_4 \neq w_4'$.

If $w_3 = w_4'$, then $e_1 \cup e_2'$ separates the two white vertices w_3' and w_4 in D_1. This means that D_1 contains three different white vertices w_3, w_3' and w_4. This contradicts $w(D_1) \leq 2$. Hence $w_3 \neq w_4'$. Similarly $w_3' \neq w_4$. Hence we have $w_3 = w_4$ and $w_3' = w_4'$.

Let e_4 be the edge of Γ_m containing w_3 but different from e_1 and e_2. Since e_4 does not contain a middle arc at w_3, we have $\partial e_4 = \{w_3, w_3'\}$ by Assumption 1. Since one of the edges a_{43} and b_{43} does not contain a middle arc at w_3, there exists an edge e_4' of Γ_{m+1} with $\partial e_4' = \{w_3, w_3'\}$ (see Figure 30c).

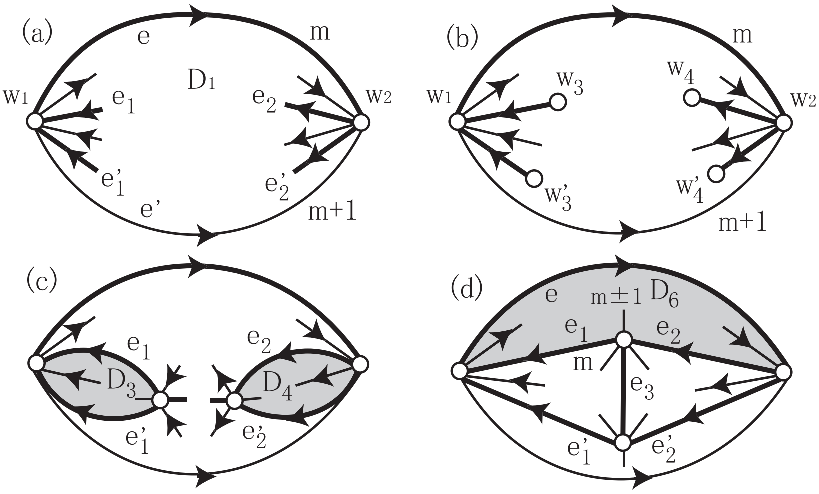

Fig. 31

However $e_4 \cup e_4'$ bounds a lens D' of type $(m, m+1)$ with $w(D') = 0$. This contradicts Lemma 3.1 (1). Therefore $w(D_1) \geq 3$.

PROPOSITION 7.5. *Let Γ be a minimal chart. For any lens D of type 2, $S^2 - D$ contains at least three white vertices.*

PROOF. Without loss of generality we can assume that D is of type $(m, m+1)$. Suppose that $S^2 - D$ contains at most two white vertices. Let $D_1 = Cl(S^2 - D)$. Then $w(D_1) \leq 2$ and D_1 is a bicolored 2-angled disk of type $(4, 4)$. We use the notations as shown in Figure 31a.

If $e_1 = e_2$, then D_1 contains a bicolored 2-angled disk of type $(2, 2)$ as shown in Figure 29a. By Lemma 7.2, $w(D_1) \geq 3$. This is a contradiction. Hence $e_1 \neq e_2$.

If $e_1 = e_2'$, then the edge e_1 splits D_1 into two disks, say D_1' and D_2'. By IO-Calculation with respect to Γ_m in D_i' for $i = 1, 2$, there exists a white vertex of Γ_m in $Int(D_i')$. Since $w(D_1) \leq 2$, we have $w(D_1') = 1$ and $w(D_2') = 1$. Hence there exists a loop of Γ_m in D_i' whose associated disk does not contain any white vertices in its interior. This contradicts Lemma 4.2. Hence $e_1 \neq e_2'$. Similarly we can show $e_1' \neq e_2$.

If $e_1' = e_2'$, then $e_1' \cup e'$ bounds a lens D_2 in D_1 with $w(D_2) \leq 2$. This contradicts Lemma 3.1 (1). Hence $e_1' \neq e_2'$.

Since for $i = 1, 2$ neither e_i nor e_i' contains a middle arc at w_i, by Assumption 1 there exist white vertices w_{i+2} and w_{i+2}' different from w_1 and w_2 with $w_{i+2} \in e_i$ and $w_{i+2}' \in e_i'$ (see Figure 31b).

If $w_3 = w_4 = w_3' = w_4'$, then the four edges e_1, e_1', e_2' and e_2 of Γ_m are situated around the white vertex w_3. This contradicts the condition (3) for charts. Hence $\{w_3, w_4, w_3', w_4'\}$ contains at least two white vertices.

Since $w(D_1) \leq 2$, the set $\{w_3, w_4, w_3', w_4'\}$ consists of two different white vertices. If three of the four vertices w_3, w_4, w_3' and w_4' are the same, then there exists a loop of Γ_m whose associated disk does not contain any white vertices in its interior. This contradicts Lemma 4.2. Hence there are three cases: (1) $w_3 = w_4'$ and $w_3' = w_4$, (2) $w_3 = w_3'$ and $w_4 = w_4'$, (3) $w_3 = w_4$ and $w_3' = w_4'$.

For the case (1), we have $w_3 = w_4' = w_3' = w_4$ or the arc $e_1 \cup e_2'$ in Γ_m intersects $e_1' \cup e_2$ in Γ_m. This is a contradiction.

For the case (2), $e_1 \cup e_1'$ bounds a 2-angled disk D_3 of Γ_m with at most one feeler such that $w(D_3) = 0$ (see Figure 31c). By Corollary 5.8, D_3 is a 2-angled disk of type (0–a). Similarly $e_2 \cup e_2'$ bounds a 2-angled disk D_4 of Γ_m of type (0–a). Since $Int(D_1) - (D_3 \cup D_4)$ does not contain any white vertices, there exists a lens D_5 of type $(m, m+1)$ with $w(D_5) = 0$. This contradicts Lemma 3.1 (1).

For the case (3), there must exist an edge e_3 of Γ_m containing w_3 and w_3' (see Figure 31d). Let D_6 be the 3-angled disk of Γ_m without feelers such that $\partial D_6 = e \cup e_1 \cup e_2$ and $w(D_6) = 0$. Since $w(D_6) = 0$, there exists a terminal edge in D_6 not containing a middle arc at the white vertex. This contradicts Assumption 1. Therefore $w(D_1) \geq 3$.

By Proposition 7.4 and 7.5, we complete the proof of Theorem 1.2.

8. Minimal charts with six white vertices

Lemma 8.1. *Let Γ be a minimal chart with a loop ℓ of label m and let $\varepsilon \in \{+1, -1\}$ be the integer such that the white vertex in ℓ is contained in $\Gamma_{m+\varepsilon}$. Let D be the associated disk of the loop ℓ. Suppose that $w(D) = 2$. If necessary we reverse the orientation of all edges, and if necessary we take the reflection of the chart Γ, then a regular neighborhood $N(D)$ contains the pseudo chart as shown in Figure* 32a *by C-moves in D keeping ∂D fixed.*

Proof. Let w_1 be the white vertex in ℓ, and e' the edge of $\Gamma_{m+\varepsilon}$ containing w_1 with $e' \subset D$. Since the edge e' of $\Gamma_{m+\varepsilon}$ does not contain a middle arc at w_1, by Assumption 1 there exists a white vertex w_2 of $\Gamma_{m+\varepsilon}$ with $\partial e' = \{w_1, w_2\}$.

If there exists a loop in $Int(D)$, then $w(D) \geq 3$ by Lemma 4.2. This is a contradiction. Hence there does not exist any loop in $Int(D)$.

Since there does not exist any loop in $Int(D)$, the white vertex w_2 is not contained in a loop. Since $w(D) = 2$, in D there exists a 2-angled disk D' of $\Gamma_{m+\varepsilon}$ with at most one feeler and $w_2 \in D'$ (see Figure 32b). Since $w(D') = 0$, a

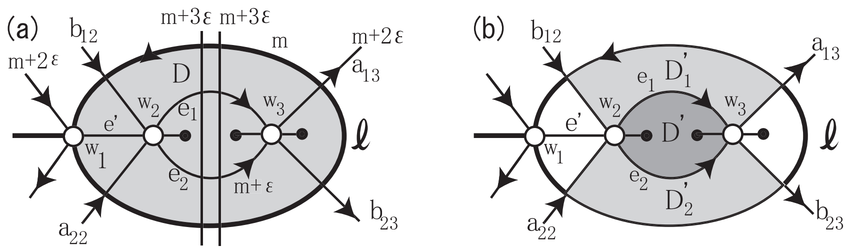

Fig. 32. The gray disk is D and the dark gray disk is D'.

regular neighborhood $N(D')$ contains one of the two pseudo charts as shown in Figure 16a and b by Corollary 5.8.

Let w_3 be the white vertex in $\partial D'$ different from w_2. Since $Int(D) - D'$ does not contain any white vertices, there is a terminal edge of $\Gamma_{m+\varepsilon}$ containing w_3 in $Cl(D - D')$. By Lemma 3.1 (1), there does not exist a lens in D. Hence all four edges b_{12}, a_{13}, a_{22} and b_{23} meet the loop ℓ. Therefore $\{w_2, w_3\} \subset \Gamma_{m+2\varepsilon}$. Thus $N(D')$ contains the pseudo chart as shown in Figure 16b.

Let E be a regular neighborhood of $e' \cup D'$ in D and $\ell' = Cl(\ell - E)$. By Disk Lemma (Lemma 3.2), we can assume that Γ is $(Cl(D - E), \ell')$-arc free (cf. Figure 17). Hence $(a_{22} \cup b_{12} \cup a_{13} \cup b_{23}) \cap D$ consists of two arcs. The two arcs split D into three disks. Let D_1 be the one of the three disks with $e_1 \subset D_1$. Now $e_1 \cup e_2$ splits the disk D_1 into three disks. One of the three disks is the 2-angled disk D'. For each $i = 1, 2$, let D'_i be the one of the three disks different from D' with $e_i \subset D'_i$ and let $\gamma_i = \ell \cap \partial D'_i$.

Since Γ is $(Cl(D - E), \ell')$-arc free, Γ is (D_1, γ_i)-arc free $(i = 1, 2)$. By applying Disk Lemma (Lemma 3.2) four times, we can assume that Γ is (D', e_i)-arc free and (D'_i, e_i)-arc free $(i = 1, 2)$. Thus if a proper arc L contained in an edge of $\Gamma_{m+3\varepsilon}$ in D separates w_2 and w_3 in D, then each of $L \cap \gamma_1$, $L \cap e_1$, $L \cap e_2$ and $L \cap \gamma_2$ consists of exactly one point. Hence $L \cap D'$ consists of a proper arc of D'. Therefore by Lemma 5.2, there must exist at least two proper arcs separating w_2 and w_3 in D each of which is contained in an edge of $\Gamma_{m+3\varepsilon}$ as shown in Figure 32a.

LEMMA 8.2. *Let Γ be a minimal chart with a loop ℓ of label m. Let D be the associated disk of the loop ℓ. Then $S^2 - D$ contains at least three white vertices.*

PROOF. If $S^2 - D$ contains at least three white vertices of Γ_m, then we have nothing to do. We may assume that $S^2 - D$ contains at most two white vertices of Γ_m. By Lemma 6.2, the loop ℓ is contained in a pair of eyeglasses or a pair of skew eyeglasses.

If ℓ is contained in a pair of eyeglasses or a pair of skew eyeglasses of type 2, then there exists a loop ℓ' of Γ_m in $S^2 - D$. By Lemma 4.2, the associated disk of ℓ' contains at least two white vertices. Thus $S^2 - D$ contains at least three white vertices.

If ℓ is contained in a pair of skew eyeglasses of type 1, then we can show that $S^2 - D$ contains at least three white vertices in a similar way to the proof of Lemma 6.9.

A subgraph of a chart is called *a solar eclipse*, if it consists of two loops and contains only one white vertex.

LEMMA 8.3. *Let Γ be a minimal chart with a loop ℓ of label m and let $\varepsilon \in \{+1, -1\}$ be the integer such that the white vertex in ℓ is contained in $\Gamma_{m+\varepsilon}$. Let D_1 be the associated disk of the loop ℓ. If there is no lens of Γ, and if $w(D_1) = 2$, then $S^2 - D_1$ contains at least two white vertices of $\Gamma_{m+2\varepsilon}$. In particular, if ℓ is contained in a solar eclipse, then $S^2 - (D_1 \cup D_2)$ contains at least two white vertices of $\Gamma_{m+2\varepsilon}$ where D_2 is the associated disk of another loop in the solar eclipse.*

PROOF. By Lemma 8.1, a regular neighborhood $N(D_1)$ contains the pseudo chart as shown in Figure 32a. We use the notations as shown in Figure 32a and b.

If there does not exist any white vertex of $\Gamma_{m+2\varepsilon}$ in $S^2 - D_1$, then $a_{13} = b_{12}$ and $a_{22} = b_{23}$. Hence $a_{13} \cup e_1$ and $a_{22} \cup e_2$ bound lenses of type $(m + \varepsilon, m + 2\varepsilon)$. This is a contradiction. Hence there exists at least one white vertex of $\Gamma_{m+2\varepsilon}$ in $S^2 - D_1$. By IO-Calculation with respect to $\Gamma_{m+2\varepsilon}$ in $Cl(S^2 - D')$, there exist at least two white vertices of $\Gamma_{m+2\varepsilon}$ in $S^2 - D_1$ where D' is the 2-angled disk of $\Gamma_{m+\varepsilon}$ in D_1.

If ℓ is contained in a solar eclipse, then none of the edges a_{13}, a_{22}, b_{12} and b_{23} intersect the disk D_2. Hence the same argument holds as the one above.

LEMMA 8.4. *Let Γ be a minimal chart with at most seven white vertices. If there exists a solar eclipse, then the associated disk of each loop of the solar eclipse contains at least three white vertices in its interior. Hence there is no solar eclipse in a minimal chart with at most six white vertices.*

PROOF. Let ℓ and ℓ' be the loops in the solar eclipse with $\ell \subset \Gamma_m$ and $\ell' \subset \Gamma_{m+\varepsilon}$ where $\varepsilon \in \{+1, -1\}$. Let w_1 be the white vertex in the solar eclipse, and let D_1 and D_2 be the associated disks of ℓ and ℓ' respectively. Then $D_1 \cap D_2 = \{w_1\}$.

Suppose that $w(D_1) \leq 2$. By Lemma 4.2 we can assume that $Int(D_1)$ contains exactly two white vertices, say w_2 and w_3. By Lemma 8.1, a regular neighborhood $N(D_1)$ contains the pseudo chart as shown in Figure 32a. We use the notations as shown in Figure 32a. Since $w_1 \in \Gamma_m \cap \Gamma_{m+\varepsilon}$ and $\ell \subset \Gamma_m$, we have $\{w_2, w_3\} \subset \Gamma_{m+\varepsilon} \cap \Gamma_{m+2\varepsilon}$.

By Corollary 1.3, there is no lens of Γ. By Lemma 8.3, there exist at least two white vertices of $\Gamma_{m+2\varepsilon}$ in $S^2 - (D_1 \cup D_2)$, say w_4 and w_5.

Since Γ contains at most seven white vertices, by Lemma 4.2 $Int(D_2)$ contains only two white vertices, say w_6 and w_7. By Lemma 8.1, a regular neighborhood $N(D_2)$ contains the pseudo chart as shown in Figure 32a. Since $w_1 \in \Gamma_m \cap \Gamma_{m+\varepsilon}$ and $\ell' \subset \Gamma_{m+\varepsilon}$, we have $\{w_6, w_7\} \subset \Gamma_m \cap \Gamma_{m-\varepsilon}$.

By Lemma 8.3, there exist at least two white vertices of $\Gamma_{m-\varepsilon}$ in $S^2 - (D_1 \cup D_2)$, say w_8 and w_9. This contradicts the fact Γ contains at most seven white vertices. Hence $w(D_1) \geq 3$ and $w(D_2) \geq 3$.

We show the third main theorem as follows:

PROOF OF THEOREM 1.4. Let ℓ be a loop in Γ_m, D_1 the associated disk of ℓ, and w_1 the white vertex in ℓ with $w_1 \in \Gamma_m \cap \Gamma_{m+\varepsilon}$ where $\varepsilon \in \{+1, -1\}$.

By Lemma 8.2, $S^2 - D_1$ contains at least three white vertices. Since Γ contains at most six white vertices, by Lemma 4.2 $Int(D_1)$ contains exactly two white vertices, say w_2 and w_3. By Lemma 8.1, a regular neighborhood $N(D_1)$ contains the pseudo chart as shown in Figure 32a.

By Corollary 1.3, there is no lens of Γ. By Lemma 8.3, there exist at least two white vertices of $\Gamma_{m+2\varepsilon}$ in $S^2 - D_1$, say w_4 and w_5. By Lemma 4.1 there exists at least one white vertex of Γ_m in the exterior of D_1, say w_6.

Since Γ_m contains only two white vertices w_1 and w_6, by Lemma 6.2 there exist a pair of eyeglasses of Γ_m. Let ℓ' be the loop of Γ_m with $w_6 \in \ell'$, and e_1 the edge of Γ_m with $\partial e_1 = \{w_1, w_6\}$. Let D_2 be the associated disk of the loop ℓ' (see Figure 33).

By Lemma 4.2, $Int(D_2)$ must contain exactly two white vertices w_4 and w_5. Without loss of generality, we can assume that a_{11} contains an outward middle arc at the white vertex w_1 (see Figure 33). For the edge b_{11}, there are three cases: $b_{11} = a_{11}$, $b_{11} = a_{16}$, or $b_{11} = b_{16}$. By Lemma 8.4 and Corollary 1.3, we have $b_{11} = b_{16}$.

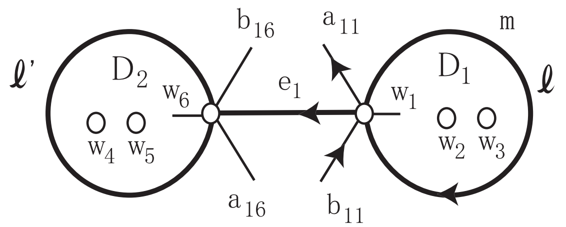

Fig. 33

Since $b_{11} = b_{16}$, we have $a_{11} \neq a_{16}$. Since the exterior of $D_1 \cup D_2$ does not contain any white vertices, the both edges a_{11} and a_{16} must be terminal edges. Since $b_{11} = b_{16}$, we have $w_6 \in \Gamma_{m+\varepsilon}$. By Lemma 8.1, a regular neighborhood $N(D_2)$ contains the pseudo chart as shown in Figure 32a.

The edge b_{11} splits the open disk $S^2 - (D_1 \cup D_2 \cup e_1)$ into two open disks. One of the two open disks contains $Int(a_{16})$, say E. Applying Disk Lemma (Lemma 3.2), we can assume Γ is $(Cl(E), e_1)$-arc free (cf. Figure 8 and 17). Hence there are four arcs in $Cl(E)$ each of which is contained in an edge of $\Gamma_{m+2\varepsilon}$ and connects a point in ℓ' and a point in e_1. Moreover there are four arcs contained in edges of $\Gamma_{m+3\varepsilon}$ in E each of which connects a point in ℓ' and a point in e_1. Therefore there are four edges of $\Gamma_{m+2\varepsilon}$ and there are at least two rings of $\Gamma_{m+3\varepsilon}$ as shown in Figure 2.

References

[1] J. S. Carter and M. Saito, "Knotted surfaces and their diagrams", Mathematical Surveys and Monographs, 55, American Mathematical Society, Providence, RI, (1998).

[2] S. Kamada, Surfaces in R^4 of braid index three are ribbon, J. Knot Theory Ramifications 1 (1992), no. 2, 137–160.

[3] S. Kamada, 2-dimensional braids and chart descriptions, Topics in knot theory (Erzurum, 1992), 277–287, NATO Adv. Sci. Inst. Ser. Math. Phys. Sci., 399, Kluwer Acad. Publ., Dordrecht, (1993).

[4] S. Kamada, "Braid and Knot Theory in Dimension Four", Mathematical Surveys and Monographs, Vol. 95, American Mathematical Society, (2002).

[5] T. Nagase and A. Shima, Properties of minimal charts and their applications I, J. Math. Sci. Univ. Tokyo, **14** (2007), 69–97.

[6] T. Nagase and A. Shima, Properties of minimal charts and their applications III–VI, in preparation.

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