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Extensions of Poisson algebras by derivations

Dedicated to the memory of Professor Shigeaki Tôgô

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Introduction

The alternating Schouten product was studied in a totally algebraic way in Bhaskara and Vismanath [3]. In this paper we shall be first concerned with this product and show that $[P, \hat{Q}] = 0$ if and only if $[P, Q] = 0$ and $(p-1)\,\mathrm{Alt}\,(P \otimes Q) = 0$ for alternating multiderivations P and Q of degree p and $q-1$ respectively, where $\hat{Q} = \mathrm{Alt}\,(q\bar{Q})$ is an alternating multilinear map of degree q (Theorem 2).

We shall then study an extension of a Poisson algebra by an derivation which is the abstract concept of a generalized Poisson algebra introduced by Berezin [2], while Kubo and Mimura [4] and Kubo [5] worked on abstract Poisson algebras, especially Poisson Lie structures on some polynomial algebras and their factor algebras. Let F be a Poisson algebra with bracket $[,]$ and D a derivation of the associative algebra F. We define a D-extension $(F, \langle,\rangle)$ of F whose bracket $\langle,\rangle$ on F is given by $\langle a, b\rangle = [a, b] + D(a)b - aD(b)$ for a, $b \in F$. By using Theorem 2 we give an equivalent condition to that an algebra $(F, \langle,\rangle)$ is a Lie algebra. Then we consider an extension of a Poisson algebra constructed from the three dimensional split simple Lie algebra.

Throughout this paper let $\mathfrak{k}$ be a field of characteristic zero and F a commutative associative algebra over $\mathfrak{k}$ with unit.

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Alternating Schouten products of multiderivations

Notations and terminology are based on Bhaskara and Viswanath [3]. For the sake of convenience we list the terms that we use here.

For $p \geqq 1$, we denote by $L_p(F)$ the set of all multilinear maps of F into itself of degree p. We define $L_0(F) = F$ and $L_{-1}(F) = 0$.

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Let u, $v \in F$, $P \in L_p(F)$ and $Q \in L_q(F)$ $(p, q \geqq 1)$. The compositions of these multilinear maps are defined as follows: (a) $u \cdot v = 0$. (b) $u \cdot P = 0$, and $P \cdot u(v_1, \dots, v_{p-1}) = P(u, v_1, \dots, v_{p-1})$ for $v_i \in F$. (c) $P \cdot Q(v_1, \dots, v_{p-1}, w_1, \dots, w_q) = P(Q(w_1, \dots, w_q), v_1, \dots, v_{p-1})$ for v_i, $w_j \in F$. (d) $P_i(v_1, \dots, v_p) = P(v_2, \dots, v_i, v_1, v_{i+1}, \dots, v_p)$. The tensor product $P \otimes Q \in L_{p+q}(F)$ is defined by $(P \otimes Q)(v_1, \dots, v_p, w_1, \dots, w_q) = P(v_1, \dots, v_p)Q(w_1, \dots, w_q)$ for v_i, $w_j \in F$.

Suppose $p \geqq 1$, $P \in L_p(F)$ and σ is a permutation of p-elements. $U_\sigma P \in L_p(F)$ is defined by $(U_\sigma P)(v_1, \dots, v_p) = P(v_{\sigma(1)}, \dots, v_{\sigma(p)})$ for $v_i \in F$. The alternating operator Alt is defined by $\mathrm{Alt}\, P = (1/p!) \sum_\sigma \mathrm{sgn}\, \sigma U_\sigma P$ for $P \in L_p(F)$ $(p \geqq 1)$ and $\mathrm{Alt}\, v = v$ for $v \in F$. Obviously we have $\mathrm{Alt}(U_\sigma P) = \mathrm{sgn}\, \sigma\, \mathrm{Alt}\, P$. When $\mathrm{Alt}\, P = P$, we call P an alternating map. We state the following

LEMMA 1 ([3; Proposition 1.4]). *Let $P \in L_p(F)$, $Q \in L_q(F)$. Then*

(1) $\mathrm{Alt}\,(P \cdot (\mathrm{Alt}\, Q)) = \mathrm{Alt}\,(P \cdot Q)$ $(p, q \geqq 0)$ *and*

(2) $p\, \mathrm{Alt}\,((\mathrm{Alt}\, P) \cdot Q) = \sum_{i=1}^{p} (-1)^{i+1}\, \mathrm{Alt}\,(P_i \cdot Q)$ $(p \geqq 1, q \geqq 0)$.

The alternating Schouten product of $P \in L_p(F)$ and $Q \in L_q(F)$ $(p, q \geqq 0)$ is defined by

$$[P, Q] = \mathrm{Alt}\,(p(\mathrm{Alt}\, P) \cdot Q + (-1)^{pq} q(\mathrm{Alt}\, Q) \cdot P)\,.$$

In [3] the following results are proved: (1) If P, Q and R are alternating maps of degree p, q and r respectively, then $(-1)^{pr}[[P, Q], R] + (-1)^{qp}[[Q, R], P] + (-1)^{rq}[[R, P], Q] = 0$ ([3; Theorem 2.7]). (2) If P and Q are multiderivations, so is $[P, Q]$.

For $Q \in L_{q-1}(F)$ $(q \geqq 1)$, we define $\bar{Q}$, $\hat{Q} \in L_q(F)$ by $\bar{Q}(v_1, \dots, v_q) = v_1 Q(v_2, \dots, v_q)$ for $v_i \in F$ and $\hat{Q} = \mathrm{Alt}\,(q\bar{Q})$. We denote by $A_p(F)$, $AD_p(F)$ the set of all alternating multilinear maps, alternating multiderivations of F of degree p respectively.

The purpose of this section is to prove the following

THEOREM 2. *Let P and Q be alternating multiderivations of degree p and $q-1$ $(p, q \geqq 1)$ respectively. Then $[P, \hat{Q}] = 0$ if and only if $[P, Q] = 0$ and $(p-1)\, \mathrm{Alt}\,(P \otimes Q) = 0$.*

To prove this theorem we need some lemmas.

LEMMA 3. (1) *If $P \in L_p(F)$ and $Q \in A_{q-1}(F)$ $(p \geqq 0, q \geqq 1)$, then*

$$(\bar{Q})_j \cdot P = \begin{cases} (-1)^j \overline{Q \cdot P} & \text{if } j \geqq 2 \\ U_\sigma(P \otimes Q) & \text{if } j = 1 \end{cases},$$

where $\mathrm{sgn}\, \sigma = (-1)^{p(q-1)}$.

(2) *If* $P \in AD_p(F)$ *and* $Q \in L_{q-1}(F)$ $(p, q \geqq 1)$, *then*

$$P \cdot \bar{Q} = U_\sigma \overline{P \cdot Q} + (-1)^{p-1} P \otimes Q\,,$$

where $\operatorname{sgn} \sigma = (-1)^{p-1}$.

(3) *If* $Q \in L_{q-1}(F)$ $(q \geqq 1)$, *then for* $u_1, \ldots, u_q \in F$,

$$\operatorname{Alt} \bar{Q}(u_1, \ldots, u_q) = \frac{1}{q} \textstyle\sum_{j=1}^{q} (-1)^{j+1} u_j \operatorname{Alt} Q(u_1, \ldots, \hat{u}_j, \ldots, u_q)\,.$$

PROOF. Let $u_1, \ldots, u_{p+q-1} \in F$. (1): If $j \geqq 2$, then

$$\begin{aligned}(\bar{Q})_j \cdot P(u_1, \ldots, u_{p+q-1}) &= (\bar{Q})_j(P(u_q, \ldots, u_{p+q-1}), u_1, \ldots, u_{q-1}) \\ &= \bar{Q}(u_1, \ldots, u_{j-1}, P(u_q, \ldots, u_{p+q-1}), u_j, \ldots, u_{q-1}) \\ &= (-1)^{j-2} u_1 Q(P(u_q, \ldots, u_{p+q-1}), u_2, \ldots, u_{q-1}) \\ &= (-1)^j \overline{Q \cdot P}(u_1, \ldots, u_{p+q-1})\,.\end{aligned}$$

Let σ be the permutation of $(p+q-1)$-elements given by $\sigma(1) = q, \ldots, \sigma(p) = p+q-1, \sigma(p+1) = 1, \ldots, \sigma(p+q-1) = q-1$. Then $\operatorname{sgn} \sigma = (-1)^{p(q-1)}$ and

$$\begin{aligned}(\bar{Q})_1 \cdot P(u_1, \ldots, u_{p+q-1}) &= \bar{Q}(P(u_q, \ldots, u_{p+q-1}), u_1, \ldots, u_{q-1}) \\ &= P(u_q, \ldots, u_{p+q-1}) Q(u_1, \ldots, u_{q-1}) \\ &= U_\sigma(P \otimes Q)(u_1, \ldots, u_{p+q-1})\,.\end{aligned}$$

(2): Let σ be the permutation of $(p+q-1)$-elements given by $\sigma(1) = p$, $\sigma(2) = 1$, $\ldots$, $\sigma(i) = i-1$, $\ldots$, $\sigma(p) = p-1$, $\sigma(j) = j$ $(p+1 \leqq j \leqq p+q-1)$. Then $\operatorname{sgn} \sigma = (-1)^{p-1}$. For $u_1, \ldots, u_{p+q-1} \in F$, we have

$$\begin{aligned}(P \cdot \bar{Q})(u_1, \ldots, u_{p+q-1}) &= P(u_p Q(u_{p+1}, \ldots, u_{p+q-1}), u_1, \ldots, u_{p-1}) \\ &= u_p P(Q(u_{p+1}, \ldots, u_{p+q-1}), u_1, \ldots, u_{p-1}) \\ &\quad + Q(u_{p+1}, \ldots, u_{p+q-1}) P(u_p, u_1, \ldots, u_{p-1}) \\ &= \overline{P \cdot Q}(u_p, u_1, \ldots, \hat{u}_p, \ldots, u_{p+q-1}) \\ &\quad + (-1)^{p-1} (P \otimes Q)(u_1, \ldots, u_{p+q-1}) \\ &= (U_\sigma \overline{P \cdot Q} + (-1)^{p-1} P \otimes Q)(u_1, \ldots, u_{p+q-1})\,.\end{aligned}$$

(3): For a permutation σ of q-elements with $\sigma(1) = j$, we denote by $\bar{\sigma}$ the permutation of $q-1$ elements such that $\bar{\sigma}(1) = \sigma(2), \ldots, \bar{\sigma}(j-1) = \sigma(j)$, $\bar{\sigma}(j+1) = \sigma(j+1)$, $\ldots$, $\bar{\sigma}(q) = \sigma(q)$. Then $\operatorname{sgn} \bar{\sigma} = (-1)^{j+1} \operatorname{sgn} \sigma$. Now we have

$$\begin{aligned}\operatorname{Alt}\bar{Q}(u_1,\dots,u_q) &= \frac{1}{q!}\sum_\sigma \operatorname{sgn}\sigma\bar{Q}(u_{\sigma(1)},\dots,u_{\sigma(q)}) \\ &= \frac{1}{q!}\sum_\sigma \operatorname{sgn}\sigma u_{\sigma(1)}Q(u_{\sigma(2)},\dots,u_{\sigma(q)}) \\ &= \frac{1}{q!}\sum_{j=1}^q \sum_{\sigma(1)=j}\operatorname{sgn}\sigma u_j Q(u_{\sigma(2)},\dots,u_{\sigma(q)}) \\ &= \frac{1}{q!}\sum_{j=1}^q u_j \sum_\sigma (-1)^{j+1}\operatorname{sgn}\sigma U_\sigma Q(u_1,\dots,\hat{u}_j,\dots,u_q) \\ &= \frac{1}{q}\sum_{j=1}^q (-1)^{j+1}u_j \operatorname{Alt} Q(u_1,\dots,\hat{u}_j,\dots,u_q). \qquad \text{Q.E.D.}\end{aligned}$$

LEMMA 4. *If $P \in AD_p(F)$ and $Q \in A_{q-1}(F)$ $(p, q \geqq 1)$, then*

$$[P, \hat{Q}] = (-1)^{p-1}q\{p \operatorname{Alt}\overline{P\cdot Q} + (-1)^{p(q-1)}(q-1)\operatorname{Alt}\overline{Q\cdot P} + (p-1)\operatorname{Alt}(P\otimes Q)\}.$$

PROOF.

$$\begin{aligned}[P, \hat{Q}] &= q[P, \bar{Q}] \\ &= q\operatorname{Alt}\{p(\operatorname{Alt}P)\cdot\bar{Q} + (-1)^{pq}q(\operatorname{Alt}\bar{Q})\cdot P\} \\ &= q\{p\operatorname{Alt}(P\cdot\bar{Q}) + (-1)^{pq}\textstyle\sum_{j=1}^q (-1)^{j+1}\operatorname{Alt}((\bar{Q})_j\cdot P)\} \quad \text{(by Lemma 1)} \\ &= q\{p\operatorname{Alt}(P\cdot\bar{Q}) + (-1)^{pq+1}(q-1)\operatorname{Alt}(\overline{Q\cdot P}) \\ &\quad + (-1)^p\operatorname{Alt}(P\otimes Q)\} \quad \text{(by Lemma 3(1))}.\end{aligned}$$

Therefore by Lemma 3 (2), we have our formula. Q.E.D.

PROOF OF THEOREM 2. Let $u_1, \dots, u_{p+q-1} \in F$. By Lemma 3 (3) and Lemma 4 we have

$$\begin{aligned}[P, \hat{Q}](u_1,\dots,u_{p+q-1}) &= (-1)^{p-1}q\{p\operatorname{Alt}\overline{P\cdot Q} + (-1)^{p(q-1)}(q-1)\operatorname{Alt}\overline{Q\cdot P} \\ &\quad + (p-1)\operatorname{Alt}(P\otimes Q)\}(u_1,\dots,u_{p+q-1}) \\ &= (-1)^{p-1}\frac{q}{P+q-1}\textstyle\sum_{j=1}^{p+q-1}(-1)^{j+1}u_j\{p\operatorname{Alt}(P\cdot Q) \\ &\quad + (-1)^{p(q-1)}(q-1)\operatorname{Alt}(Q\cdot P)\}(u_1,\dots,\hat{u}_j,\dots,u_{p+q-1}) \\ &\quad + (-1)^{p-1}q(p-1)\operatorname{Alt}(P\otimes Q)(u_1,\dots,u_{p+q-1})\end{aligned}$$

$$= (-1)^{p-1}\frac{q}{p+q-1}\sum_{j=1}^{p+q-1}(-1)^{j+1}u_j[P, Q](u_1, \dots, \hat{u}_j, \dots, u_{p+q-1})$$
$$+ (-1)^{p-1}q(p-1)\,\mathrm{Alt}\,(P \otimes Q)(u_1, \dots, u_{p+q-1}).$$

Put $u_1 = 1$. Since P, Q and $[P, Q]$ are multiderivations, we have

$$[P, \hat{Q}](1, u_2, \dots, u_{p+q-1}) = (-1)^{p-1}\frac{q}{p+q-1}[P, Q](u_2, \dots, u_{p+q-1}).$$

This shows that $[P, \hat{Q}] = 0$ implies $[P, Q] = 0$. Therefore we have $(p-1)\,\mathrm{Alt}\,(P \otimes Q) = 0$. Q.E.D.

We shall prove the following

PROPOSITION 5. *Let P and Q be alternating multiderivations of degree $p-1$, $q-1$ respectively ($p, q \geqq 1$). If $p \neq q$, then $[\hat{P}, \hat{Q}] = 0$ if and only if* $\mathrm{Alt}\,(P \otimes Q) = 0$. *If $p = q$, then $[\hat{P}, \hat{Q}] = 0$.*

PROOF. Let $u_1, \dots, u_{p+q-1} \in F$ and σ be the permutation given by $\sigma(1) = p$, $\sigma(2) = 1, \dots, \sigma(p) = p-1$, $\sigma(j) = j$ $(p+1 \leqq j \leqq p+q-1)$. Then

$$\begin{aligned}\bar{P}\cdot\bar{Q}(u_1, \dots, u_{p+q-1}) &= \bar{P}(u_pQ(u_{p+1}, \dots, u_{p+q-1}), u_1, \dots, u_{p-1})\\ &= u_pQ(u_{p+1}, \dots, u_{p+q-1})P(u_1, \dots, u_{p-1})\\ &= \overline{P \otimes Q}(u_p, u_1, \dots, \hat{u}_p, \dots, u_{p+q-1})\\ &= U_\sigma\overline{P \otimes Q}(u_1, \dots, u_{p+q-1}).\end{aligned}$$

Therefore by Lemma 3 (3),

$$\begin{aligned}\mathrm{Alt}\,(\bar{P}\cdot\bar{Q})(u_1, \dots, u_{p+q-1}) &= \mathrm{Alt}\,(U_\sigma\overline{P \otimes Q})(u_1, \dots, u_{p+q-1})\\ &= (-1)^{p-1}\,\mathrm{Alt}\,\overline{P \otimes Q}(u_1, \dots, u_{p+q-1})\\ &= (-1)^{p-1}\frac{1}{p+q-1}\sum_{j=1}^{p+q-1}(-1)^{j+1}u_j\,\mathrm{Alt}\,(P \otimes Q)(u_1, \dots, \hat{u}_j, \dots, u_{p+q-1}).\end{aligned}$$

Hence

$$\begin{aligned}[\hat{P}, \hat{Q}](u_1, \dots, u_{p+q-1}) &= pq\{p\,\mathrm{Alt}\,(\bar{P}\cdot\bar{Q}) + (-1)^{pq}q\,\mathrm{Alt}\,(\bar{Q}\cdot\bar{P})\}(u_1, \dots, u_{p+q-1})\\ &= \frac{pq}{p+q-1}\sum_{j=1}^{p+q-1}u_j\{(-1)^{p+j}p\,\mathrm{Alt}\,(P \otimes Q)\\ &\quad + (-1)^{pq+q+j}q\,\mathrm{Alt}\,(Q \otimes P)\}(u_1, \dots, \hat{u}_j, \dots, u_{p+q-1})\end{aligned}$$

$$= \frac{pq}{p+q-1} \sum_{j=1}^{p+q-1} (-1)^{p+j}(p-q)u_j \operatorname{Alt}(P \otimes Q)(u_1, \ldots, \hat{u}_j, \ldots, u_{p+q-1}).$$

Then the proof will be done similarly to that of Theorem 2. Q.E.D.

Extensions of Poisson algebras by derivations

Assume that F has a Lie bracket $[,]$. An algebra $(F, [,])$ is called a Poisson algebra if $[ab, c] = a[b, c] + b[a, c]$ for $a, b, c \in F$. Let D be a derivation of an associative algebra F. Then we define a new bracket $\langle,\rangle$ on F by

$$\langle a, b\rangle = [a, b] + D(a)b - aD(b) \qquad \text{for} \quad a, b \in F.$$

Let us denote by $(F, \langle,\rangle)$ the algebra F with a product given by $\langle,\rangle$, and call this algebra a *D-extension* of a Poisson algebra $(F, [,])$.

It is easy to see the following two propositions.

PROPOSITION 6. *Let $(F, \langle,\rangle)$ be a D-extension of a Poisson algebra $(F, [,])$. Then for $u_1, \ldots, u_n, v \in F$,*

$$\langle u_1 \ldots u_n, v\rangle = \sum_{i=1}^{n} u_1 \ldots u_{i-1}\langle u_i, v\rangle u_{i+1} \ldots u_n + (n-1)u_1 \ldots u_n D(v).$$

In particular for $a, b, c \in F$,

$$\langle ab, c\rangle = \langle a, c\rangle b + a\langle b, c\rangle + abD(c).$$

PROPOSITION 7. *Let A_D, B_d be D, d-extensions of Poisson algebras A, B respectively and ϕ a Poisson isomorphism of A onto B. Then ϕ is an isomorphism of A_D onto B_d if and only if $d\phi = \phi D$.*

We shall give an equivalent condition to that a D-extension $(F, \langle,\rangle)$ is a Lie algebra. Let $G \in AD_2(F)$ be defined by $G(a, b) = [a, b]$ for $a, b \in F$. Observing $\hat{D}(a, b) = aD(b) - bD(a)$, we have

$$\langle a, b\rangle = (G - \hat{D})(a, b) \qquad \text{for} \quad a, b \in F.$$

Therefore $(F, \langle,\rangle)$ is a Lie algebra iff $[G - \hat{D}, G - \hat{D}] = 0$ ([3; Proposition 2.9]) which is equivalent to $[G, \hat{D}] = 0$ because $[\hat{D}, G] = (-1)^4[G, \hat{D}]$ and $[\hat{D}, \hat{D}] = 0$ (Proposition 5). Now we shall prove the following

THEOREM 8. *Let $(F, \langle,\rangle)$ be a D-extension of a Poisson algebra $(F, [,])$. Then an algebra $(F, \langle,\rangle)$ is a Lie algebra if and only if for any elements $a, b, c \in F$ the following equations hold*

$$(*)\qquad D([a, b]) = [D(a), b] + [a, D(b)] \quad \textit{and}$$
$$[a, b]D(c) + [b, c]D(a) + [c, a]D(b) = 0\,.$$

PROOF. Let $G \in AD_2(F)$ be given above. By Theorem 2, $[G, \hat{D}] = 0$ iff $[G, D] = 0$ and $\mathrm{Alt}\,(G \otimes D) = 0$. This theorem follows from the following computation:

$$\begin{aligned}
[G, D](a, b) &= \{2\,\mathrm{Alt}\,(G \cdot D) + \mathrm{Alt}\,(D \cdot G)\}(a, b) \\
&= G(D(b), a) - G(D(a), b) + D(G(a, b)) \\
&= -[a, D(b)] - [D(a), b] + D([a, b])\,, \\
\mathrm{Alt}\,(G \otimes D)(a, b, c) &= 3^{-1}(G(a, b)D(c) + G(b, c)D(a) + G(c, a)D(b)) \\
&= 3^{-1}([a, b]D(c) + [b, c]D(a) + [c, a]D(b))\,. \qquad \text{Q.E.D.}
\end{aligned}$$

Let $J(a, b, c) = [a, b]D(c) + [b, c]D(a) + [c, a]D(b)$ for $a, b, c \in F$. By the proof of Theorem 8, $J = 3\,\mathrm{Alt}\,(G \otimes D)$. This says that J is a multiderivation. Therefore to verify the condition that $J = 0$ on F, it is enough to check this for only generators of an associative algebra F.

PROPOSITION 9. *Assume that F is associatively generated by S. If a derivation D of F satisfies the conditions $(*)$ on S, then so does D on F.*

PROOF. We shall prove our assertion for the first condition of $(*)$. The second one is already seen just above.

$$\begin{aligned}
D([ab, c]) &= D(a[b, c] + b[a, c]) \\
&= ([a, c]D(b) + a[D(b), c] + D(a)[b, c] + b[D(a), c]) \\
&\quad + (a[b, D(c)] + b[a, D(c)]) \\
&= [D(ab), c] + [ab, D(c)] \qquad \text{for} \quad a, b, c \in S.
\end{aligned}$$

By this formula and an induction the proof will be completed. Q.E.D.

EXAMPLE. Let L be a finite-dimensional Lie algebra over $\mathfrak{k}$ with a basis $\{x_1, \dots, x_n\}$ and R the polynomial algebra $\mathfrak{k}[x_1, \dots, x_n]$. We consider the Possion algebra $G = \mathrm{L}(L; R, \{\partial/\partial x_i\})$ defined in [4], whose Poisson bracket $[\,,]$ on R is given by

$$[a, b] = \textstyle\sum_{i,j} [x_i, x_j] \dfrac{\partial a}{\partial x_i} \dfrac{\partial b}{\partial x_j} \qquad \text{for} \quad a, b \in R\,.$$

Let D be a derivation of R and G_D its D-extension. Then an algebra G_D is a Lie algebra by Theorem 8 and Proposition 9 if D satisfies the following conditions: For $i, j, k = 1, \dots, n$,

$$(**) \qquad D([x_i, x_j]) = [D(x_i), x_j] + [x_i, D(x_j)] \quad \text{and}$$

$$[x_i, x_j]D(x_k) + [x_j, x_k]D(x_i) + [x_k, x_i]D(x_j) = 0 .$$

We write $D(x_i) = \sum_m a_{im}$, where a_{im} is homogeneous of degree m, and define a derivation D_m of R by $D_m(x_i) = a_{im}$ for $i = 1, \dots, n$ and $m = 0, 1 \dots$. We can easily see that D satisfies (**) iff D_m satisfies (**) for $m = 0, 1, \dots$. Under this condition D_1 is a derivation of the Lie algebra L. Therefore if L is split simple, there exists an element $z \in L$ such that $D_1(w) = \operatorname{ad} z(w)$ for $w \in L$.

For the three dimensional Lie algebras, Mimura and Ikushima [6] computed all of the D_0-extensions of the Poisson algebras of all C^∞-functions on C^∞-manifolds.

We can give a Lie algebra L such that $\operatorname{ad} z$ does not satisfy (**) on an $\operatorname{ad} z$-extension of a Poisson algebra $\mathrm{L}(L; \mathfrak{k}[x_1, \dots, x_n], \{\partial/\partial x_i\})$ for some $z \in L$. Let L be the Lie algebra over $\mathfrak{k}$ described in terms of a basis $\{x_1, \dots, x_5\}$ by the following multiplication table:

$$[x_1, x_2] = x_2 , \qquad [x_1, x_3] = x_3 , \qquad [x_1, x_4] = 2x_4 ,$$

$$[x_1, x_5] = 3x_5 , \qquad [x_2, x_3] = x_4 , \qquad [x_2, x_4] = x_5 ,$$

$[x_i, x_j] = 0$ if it is not in the table above ([7; Example 2]). Then

$$[x_2, x_3] \operatorname{ad} x_1(x_4) + [x_3, x_4] \operatorname{ad} x_1(x_2) + [x_4, x_2] \operatorname{ad} x_1(x_3) = 2x_4^2 - x_3x_5 \neq 0 .$$

$\mathrm{L}(SL(2, \mathfrak{k}); \mathfrak{k}[x, y, h], \{\partial/\partial x, \partial/\partial y, \partial/\partial h\})$. Let L be a Lie algebra over $\mathfrak{k}$ with a basis $\{x, y, z\}$ and multiplications $[x, y] = h$, $[h, x] = 2x$, $[h, y] = -2y$. Put $A = \mathfrak{k}[x, y, z]$. We consider the $\operatorname{ad} h$-extension $(A, \langle , \rangle)$ of the Poisson algebra $\mathrm{L}(SL(2, \mathfrak{k}); A, \{\partial/\partial x, \partial/\partial y, \partial/\partial h\})$. We note that $(A, \langle , \rangle)$ is a Lie algebra because $\operatorname{ad} h$ satisfies (**).

Let A_m be a weight space $\{a \in A: [h, a] = ma\}$, $A_* = \sum_{m \neq 0} A_m$, and write $\langle a, {}_n b\rangle = \langle a, b, \dots, b\rangle$ where b appears n times in the right hand side. We have the following formulas:

1) $\langle x, y\rangle = h + 4xy$, $\langle h, x\rangle = 2x - 2hx$, $\langle h, y\rangle = -2y + 2hy$.
2) $\langle h, {}_n x\rangle = -2^n(n-2)!x^n$, $\langle h, {}_n y\rangle = 2(-2)^{n-1}(n-2)!y^n$ $(n \geqq 2)$.
3) $\langle a, h\rangle = m(ah - a)$, $\langle a, x\rangle = [a, x] + (m-2)ax$,
   $\langle a, y\rangle = [a, y] + (m+2)ay$ for $a \in A_m$.
4) $\langle h^p x^q, y^r\rangle = -2prh^{p-1}x^q y^r + qrh^{p+1}x^{q-1}y^{r-1} + 2(q+r)h^p x^q y^r$.
5) $\langle h^p y^r, x^q\rangle = 2pqh^{p-1}x^q y^r - qrh^{p+1}x^{q-1}y^{r-1} - 2(q+r)h^p x^q y^r$.
6) $\langle x^q, y^r\rangle = qrhx^{q-1}y^{r-1} + 2(q+r)x^q y^r$.

LEMMA 10. *Assume that* $a \in A_m$. *Then* $\langle a, x\rangle = A_{m+2}$, $\langle a, y\rangle \in A_{m-2}$, $\langle a, h\rangle \in A_m$.

Let B be the subalgebra of the ad h-extension $(A, \langle\,,\rangle)$ generated by x, y, h.

PROPOSITION 11. A_*, *B as above.*

(1) $A_* \subseteq B$.
(2) $h^2 \notin B$, *hence* $B \subsetneqq A$.

PROOF. (1): For q, $r \geqq 2$, x^q and y^r belong to B by the formula 2). By $\langle h^p x^q, h\rangle = 2q(h^{p+1}x^q - h^p x^q)$ and induction on p we have $h^p x^q \in B$ ($p \geqq 0$, $q \geqq 1$). Similarly $h^p y^r \in B$ ($p \geqq 0, r \geqq 1$). By 4) and 5)

$$\langle h^p x^q, y^r\rangle + \langle h^p y^r, x^q\rangle = 2p(q-r)h^{p-1}x^q y^r .$$

Therefore if $p \geqq 0$, q, $r \geqq 1$ and $q \neq r$, then $h^p x^q y^r \in B$. These show that $A_* \subseteq B$.

(2): Assume that $h^2 \in B$ and write $h^2 = \langle f_1, x\rangle + \langle f_2, y\rangle + \langle f_3, h\rangle$, $f_i \in A$. By Lemma 10 we may assume that $f_1 \in A_{-2}$, $f_2 \in A_2$ and $\langle f_3, h\rangle = 0$. Then we put

$$f_1 = \textstyle\sum_{p,n} a_{p,n} h^p x^n y^{n+1} , \qquad f_2 = \textstyle\sum_{p,n} b_{p,n} h^p x^{n+1} y^n ,$$

where $a_{p,n}$, $b_{p,n} \in \mathfrak{k}$. By 3) we have

$$h^2 = \textstyle\sum_{p,n} (a_{p,n} - b_{p,n})(2ph^{p-1}(xy)^{n+1} - 4h^p(xy)^{n+1} - (n+1)h^{p+1}(xy)^n) .$$

In this formula, putting $xy = 0$, we have $h^2 = \sum_p (b_{p,0} - a_{p,0})h^{p+1}$ and $b_{1,0} - a_{1,0} = 1$, $b_{0,0} - a_{0,0} = 0$. On the other hand, putting $h = 0$, we have

$$\textstyle\sum_n \{2(b_{0,n} - a_{0,n}) - (b_{1,n} - a_{1,n})\}(xy)^{n+1} = 0 .$$

Then $2(b_{0,0} - a_{0,0}) - (b_{1,0} - a_{1,0}) = 0$, which is a contradiction. Q.E.D.

Let h^A be the smallest ideal of a Lie algebra A containing h. We can write $h^A = \sum_n \langle h, {}_n A\rangle$ ([1; p. 29]). We have the following

COROLLARY 12. (1) $A = B + \mathfrak{k}[h]$. (2) $B = h^A$.

PROOF. (1): Put $C = B + \mathfrak{k}[h]$. Then

$$\langle h^p x^n, y^n\rangle = n^2 h^{p+1}(xy)^{n-1} - 2nph^{p-1}(xy)^n + 4nh^p(xy)^n \in B \qquad (n \geqq 1)$$

by 4) and the proof of Proposition 11 (1). Putting $n = 1$ and induction on p we have $h^p xy \in C$. Then by induction on n we see $h^p(xy)^n \in C$ ($p \geqq 0$, $n \geqq 1$). Therefore $A_0 \subseteq C$. Hence by Proposition 11, $A = A_0 + A_* = C$.

(2): Put $H = h^A$. Since $\langle h, {}_2x\rangle$ and $\langle x^2, h\rangle$ belong to H, so do x^2 and hx^2. Therefore $x^2y = (\langle hx^2, y\rangle + \langle hy, x^2\rangle)/2 \in H$. Furthermore $\langle x^2, y\rangle = 2hx + 6x^2y \in H$. Hence $hx \in H$. By 1), $x \in H$. Similarly we have $y \in H$.

Conversely by Lemma 2.3 in [1; Chapter 2] we have $h^A = (h^{\mathfrak{k}[h]})^B = h^B \subseteq B$.

Q.E.D.

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