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Kirillov-Kostant theory and Feynman path integrals on coadjoint orbits II

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Introduction

Let G be a connected and simply connected solvable Lie group. In this paper we construct irreducible unitary representations of G by using the Feynman path integrals on coadjoint orbits [1] [13].

In § 1, we compute the path integrals on $M = \mathbf{R}^n \times \mathbf{R}^n$. Let θ be a 1-form on M and H a C^∞-function on M which satisfies certain conditions. The path integral $K_{\theta,H}(x'', x'; T)$ $(x', x'' \in \mathbf{R}^n)$ computed by using the action $\int_0^T \gamma^* \theta - H(\gamma(t))\, dt$ (where γ runs over a certain set of paths on M) can be written by the solution of differential equations defined by θ and H.

In § 2, we investigate the path integrals on coadjoint orbits. Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{g}^*$ the dual space of $\mathfrak{g}$. Fix an element λ of $\mathfrak{g}^*$ and choose a real polarization $\mathfrak{p}$. Following the Kirillov-Kostant theory [4] [5] [14], we construct an irreducible unitary representation $\pi_\lambda^{\mathfrak{p}}$ of G. We put $\theta_\lambda = \langle \lambda, g^{-1} dg \rangle$ and $H_Y = \langle \lambda, g^{-1} Y g \rangle$ for any $Y \in \mathfrak{g}$. We show that the integral operator of K_{θ_λ, H_Y} corresponds to $\pi_\lambda^{\mathfrak{p}}(exp\, TY)$.

§ 1 Path integrals on $\mathbf{R}^n \times \mathbf{R}^n$

In this section, we shall compute the Feynman path integrals on $M = \mathbf{R}^n \times \mathbf{R}^n$. Let $n_1, \ldots, n_m$ be natural numbers such that $\sum_{i=1}^m n_i = n$. We put $U^i = \mathbf{R}^{n_i}$ and $V^i = \mathbf{R}^{n_i}$ for $i = 1 \cdots m$. Let ${}^t(x, y) = {}^t(x^1 \cdots x^m y^1 \cdots y^m)$ be the normal coordinates on $M = U^1 \times \cdots \times U^m \times V^1 \times \cdots \times V^m$ where $x^i = {}^t(x^{i,1} \cdots x^{i,n_i}) \in U^i$ and $y^i = {}^t(y^{i,1} \cdots y^{i,n_i}) \in V^i$ for $i = 1 \cdots m$. Let θ be a 1-form on M and H a C^∞-function on M. Suppose that θ and H are expressed in the following forms respectively:

$$\theta = \sum_{i=1}^m {}^t y^i (dx^i + \sum_{j=1}^{i-1} f^{ij} dx^j) + {}^t a^i dx^i + {}^t b^i dy^i \tag{1.1}$$

where

$$f^{ij} = f^{ij0} + \sum_{k=1}^{n_i} x^{i,k} f^{ijk},$$
$$f^{ijk} \in C^\infty(U^1 \times \cdots \times U^{i-1}, \mathfrak{M}(n_i, n_j, \mathbf{R})), \qquad k = 0, 1, \dots, n_i,$$
$$a^i \in C^\infty(U^1 \times \cdots \times U^m, U^i),$$
$$b^i \in C^\infty(M, U^i)$$

and

$$H = \sum_{i=1}^{m} {}^t y^i (h^{i0} + \sum_{k=1}^{n_i} x^{ik} h^{ik}) + c \tag{1.2}$$

where

$$h^{ik} \in C^\infty(U^1 \times \cdots \times U^{i-1}, U^i), \qquad k = 0, 1, \dots, n_i,$$
$$c \in C^\infty(U^1 \times \cdots \times U^m, \mathbf{R}).$$

For ${}^t(x, y) \in M$, we put $dx = dx^{1,1} \wedge \cdots \wedge dx^{1,n_1} \wedge dx^{2,1} \wedge \cdots \wedge dx^{m,n_m}$ and $d(y/2\pi) = d(y^{1,1}/2\pi) \wedge \cdots \wedge d(y^{1,n_1}/2\pi) \wedge d(y^{2,1}/2\pi) \wedge \cdots \wedge d(y^{m,n_m}/2\pi)$. Now for $x', x'' \in \mathbf{R}^n$ and $T \in \mathbf{R}$ define the path integral $K_{\theta,H}(x'', x'; T)$ by

$$\begin{aligned} &K_{\theta,H}(x'', x'; T) \\ &= \lim_{N\to\infty} \int dx_1 \cdots dx_{N-1}\, d\frac{y_1}{2\pi} \cdots d\frac{y_N}{2\pi} exp\left\{\sqrt{-1} \int_0^T \gamma^*\theta - H(\gamma(t))\, dt\right\} \end{aligned} \tag{1.3}$$

where

$$\gamma(t) = {}^t\left(x_{k-1} + \frac{x_k - x_{k-1}}{T/N}\left(t - \frac{k-1}{N}T\right), y_{k-1}\right) \in M \quad \text{for} \quad t \in \left[\frac{k-1}{N}T, \frac{k}{N}T\right),$$

$x_N = x''$ and $x_0 = x'$.

For simplicity, we put

$$\kappa(x, \dot{x}) = \sum_{j=1}^{m} {}^t a^j(x) \dot{x}^j + c(x),$$
$$\tau^i(x, \dot{x}) = \sum_{j=1}^{i-1} f^{ij0}(x^1, \dots, x^{i-1}) \dot{x}^j - h^i(x^1, \dots, x^{i-1}),$$
$$\begin{aligned} v^i(x, \dot{x}) = &\sum_{j=1}^{i-1} (f^{ij1}(x^1, \dots, x^{i-1}) \dot{x}^j, \dots, f^{ijn_i}(x^1, \dots, x^{i-1}) \dot{x}^j) \\ &- (h^{i1}(x^1, \dots, x^{i-1}), \dots, h^{in_i}(x^1, \dots, x^{i-1})) \end{aligned}$$

and

$$\mu^i(x, \dot{x}) = \tau^i(x, \dot{x}) + v^i(x, \dot{x})x^i.$$

From the form of μ^i, for $x \in \mathbf{R}^n = U^1 \times \cdots \times U^m$, the differential equation

$$\begin{cases} \dot{w} + \mu(w, \dot{w}) = 0, \\ w(0) = x \end{cases} \tag{1.4}$$

has a unique solution $w(x, t)$. As the next theorem shows, the path integral $K_{\theta,H}(x'', x'; T)$ is described by using the solution w.

THEOREM 1.

$$K_{\theta,H}(x'', x'; T)$$
$$= \delta(x'' - w(x', T))\, exp\left\{\sqrt{-1}\int_0^T \kappa(w(x', t), \dot{w}(x', t))\, dt\right\}\left|\frac{dw(x', T)}{dx'}\right|^{\frac{1}{2}}$$

where $\dfrac{dw(x', T)}{dx'}$ denotes the Jacobian.

PROOF. By the definition of γ in (1.3), we can assume that $b^i = 0$ and by integrating with respect to y, we obtain

$$K_{\theta,H}(x'', x'; T)$$
$$= \lim_{N \to \infty} \int dx_1 \cdots dx_{N-1} \prod_{k=1}^{N} \delta\left(x_k - x_{k-1} + \int_{\frac{k-1}{N}T}^{\frac{k}{N}T} \mu(x(t), \dot{x}(t))\, dt\right)$$
$$\times\, exp\left\{\sqrt{-1}\int_0^T \kappa(x(t), \dot{x}(t))\, dt\right\}$$

where

$$x(t) = x_{k-1} + \frac{x_k - x_{k-1}}{T/N}\left(t - \frac{k-1}{N}T\right) \qquad \text{for } t \in \left[\frac{k-1}{N}T, \frac{k}{N}T\right],$$

$x_N = x''$ and $x_0 = x'$.

In order to integrate with respect to x, we define $z_{N,k} \in \mathbf{R}^n$ by the equations

$$\begin{cases} z_{N,k} - z_{N,k-1} \\ = -\displaystyle\int_{\frac{k-1}{N}T}^{\frac{k}{N}T} \mu\left(z_{N,k-1} + \frac{z_{N,k} - z_{N,k-1}}{T/N}\left(t - \frac{k-1}{N}T\right), \frac{z_{N,k} - z_{N,k-1}}{T/N}\right) dt, \\ z_{N,0} = x'. \end{cases} \tag{1.5}$$

Now we suppose that (1.5) is well-defined when N is sufficiently large. We put

$$z_N(t) = z_{N,k-1} + \frac{z_{N,k} - z_{N,k-1}}{T/N}\left(t - \frac{k-1}{N}T\right) \qquad \text{for } t \in \left[\frac{k-1}{N}T, \frac{k}{N}T\right].$$

Then we obtain

$$\begin{aligned} &K_{\theta,H}(x'', x'; T) \\ &= \lim_{N\to\infty} \delta(x'' - z_N(T))\, exp\left\{\sqrt{-1}\int_0^T \kappa(z_N(t), \dot z_N(t))\,dt\right\} \\ &\times \prod_{k=1}^{N}\prod_{i=1}^{m}\left|I_{n_i} - \int_{\frac{k-1}{N}T}^{\frac{k}{N}T} v^i(z_N(t), \dot z_N(t))\,\frac{\frac{k}{N}T - t}{T/N}\,dt\right|^{-1} \end{aligned}$$

Hence, to complete the proof, we have only to show that the following (i) and (ii) hold.

(i) $z_N^i(t)$ is well-defined when N is sufficiently large and $z_N^i(t)$ and $\dot z_N^i(t)$ converge to $w^i(x', t)$ and $\dot w^i(x', t)$ respectively uniformly on $[0, T]$ for $i = 1 \cdots m$.

(ii)

$$\lim_{N\to\infty} \prod_{k=1}^{N}\prod_{i=1}^{m}\left|I_{n_i} - \int_{\frac{k-1}{N}T}^{\frac{k}{N}T} v^i(z_N(t), \dot z_N(t))\,\frac{\frac{k}{N}T - t}{T/N}\,dt\right|^{-1} = \left|\frac{dw(x', T)}{dx'}\right|^{\frac{1}{2}}.$$

Since $\mu^i(x, \dot x)$ is independent of $x^{i+1},\dots,x^m, \dot x^i,\dots,\dot x^m$, we can use the induction with respect to i to show (i). The following discussion shows the facts that $z_N^1(t)$ is well-defined when N is sufficiently large and $z_N^1(t)$ and $\dot z_N^1(t)$ converge to $w^i(x', t)$ and $\dot w^i(x', t)$ respectively uniformly on $[0, T]$.

Now for $i = 1 \cdots l-1$ suppose that $z_N^i(t)$ is well-defined when N is sufficiently large and that $z_N^i(t)$ and $\dot z_N^i(t)$ converge to $w^i(x', t)$ and $\dot w^i(x', t)$ respectively uniformly on $[0, T]$.

For simplicity, we put α_N, α, β_N and β as follows:

$$\begin{aligned} \alpha_N(t) &= v^l(z_N(t), \dot z_N(t)), \\ \alpha(t) &= v^l(w(x', t), \dot w(x', t)), \\ \beta_N(t) &= \tau^l(z_N(t), \dot z_N(t)) \end{aligned}$$

and

$$\beta(t) = \tau^l(w(x', t), \dot w(x', t)).$$

Now since $\dot w^l(x', t) = \alpha(t) w^l(x', t) + \beta(t)$, we obtain

$$w^l\left(x', \frac{k}{N}T\right) = w^l\left(x', \frac{k-1}{N}T\right) + \int_{\frac{k-1}{N}T}^{\frac{k}{N}T} \alpha(t) w(x', t)\,dt + \int_{\frac{k-1}{N}T}^{\frac{k}{N}T} \beta(t)\,dt.$$

We put

$$A_{N,k} = \left(I_{n_l} - \int_{\frac{k-1}{N}T}^{\frac{k}{N}T} \alpha_N(t) \frac{\frac{k}{N}T - t}{T/N} dt \right),$$

$$B_{N,k} = \left(I_{n_l} + \int_{\frac{k-1}{N}T}^{\frac{k}{N}T} \alpha_N(t) \frac{t - \frac{k-1}{N}T}{T/N} dt \right)$$

and

$$C_{N,k} = \int_{\frac{k-1}{N}T}^{\frac{k}{N}T} \beta_N(t)\, dt.$$

Then, from the definition of $z_{N,k}$, we have

$$A_{N,k} z^l_{N,k} = B_{N,k} z^l_{N,k-1} + C_{N,k}.$$

From the hypothesis of the induction, α_N and β_N converge to α and β respectively uniformly on $[0, T]$. When T/N is sufficiently small, $A_{N,k}$ has the inverse matrix. Hence when N is sufficiently large, $z_{N,k}$ is well-defined and

$$z^l_{N,k} = A^{-1}_{N,k} B_{N,k} z^l_{N,k-1} + A^{-1}_{N,k} C_{N,k}.$$

Now we can find a positive number Γ such that for any positive number ε there exists a number N_0 for which

$$\begin{aligned}
&\left\| z^l_{N,k} - w^l\left(x', \frac{k}{N} T\right) \right\| \\
&\leq \left\| A^{-1}_{N,k} B_{N,k} \left(z^l_{N,k-1} - w^l\left(x', \frac{k-1}{N} T\right)\right) \right\| \\
&+ \left\| A^{-1}_{N,k} B_{N,k} w^l\left(x', \frac{k-1}{N} T\right) - w^l\left(x', \frac{k-1}{N} T\right) - \int_{\frac{k-1}{N}T}^{\frac{k}{N}T} \alpha(t) w(x', t)\, dt \right\| \\
&+ \left\| A^{-1}_{N,k} B_{N,k} - \int_{\frac{k-1}{N}T}^{\frac{k}{N}T} \beta(t)\, dt \right\| \\
&\leq \left(1 + \frac{\Gamma}{N}\right) \left\| \left(z^l_{N,k-1} - w^l\left(x', \frac{k-1}{N} T\right)\right) \right\| + \frac{1}{2}\frac{\varepsilon}{N} + \frac{1}{2}\frac{\varepsilon}{N}
\end{aligned}$$

for any $N > N_0$. Using the above inequality repeatedly and using $z^l_{N,0} = w^l(x', 0)$, we obtain

$$\left\| z^l_{N,k} - w^l\left(x', \frac{k}{N} T\right) \right\| \leq \sum_{i=1}^{l} \left(1 + \frac{\Gamma}{N}\right)^l \frac{\varepsilon}{N} \leq (e^{(1+\frac{1}{N})\Gamma - 1}) \frac{\varepsilon}{\Gamma}.$$

Therefore

$$\lim_{N\to\infty} \max_{k=0\cdots N} \left\| z^l_{N,k} - w^l\left(x', \frac{k}{N}T\right)\right\| = 0.$$

This shows that $z^l_N(t)$ converges to $w^l(t)$ uniformly on $[0, T]$. And from the following inequality

$$\begin{aligned}
&\left\| \frac{z^l_{N,k} - z^l_{N,k-1}}{T/N} - \dot{w}^l\left(x', \frac{k-1}{N}T\right)\right\| \\
&= \left\| \frac{(A_{N,k} - I_{n_l}) z^l_{N,k-1} + B_{N,k}}{T/N} - \alpha\left(\frac{k-1}{N}T\right) w^l\left(x', \frac{k-1}{N}T\right) - \beta\left(\frac{k-1}{N}T\right)\right\| \\
&\leq \left\| \frac{A_{N,k} - I_{n_l}}{T/N}\left(z^l_{N,k-1} - w^l\left(x', \frac{k-1}{N}T\right)\right)\right\| \\
&+ \left\| \left(\frac{A_{N,k} - I_{n_l}}{T/N} - \alpha\left(\frac{k-1}{N}T\right)\right) w^l\left(x', \frac{k-1}{N}T\right)\right\| \\
&+ \left\| \frac{B_{n,k}}{T/N} - \beta\left(\frac{k-1}{N}T\right)\right\|
\end{aligned}$$

we obtain

$$\lim_{N\to\infty} \max_{k=0\cdots N} \left\| \frac{z^l_{N,k} - z^l_{N,k-1}}{T/N} - \dot{w}^l\left(x', \frac{k-1}{N}T\right)\right\| = 0.$$

This shows that $\dot{z}^l_N(t)$ converges to $\dot{w}^l(t)$ uniformly on $[0, T]$. Thus we have proved (i).

Now we have

$$\lim_{N\to\infty} \max_{k=0\cdots N} \left\| I_{n_l} - \int_{\frac{k-1}{N}T}^{\frac{k}{N}T} \alpha_N(t) \frac{\frac{k}{N}T - t}{T/N} dt - exp\left(-\frac{1}{2}\int_{\frac{k-1}{N}T}^{\frac{k}{N}T} \alpha(t)\, dt\right)\right\| \frac{N}{T} = 0.$$

Hence we obtain

$$\begin{aligned}
\lim_{N\to\infty} \prod_{k=1}^{N} \left| I_{n_l} - \int_{\frac{k-1}{N}T}^{\frac{k}{N}T} \alpha_N(t) \frac{\frac{k}{N}T - t}{T/N} dt\right|^{-1} &= \lim_{N\to\infty} \prod_{k=1}^{N} \left| exp\left(-\frac{1}{2}\int_{\frac{k-1}{N}T}^{\frac{k}{N}T} \alpha(t)\, dt\right)\right|^{-1} \\
&= exp\frac{1}{2}\int_0^T \alpha(t)\, dt \\
&= \left|\frac{dw^l(x', T)}{dx'^l}\right|^{\frac{1}{2}}.
\end{aligned}$$

Since $dw^l(x', T)/dx'^i = 0$ for $i > l$, consequently we obtain (ii). $\square$

§2 Path integrals on coadjoint orbits

Let G be a connected and simply connected solvable Lie group, $\mathfrak{g}$ the Lie algebra of G, $\mathfrak{g}^*$ the dual space of $\mathfrak{g}$. For $\lambda \in \mathfrak{g}^*$, we put $G_\lambda = \{g \in G | Ad^*(g)\lambda = \lambda\}$. Then the coadjoint orbit O_λ is canonically identified with the homogeneous space G/G_λ. Let $\mathfrak{g}_\lambda$ be the Lie algebra of G_λ and we consider a real polarization $\mathfrak{p}\,(\mathfrak{g}_\lambda \subset \mathfrak{p} \subset \mathfrak{g})$. We fix a Lie subgroup P of G the Lie algebra of which coincides with $\mathfrak{p}$. Here we suppose that $Ad^*(P)\lambda = \lambda + \mathfrak{p}^\perp$ where $\mathfrak{p}^\perp = \{\zeta \in \mathfrak{g}^* | \zeta(X) = 0 \text{ for any } X \in \mathfrak{p}\}$.

First we choose the coordinates on O_λ to define the path integrals on O_λ. We take a chain $\mathfrak{g}_1 \supset \mathfrak{g}_2 \supset \cdots \supset \mathfrak{g}_m \supset \mathfrak{g}_{m+1}$ of ideals in $\mathfrak{g}$, beginning with $\mathfrak{g}_1 = \mathfrak{g}$, ending with $\mathfrak{g}_{m+1} = \{0\}$, such that the factor algebras $\mathfrak{g}_i/\mathfrak{g}_{i+1}$ $(i = 1,\dots,m)$ are all abelian. We put $n = dim\, G/P$ and $n_i = dim\, \mathfrak{g}_i/(\mathfrak{g}_{i+1} + \mathfrak{g}_i \cap \mathfrak{p})$. Then we have $n = \sum_{i=1}^m n_i$. Now $\mathfrak{g}_i/(\mathfrak{g}_{i+1} + \mathfrak{g}_i \cap \mathfrak{p}) \simeq (\mathfrak{g}_i \cap \mathfrak{p})/(\mathfrak{g}_{i+1} \cap \mathfrak{p})$. Therefore we can take $X_{i,j} \in \mathfrak{g}_i (i = 1 \cdots m, j = 1 \cdots n_i)$ such that

$$\sum_{j=1}^{n_i} \mathbf{R} X_{i,j} \oplus (\mathfrak{g}_{i+1} + \mathfrak{g}_i \cap \mathfrak{p}) = \mathfrak{g}_i.$$

We define the mapping $\phi\colon \mathbf{R}^n = U^1 \times \cdots \times U^m \to G$ by

$$\phi(x) = exp(x^{1,1} X_{1,1}) \cdots exp(x^{1,n_1} X_{1,n_1}) \cdots exp(x^{m,n_m} X_{m,n_m})$$

where $x = {}^t(x^1 \cdots x^m)$ and $x^i = {}^t(x^{i,1} \cdots x^{i,n_i})$. Let G_i be the analytic subgroup of G corresponding to $\mathfrak{g}_i$. Then the analytic subgroup of G corresponding to $\mathfrak{g}_{i+1} + \mathfrak{g}_i \cap \mathfrak{p}$ can be written as $G_{i+1}(G_i \cap P)$. This shows that the mapping $\mathbf{R}^n \ni x \mapsto \phi(x)P \in G/P$ is an onto-diffeomorphism.

For $i = 1 \cdots m$ and $j = 1 \cdots m_i$, we take $\zeta_{ij} \in \mathfrak{p}^\perp$ such that $\zeta_{ij}(X_{k,l}) = \delta_{ik}\delta_{jl}$ and take an immersion map $\psi\colon \mathbf{R}^n = V^1 \times \cdots \times V^m \subsetneq P$ such that

$$Ad^*(\psi(y))\lambda = \lambda + \sum_{i=1}^m \sum_{j=1}^{n_i} y^{i,j} \zeta_{ij}$$

where $y = {}^t(y^1 \cdots y^m)$ and $y^i = {}^t(y^{i,1} \cdots y^{i,n_i})$. Then the mapping $\mathbf{R}^n \ni y \mapsto \psi(y)\, G_\lambda \in P/G_\lambda$ and the mapping $M \ni {}^t(x, y) \mapsto \phi(x)\psi(y)\, G_\lambda \in G/G_\lambda$ are onto-diffeomorphisms.

We take a 1-form θ_λ on M and a C^∞-function on M for $Y \in \mathfrak{g}$ as follows:

$$\theta_\lambda = \langle \lambda, g^{-1} dg \rangle,$$
$$H_Y = \langle \lambda, g^{-1} Y g \rangle$$

where $g = \phi(x)\psi(y)$ and ${}^t(x, y) \in M$. From the definition of ψ, we have

$$\theta_\lambda = \sum_{\substack{i=1\dots m\\ j=1\dots n_i}} y^{i,j}\langle \zeta_{ij}, \phi^{-1}(x)\,d\phi(x)\rangle$$
$$+ \langle \lambda, \phi^{-1}(x)\,d\phi(x)\rangle + \langle \lambda, \psi^{-1}(x)\,d\psi(x)\rangle$$

and from the definition of ϕ, we have

$$\langle \zeta_{ij}, \phi^{-1}(x)\,d\phi(x)\rangle$$
$$= dx^{i,j} + \langle \zeta_{ij}, g^{-1}dg\rangle + \sum_{k=1}^{n_i} x^{i,k}\langle \zeta_{ij}, [X_{i,k}, g^{-1}dg]\rangle$$

where

$$g = exp(x^{1,1}X_{1,1})\cdots exp(x^{1,n_1}X_{1,n_1})\cdots exp(x^{i-1,n_{i-1}}X_{i-1,n_{i-1}}).$$

Hence θ_λ is expressed as in the form (1.1) and similarly we can see that H_Y is expressed as in the form (1.2).

Now following the Kirillov-Kostant theory, we construct a unitary representation of G. We assume that the Lie algebra homomorphism

$$\mathfrak{p} \ni X \longmapsto -\sqrt{-1}\langle \lambda, X\rangle \in \sqrt{-1}\mathbf{R}$$

lifts to a unitary character η_λ of P. We denote by η_ρ the character of P such that $|\Omega|^{\frac{1}{2}}$ is the line bundle associated with η_ρ, where $|\Omega|^{\frac{1}{2}}$ denotes the square root of absolute value of the volume bundle on $G/(G\cap P)$. We put $\xi_\lambda = \eta_\lambda\eta_\rho$.

Let L_{ξ_λ} denote the line bundle associated with ξ_λ over the homogeneous space G/P. Then the space $C^\infty(L_{\xi_\lambda})$ of all complex valued C^∞-sections of L_{ξ_λ} can be identified with

$$\{f \in C^\infty(G); f(gp) = \xi_\lambda(p)^{-1}f(g)\ (g \in G, p \in P)\}.$$

For any $g \in G$ we define an operator $\pi_\lambda^{\mathfrak{p}}(g)$ on $C^\infty(L_{\xi_\lambda})$: For $f \in C^\infty(L_{\xi_\lambda})$

$$(\pi_\lambda^{\mathfrak{p}}(g)f)(x) = f(g^{-1}x) \qquad (x \in G).$$

By using the diffeomorphism $\mathbf{R}^n \ni x \mapsto \phi(x) \in G/P$, we can regard $\pi_\lambda^{\mathfrak{p}}(g)$ as an operator on $C^\infty(\mathbf{R}^n)$. Then $\pi_\lambda^{\mathfrak{p}}(g)$ is an isometry on $L^2(\mathbf{R}^n)$ so that we obtain a unitary representation of G on $L^2(\mathbf{R}^n)$.

The next theorem shows that the integral operator whose kernel function is the path integral $K_{\theta_\lambda, H_Y}(x'', x'; T)$ coincides with the operator $\pi_\lambda^{\mathfrak{p}}(exp\,TY)$.

THEOREM 2.

$$\int K_{\theta_\lambda, H_Y}(x'', x'; T) f(x') dx' = (\pi_\lambda^{\mathfrak{p}}(exp\, TY) f)(x'')$$

where

$$f \in C_c^\infty(\mathbf{R}^n).$$

PROOF. First for $x \in \mathbf{R}^n$ and $t \in \mathbf{R}$, we define $v(x, t) \in \mathbf{R}^n$ and $p(x, t) \in P$ by

$$exp(-tY)\phi(x) = \phi(v(x, t)) p(x, t).$$

Then we have

$$(\pi_\lambda^{\mathfrak{p}}(exp\, TY) f)(x) = \xi_\lambda^{-1}(p(x, T)) f(v(x, T)) \left| \frac{dv(x, T)}{dx} \right|^{\frac{1}{2}}. \tag{2.1}$$

The differential equations (1.4) corresponding to θ_λ and H_Y is written as the system

$$\begin{cases} \left\langle \zeta_{ij}, \phi^{-1}(w) \dfrac{d\phi(w)}{dt} - \phi^{-1}(w) Y \phi(w) \right\rangle = 0, \qquad i = 1 \cdots m,\ j = 1 \cdots n_i, \\ w(0) = x. \end{cases}$$

By the uniqueness of the solution $w(x, t)$, we have $w(x, t) = v(x, -t)$. Hence from Theorem 1, we get

$$K_{\theta_\lambda, H_Y}(x'', x'; T)$$
$$= \delta(x'' - v(x', -T)) exp\left\{ \sqrt{-1} \int_0^T \left\langle \lambda, p(x', -t) \frac{dp^{-1}(x', -t)}{dt} \right\rangle dt \right\} \left| \frac{dv(x', -T)}{dx'} \right|^{\frac{1}{2}}.$$

By using the second kind coordinates on P, it is easy to show that

$$exp\left\{ \sqrt{-1} \int_0^T \left\langle \lambda, p(x', -t) \frac{dp^{-1}(x', -t)}{dt} \right\rangle dt \right\} = \xi_\lambda(p(x', -T)).$$

Hence we get

$$K_{\theta_\lambda, H_Y}(x'', x'; T) = \delta(x'' - v(x', -T)) \xi_\lambda(p(x', -T)) \left| \frac{dv(x', -T)}{dx'} \right|^{\frac{1}{2}}$$
$$= \delta(x' - v(x'', T)) \xi_\lambda^{-1}(p(x'', T)) \left| \frac{dv(x'', T)}{dx''} \right|^{\frac{1}{2}}. \tag{2.2}$$

Now Theorem 2 follows from (2.1) and (2.2). □

References

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