A Stroboscopic Method for the Critical Case where the Jacobian Vanishes

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1. Introduction.

In this paper, we are concerned with a real system of \( n \) nonlinear differential equations of the form as follows:

\[
\frac{dx_i}{dt} = \varepsilon f_i(x, t, \varepsilon) \quad (i = 1, 2, \ldots, n),
\]

where

1° \( \varepsilon \) is a parameter such that \( |\varepsilon| < \delta \) (\( \delta > 0 \));
2° the functions \( f_i(x, t, \varepsilon) \ (i = 1, 2, \ldots, n) \) are periodic in \( t \) with period \( T > 0 \) and are continuous in the domain

\[
D: |x| = \sum_{i=1}^{n} |x_i| < L, \quad -\infty < t < +\infty, \quad |\varepsilon| < \delta
\]

together with \( \frac{\partial f_i(x, t, \varepsilon)}{\partial x_j}, \frac{\partial f_i(x, t, \varepsilon)}{\partial \varepsilon} \) \((i, j = 1, 2, \ldots, n)\).

Let us consider the functions

\[
F_i(x) = \frac{1}{T} \int_{0}^{T} f_i(x, t, 0)dt \quad (i = 1, 2, \ldots, n).
\]

Then, as is well known, there exists a periodic solution of (1.1) provided there exists a real solution \( x_i = c_i \) \((i = 1, 2, \ldots, n)\) of the system of equations

\[
F_i(x) = 0 \quad (i = 1, 2, \ldots, n)
\]

and the Jacobian \( J \) of \( F_i(x) \) with respect to \( x_j \) does not vanish for \( x_i = c_i \) \((i = 1, 2, \ldots, n)\). In this case, as is well known, the stability of the assured periodic solution of (1.1) is decided according to the signs of the eigenvalues of \( J \).

But, if the Jacobian \( J \) vanishes for \( x_i = c_i \) \((i = 1, 2, \ldots, n)\), the periodic solution of (1.1) does not necessarily exist even if there exists a real solution of (1.3).

In the present paper, we investigate some cases where the Jacobian \( J \) vanishes for \( x_i = c_i \) \((i = 1, 2, \ldots, n)\) but nevertheless the equation (1.1) has a periodic solution.
For our discussions, the assumption 2° is not strong enough, because our investigation needs more minute computation than in the ordinary case, i.e. the case where the Jacobian \( J \) does not vanish for \( x_i = c_i \) (i = 1, 2, ..., n). Thus, in the present paper, the condition 2° is replaced by the stronger one as follows:

2° the functions \( f_i(x, t, \varepsilon) \) (i = 1, 2, ..., n) are periodic in \( t \) with period \( T \) (>0) and are continuous in the domain

\[
D: |x| = \sum_{i=1}^{n} |x_i| < L, -\infty < t < +\infty, |\varepsilon| < \delta
\]

together with their derivatives with respect to \((x, \varepsilon)\) up to the 3rd order.

2. Preliminary calculations.

Let

\[
(2.1) \quad x_i = \phi_i(u, t, \varepsilon) \quad (i = 1, 2, \ldots, n)
\]

be the solution of (1.1) such that

\[
(2.2) \quad \phi_i(u, 0, \varepsilon) = u_i \quad (i = 1, 2, \ldots, n),
\]

where \( |u| = \sum_{i=1}^{n} |u_i| < L \). From the form of (1.1) and the assumptions on \( f_i(x, t, \varepsilon) \) (i = 1, 2, ..., n), it is readily seen that, if \( |\varepsilon| \) is sufficiently small, the functions \( \phi_i(u, t, \varepsilon) \) (i = 1, 2, ..., n) are expanded as

\[
(2.3) \quad \phi_i(u, t, \varepsilon) = \phi_i^{(0)}(u, t) + \varepsilon \phi_i^{(1)}(u, t) + \varepsilon^2 \phi_i^{(2)}(u, t) + \varepsilon^3 \phi_i^{(3)}(u, t) + q_i(u, t, \varepsilon) \quad (i = 1, 2, \ldots, n)
\]

for any finite value of \( t \), where \( q_i(u, t, \varepsilon) = O(\varepsilon^4) \) as \( \varepsilon \to 0 \).

Now, by the initial condition (2.2), it is evident that

\[
(2.4) \quad \begin{cases} 
\phi_i^{(0)}(u, 0) = u_i, \\
\phi_i^{(1)}(u, 0) = \phi_i^{(2)}(u, 0) = \phi_i^{(3)}(u, 0) = q_i(u, 0, \varepsilon) = 0 \quad (i = 1, 2, \ldots, n).
\end{cases}
\]

If we substitute (2.3) into the initial equation (1.1) and compare the coefficients of powers of \( \varepsilon \), we have the system of the linear differential equations with respect to \( \phi_i^{(0)}, \phi_i^{(1)}, \phi_i^{(2)}, \phi_i^{(3)} (i = 1, 2, \ldots, n) \). These equations are solved successively under the initial conditions (2.4) as follows:

\[
(2.5) \quad \begin{cases} 
\phi_i^{(0)}(u, t) = u_i, \\
\phi_i^{(1)}(u, t) = \int_{0}^{t} f_i(u, t_1, 0) dt_1, \\
\phi_i^{(2)}(u, t) = \int_{0}^{t} \left[ \sum_{j=1}^{n} f_{ij}(u, t_1, 0) \int_{0}^{t_1} f_j(u, t_2, 0) dt_2 + f_i(u, t_1, 0) \right] dt_1,
\end{cases}
\]
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\[ \varphi_i^{(3)}(u, t) = \int_0^t \left[ \sum_{j=1}^n f_{ij}(u, t_1, 0) \right] dt_1 \left\{ \sum_{k=1}^n f_{ik}(u, t_2, 0) \right\} dt_2 \times \left[ \int_0^{t_1} f_k(u, t_3, 0) dt_3 + f'_j(u, t_2, 0) \right] dt_2 + \frac{1}{2} \sum_{j=1}^n \sum_{k=1}^n f_{ijk}(u, t_1, 0) \left( \int_0^{t_1} f_j(u, t_2, 0) dt_2 \right) \left( \int_0^{t_1} f_k(u, t_2, 0) dt_2 \right) + \sum_{j=1}^n f'_{ij}(u, t_1, 0) \int_0^{t_1} f_j(u, t_2, 0) dt_2 + \frac{1}{2} f''_j(u, t_1, 0) \right] dt_1 \]

where

\[ f_{ij}(x, t, \varepsilon) = \frac{\partial f_i}{\partial x_j}(x, t, \varepsilon), \quad f'_i(x, t, \varepsilon) = \frac{\partial f_i}{\partial \varepsilon}(x, t, \varepsilon), \]

\[ f_{ijk}(x, t, \varepsilon) = \frac{\partial^2 f_i}{\partial x_j \partial x_k}(x, t, \varepsilon), \quad f''_i(x, t, \varepsilon) = \frac{\partial^2 f_i}{\partial x_i \partial \varepsilon}(x, t, \varepsilon), \]

As is readily seen, the necessary and sufficient condition that the solution \( x_i = \psi_i(u, 0) \) (\( i = 1, 2, \ldots, n \)) is periodic in \( t \) with period \( T \), is that

\[ (2.6) \quad \varphi_i(u, 0, \varepsilon) = \varphi_i(u, T, \varepsilon) \quad (i = 1, 2, \ldots, n). \]

This condition can be written by (2.3) and (2.5) as follows:

\[ (2.7) \quad \varphi_i^{(3)}(u, T) + \varepsilon \varphi_i^{(2)}(u, T) + \varepsilon^2 \varphi_i^{(3)}(u, T) + O(\varepsilon^3) = 0 \quad (i = 1, 2, \ldots, n). \]

Now, we assume that

1° the equation

\[ (2.8) \quad \varphi_i^{(1)}(u, T) = 0 \quad (i = 1, 2, \ldots, n) \]

has a real solution \( u_i = c_i \) (\( i = 1, 2, \ldots, n \)) such that \( |c| = \sum_{i=1}^n |c_i| < L \);

2° the Jacobian \( J_0 = \det \left( \frac{\partial \varphi_i^{(1)}}{\partial u_j} (c, T) \right) \) (\( i, j = 1, 2, \ldots, n \)) vanishes.

In the present paper, we shall investigate the case where the rank \( k \) of the Jacobian matrix \( \left( \frac{\partial \varphi_i^{(1)}}{\partial u_j} (c, T) \right) \) (\( i, j = 1, 2, \ldots, n \)) is not zero.

Let \( l \) be the rank of the matrix \( \left( \frac{\partial \varphi_i^{(1)}}{\partial u_j} (c, T), \varphi_i^{(3)}(c, T) \right) \), then, evidently \( l \geq k \geq 1 \).

The case where \( k < l \) and the case where \( k = l \) shall be studied separately in the sequel.

3. Existence of a periodic solution: Case I where \( k < l \).

By our assumptions, we may assume, without loss of generality, that
Then there exist numbers $\xi_{va} (\alpha = 1, 2, \ldots, k; \nu = k + 1, \ldots, n)$ such that

$$
(3.2) \quad \frac{\partial \xi_{va}}{\partial u_j}(c, T) + \sum_{\alpha=1}^{k} \xi_{va} \frac{\partial \xi_{va}}{\partial u_j}(c, T) = 0 \quad (j = 1, 2, \ldots, n; \nu = k + 1, \ldots, n).
$$

Making use of these $\xi_{va} (\alpha = 1, 2, \ldots, k; \nu = k + 1, \ldots, n)$, let us rewrite the equations (2.7) as follows:

$$
(3.3) \quad \begin{cases}
\frac{\partial \phi_1}{\partial u}(u, T) + \varepsilon \frac{\partial \phi_2}{\partial u}(u, T) + o(\varepsilon) = 0, \\
\frac{\partial \phi_1}{\partial u}(u, T) + \sum_{\beta=1}^{k} \xi_{va} \frac{\partial \phi_2}{\partial u}(u, T) \\
+ \varepsilon \left( \frac{\partial \phi_2}{\partial u}(u, T) + \sum_{\beta=1}^{k} \xi_{va} \frac{\partial \phi_2}{\partial u}(u, T) \right) + o(\varepsilon) = 0,
\end{cases}
$$

(\alpha = 1, 2, \ldots, k; \nu = k + 1, \ldots, n).

Since the Jacobian $J_1$ does not vanish, for sufficiently small $|\varepsilon|$, the first $k$ equations of (3.3) can be solved with respect to $u_{va} (\alpha = 1, 2, \ldots, k)$ in the neighborhood of $u_i = c_i (i = 1, 2, \ldots, n)$ as follows:

$$
(3.4) \quad u_{va} = u_{va}(u_{k+1}, \ldots, u_n, \varepsilon) \quad (\alpha = 1, 2, \ldots, k),
$$

where

$$
(3.5) \quad u_{va}(u_{k+1}, \ldots, u_n, 0) = c_{va} \quad (\alpha = 1, 2, \ldots, k).
$$

For brevity, let us write the functions (3.4) as $u_{va} = u_{va}(u_{va}, \varepsilon)$. Such a notation is used in the sequel without any comment.

For $(u_{va}, \varepsilon) = (c_{va}, 0)$, the derivatives of the functions $u_{va} = u_{va}(u_{va}, \varepsilon) (\alpha = 1, 2, \ldots, k)$ are obtained readily as follows:

$$
(3.6) \quad \begin{cases}
\frac{\partial u_{va}}{\partial u_{va}} = - \frac{1}{J_1} \sum_{\beta=1}^{k} \frac{\partial \phi_{1\beta}}{\partial u_{va}} D_{\beta va}, \\
\frac{\partial u_{va}}{\partial \varepsilon} = - \frac{1}{J_1} \sum_{\beta=1}^{k} \frac{\partial \phi_{2\beta}}{\partial u_{va}} D_{\beta va},
\end{cases}
$$

(\alpha = 1, 2, \ldots, k; \nu = k + 1, \ldots, n),

where $D_{\beta va}$ are the cofactors of the elements $\frac{\partial \phi_{1\beta}}{\partial u_{va}} (\alpha, \beta = 1, 2, \ldots, k)$ in $J_1$.

In order to solve the equations (3.3), let us substitute $u_{va} = u_{va}(u_{va}, \varepsilon) (\alpha = 1, 2, \ldots, k)$ into the last $n - k$ equations of (3.3). The resulting equations are written as follows:

$$
(3.7) \quad \psi_\mu(u_{va}, \varepsilon) \overset{\text{def}}{=} \psi^{(1)}_\mu(u_{va}, \varepsilon) + \varepsilon \psi^{(2)}_\mu(u_{va}, \varepsilon) + o(\varepsilon) = 0 \quad (\mu = k + 1, \ldots, n),
$$
where

\[
\psi_{\mu}(u, \varepsilon) = \phi(u) + \sum_{i=1}^{k} \xi_{i, \varepsilon} \phi^{(i)}(u, \varepsilon) + \left( i = 1, 2, \ldots; \mu = k + 1, \ldots, n \right).
\]

Now, by (3.5) and (3.2), it holds that

\[
\psi_{\mu}(u, 0) = 0, \quad \frac{\partial \psi_{\mu}}{\partial u}(c, 0) = 0, \quad \frac{\partial \psi_{\mu}}{\partial \varepsilon}(c, 0) = 0
\]

(3.9)

(\mu, \lambda = k + 1, \ldots, n).

Hence the equations (3.7) are of the forms as follows:

\[
\psi_{\mu}^{(1)}(c, 0) = \mu_{\lambda}^{(1)}(c, 0) (u_\lambda - c_\lambda) (u_\lambda - c_\lambda)
\]

\[
+ \psi_{\mu}^{(2)}(c, 0) + \psi_{\mu}^{(3)}(c, 0) + o(\varepsilon) = 0
\]

(\mu = k + 1, \ldots, n),

where

\[
\psi_{\mu}^{(1)}(c, 0) = \mu_{\lambda}^{(1)}(c, 0) - \frac{1}{2} \sum_{\kappa=k+1}^{n} \sum_{\lambda=k+1}^{n} \frac{\partial^2 \psi_{\mu}^{(1)}}{\partial u_\kappa \partial u_\lambda}(c, 0) (u_\kappa - c_\kappa) (u_\lambda - c_\lambda)
\]

\[
= \psi_{\mu}^{(2)}(c, 0) - \psi_{\mu}^{(3)}(c, 0)
\]

(3.10)

(\mu = k + 1, \ldots, n).

Here, by the assumption that \( k < l \), at least one of \( \psi_{\mu}^{(2)}(c, 0) \)'s (\( \mu = k + 1, \ldots, n \)) does not vanish.

Let us investigate the case where \( \varepsilon > 0 \). The case where \( \varepsilon < 0 \) can be reduced to the former case by the substitution \( \varepsilon = -\varepsilon' \).

Put

\[
u_{\mu} - c_{\mu} = \varepsilon^{1/2} v_{\mu} \quad (\mu = k + 1, \ldots, n),
\]

then the functions \( \psi_{\mu}(u, \varepsilon) = \psi_{\mu}(c, \varepsilon^{1/2} v_{\mu}, \varepsilon) (\mu = k + 1, \ldots, n) \) are of the forms

\[
\psi_{\mu}(c, \varepsilon^{1/2} v_{\mu}, \varepsilon) = \varepsilon \Omega_{\mu}(v_{\mu}, \varepsilon)
\]

(3.11)

\[
= \varepsilon \left[ \frac{1}{2} \sum_{\kappa=k+1}^{n} \sum_{\lambda=k+1}^{n} \frac{\partial^2 \psi_{\mu}^{(1)}}{\partial u_\kappa \partial u_\lambda}(c, 0) v_{\kappa} v_{\lambda} + \psi_{\mu}^{(2)}(c, 0) + o(1) \right]
\]

as \( \varepsilon \to 0 \).

Now, let us consider the quadratic equations
and suppose these equations have a real solution \( v = d \) \( (v = k + 1, \ldots, n) \).

Then, if

\[
J_2 = \det \left( \sum_{\lambda=k+1}^{n} \frac{\partial^2 \phi^{(1)}_{\mu}}{\partial u_{\lambda} \partial u_{\lambda}} (c, 0) d_\lambda \right) \neq 0
\]

\( (\mu, \kappa = k + 1, \ldots, n) \),

the equations \( \Omega_{\mu}(v, \varepsilon) = 0 \) \( (\mu = k + 1, \ldots, n) \) have certainly a unique real solution \( v \) \( (v = k + 1, \ldots, n) \) which tends to \( d \) as \( \varepsilon \to 0 \). Evidently such a solution is continuously differentiable with respect to \( \varepsilon^{1/2} \), consequently it is of the form

\[
v = d + O(\varepsilon^{1/2}) \quad (v = k + 1, \ldots, n).
\]

By (3.11), the solution \( v \) of the above form yields the solution \( u \) of the equations (3.7) which is of the form

\[
u = c + \varepsilon^{1/2} d + o(\varepsilon^{1/2}) \quad (v = k + 1, \ldots, n).
\]

If we substitute (3.16) into (3.4) and make use of the first of (3.6), we see that

\[
u = c + \varepsilon^{1/2} \frac{1}{k} \sum_{\beta=1}^{n} \sum_{\gamma=k+1}^{n} \frac{\partial \phi^{(1)}_{\mu}}{\partial u_{\gamma}} (c, T) D_{\beta \gamma} d + o(\varepsilon^{1/2})
\]

\( (\alpha = 1, 2, \ldots, k) \).

The results obtained above are stated as

\textbf{Theorem 1.} In the case where \( k < l \), if the quadratic equations (3.13) have a real solution \( v = d \) \( (v = k + 1, \ldots, n) \) and the Jacobian \( J_2 \) defined by (3.14) does not vanish, then there exists a periodic solution of (1.1) corresponding to \( u \) \( (i = 1, 2, \ldots, n) \) given by (3.17) and (3.16).

### 4. Existence of a periodic solution: Case II where \( k = l \).

As in the case I, we may assume (3.1) without loss of generality. In the present case, due to the assumption that \( k = l \), the equalities

\[
\phi^{(2)}_{\nu}(c, T) + \sum_{\alpha=1}^{k} \xi_{\nu} \phi^{(2)}_{\alpha}(c, T) = 0 \quad (v = k + 1, \ldots, n)
\]

hold at the same time as (3.2).

As in the case I, we rewrite the equations (2.7) as follows:

\[
\begin{cases}
\phi^{(3)}_{\nu}(u, T) + \varepsilon \phi^{(2)}_{\nu}(u, T) + \varepsilon^2 \phi^{(1)}_{\nu}(u, T) + o(\varepsilon^2) = 0, \\
\sum_{\beta=1}^{k} \xi_{\nu} \phi^{(1)}_{\beta}(u, T) + \varepsilon \phi^{(2)}(u, T) + \sum_{\beta=1}^{k} \xi_{\nu} \phi^{(2)}_{\beta}(u, T) = 0
\end{cases}
\]
and we substitute the solution

\[(4.3) \quad u_\alpha = u_\alpha (u_\nu, \varepsilon) \quad (\alpha = 1, 2, \ldots, k)\]

of the first \(k\) equations into the last \((n-k)\) equations. Then the resulting equations are of the same form as (3.7), but, in the present case, due to (4.1),

\[(4.4) \quad \psi^{(2)}_\mu (c_\nu, 0) = 0 \quad (\nu = k + 1, \ldots, n)\]

in addition to (3.9).

Thus the equations (3.7) are written in the present case as follows:

\[
\begin{align*}
\psi_\mu (u_\nu, \varepsilon) &= \frac{1}{2} \sum_{\kappa = -\lambda + 1}^{\lambda - 1} \frac{\partial^2 \psi^{(1)}_\mu (c_\nu, 0)}{\partial u_\kappa \partial u_\lambda} (c_\nu - c_\kappa)(u_\kappa - c_\kappa) + \varepsilon^2 \frac{\partial^2 \psi^{(1)}_\mu (c_\nu, 0)}{\partial \varepsilon^2} (c_\nu - c_\kappa) + \psi^{(3)}_\mu (u_\nu, \varepsilon) \\
&+ \varepsilon \left[ \sum_{\kappa = -\lambda + 1}^{\lambda - 1} \frac{\partial \psi^{(2)}_\mu (c_\nu, 0)}{\partial u_\kappa} (c_\nu - c_\kappa) + \varepsilon \frac{\partial \psi^{(2)}_\mu (c_\nu, 0)}{\partial \varepsilon} (c_\nu - c_\kappa) + \psi^{(3)}_\mu (u_\nu, \varepsilon) \right] \\
&+ \varepsilon^2 \left[ \psi^{(3)}_\mu (c_\nu, 0) + \psi^{(3)}_\mu (u_\nu, \varepsilon) \right] + o(\varepsilon^2) \\
(\mu = k + 1, \ldots, n),
\end{align*}
\]

where \(\psi^{(i)}_\mu (u_\nu, \varepsilon) (i = 1, 2, 3; \mu = k + 1, \ldots, n)\) are respectively the remainders in \(\psi^{(i)}_\mu (u_\nu, \varepsilon)\) from which the terms written explicitly are subtracted.

Let us put

\[(4.5) \quad u_\nu - c_\nu = \varepsilon v_\nu \quad (\nu = k + 1, \ldots, n).\]

Then the functions \(\psi_\mu (u_\nu, \varepsilon) = \psi_\mu (c_\nu + \varepsilon v_\nu, \varepsilon) (\mu = k + 1, \ldots, n)\) can be written as follows:

\[
\begin{align*}
\psi_\mu (c_\nu + \varepsilon v_\nu, \varepsilon) &= \varepsilon^2 \psi_\mu (v_\nu, \varepsilon) \\
&= \varepsilon^2 \left[ \frac{1}{2} \sum_{\kappa = -\lambda + 1}^{\lambda - 1} \sum_{\lambda = -\lambda + 1}^{\lambda - 1} \frac{\partial^2 \psi^{(1)}_\mu (c_\nu, 0)}{\partial u_\kappa \partial u_\lambda} v_\kappa v_\lambda \\
&+ \sum_{\kappa = -\lambda + 1}^{\lambda - 1} \left\{ \frac{\partial^2 \psi^{(1)}_\mu (c_\nu, 0)}{\partial u_\kappa \partial \varepsilon} (c_\nu, 0) + \frac{\partial \psi^{(2)}_\mu (c_\nu, 0)}{\partial u_\kappa} \right\} v_\kappa \\
&+ \frac{1}{2} \frac{\partial^2 \psi^{(1)}_\mu (c_\nu, 0)}{\partial \varepsilon^2} (c_\nu, 0) + \frac{\partial \psi^{(2)}_\mu (c_\nu, 0)}{\partial \varepsilon} (c_\nu, 0) + \psi^{(3)}_\mu (c_\nu, 0) + o(1) \right] \\
(\mu = k + 1, \ldots, n).
\end{align*}
\]

Therefore, if the quadratic equations

\[(4.7) \quad \psi^{(2)}_\mu (c_\nu, 0) = 0 \quad (\mu = k + 1, \ldots, n)\]
have a real solution \( v_\nu = d_\nu \) (\( \nu = k+1, \ldots, n \)), the equations \( \Omega_\mu(v_\nu, \varepsilon) = 0 \) (\( \mu = k+1, \ldots, n \)) have a unique real solution such that \( v_\nu = d_\nu + O(\varepsilon) \) (\( \nu = k+1, \ldots, n \)) as \( \varepsilon \to 0 \), provided the Jacobian

\[
J_2 = \det \left( \sum_{\lambda=k+1}^{n} \frac{\partial^2 \psi^{(1)}(c_\nu, 0)}{\partial u_\mu \partial u_\lambda} (c_\nu, 0) d_\lambda + \frac{\partial^2 \psi^{(1)}(c_\nu, 0)}{\partial u_\mu \partial \varepsilon} (c_\nu, 0) \right) \neq 0
\]

(\( \mu, \lambda = k+1, \ldots, n \)).

The solution \( v_\nu = d_\nu + O(\varepsilon) \) (\( \nu = k+1, \ldots, n \)) of \( \Omega_\mu(v_\nu, \varepsilon) = 0 \) (\( \mu = k+1, \ldots, n \)) yields the solution of (4.2) of the forms as follows:

\[
\begin{align*}
\begin{cases}
  u_\alpha = c_\alpha - \varepsilon \frac{1}{J_1} \sum_{\beta=1}^{n} D_{\beta\alpha} \left( \sum_{\mu=k+1}^{n} \frac{\partial \psi^{(1)}(c_\nu, 0)}{\partial u_\mu} (c_\nu, 0) d_\mu + \psi^{(2)}(c_\nu, 0) \right) + O(\varepsilon), \\
  u_\nu = c_\nu + \varepsilon d_\nu + O(\varepsilon)
\end{cases}
\end{align*}
\]

(\( \nu = k+1, \ldots, n \)).

The results obtained above are stated as

**Theorem 2.** In the case II where \( k = 1 \), if the quadratic equations (4.7) have a real solution \( v_\nu = d_\nu \) (\( \nu = k+1, \ldots, n \)) and the Jacobian \( J_2 \) defined by (4.8) does not vanish, then there exists a periodic solution of (1.1) corresponding to \( u_i \) (\( i = 1, 2, \ldots, n \)) given by (4.9).

### 5. Stability of the periodic solution.

Let us consider the real transformation

\[
r_i = q_\alpha(\tilde{u} + r, T, \varepsilon) - \tilde{u}_i \quad (i = 1, 2, \ldots, n),
\]

where \( \tilde{u}_i \) is a real solution of (2.7). Then, as is well known, the stability of the periodic solution \( x_i = q_\alpha(\tilde{u}, t, \varepsilon) \) is decided according to the convergency of iteration of the transformation (5.1).

In order to simplify the calculation, let us transform \( r \) to \( s \) by the linear transformation

\[
s = Pr.
\]

Here \( P = \begin{pmatrix} E_k & 0 \\ \Xi & E_{n-k} \end{pmatrix} \), where \( E_k \) and \( E_{n-k} \) are the unit matrices of order \( k \) and \( n-k \) respectively and \( \Xi = (\xi_{\alpha\beta}) \). By (5.2), the transformation (5.1) is rewritten in terms of \( s \) as follows:

\[
\begin{align*}
\begin{cases}
  s'_\alpha = q_\alpha(\tilde{u} + P^{-1}s, T, \varepsilon) - \tilde{u}_\alpha \quad (\alpha = 1, 2, \ldots, k), \\
  s'_\nu = q_\nu(\tilde{u} + P^{-1}s, T, \varepsilon) + \sum_{\beta=1}^{k} \xi_{\nu\beta} \tilde{u}_\beta (\nu = k+1, \ldots, n).
\end{cases}
\end{align*}
\]
Here it is evident that \( P^{-1} = \begin{pmatrix} E_k & 0 \\ -\varepsilon E_{n-k} \end{pmatrix} \). Since \( \tilde{u}_i (i = 1, 2, \ldots, n) \) is a solution of (2.7), the transformation (5.3) can be rewritten as follows:

\[
\begin{pmatrix} s'_a \\ s'_\nu \end{pmatrix} = \sum_{\beta = 1}^{k} \left\{ \sum_{\mu = -k+1}^{n} \frac{\partial q_{\alpha}}{\partial u_{\beta}} \left( \tilde{u}, T, \varepsilon \right) s_{\mu} + o(|s|) \right\} s_{\beta} \\
- \sum_{\mu = -k+1}^{n} \left\{ \xi_{\mu \beta} \left( \tilde{u}, T, \varepsilon \right) s_{\mu} + o(|s|) \right\} s_{\beta} \\
+ \sum_{\mu = -k+1}^{n} \left\{ \sum_{\gamma = 1}^{k} \xi_{\gamma \beta} \left( \tilde{u}, T, \varepsilon \right) s_{\mu} + o(|s|) \right\} s_{\gamma} \\
+ \sum_{\mu = -k+1}^{n} \left\{ \sum_{\gamma = 1}^{k} \xi_{\gamma \beta} \left( \tilde{u}, T, \varepsilon \right) s_{\mu} + o(|s|) \right\} s_{\gamma}
\]

(5.4)

where \(|s| = \sum_{i=1}^{n} |s_i|\).

In the sequel, the case I where \( k < l \) and the case II where \( k = l \) are investigated separately.

**Case I.** In this case, by (3.16) and (3.17), the partial derivatives \( \frac{\partial q_{i}}{\partial u_{j}} \left( \tilde{u}, T, \varepsilon \right) (i, j = 1, 2, \ldots, n) \) can be written as follows:

\[
\frac{\partial q_{i}}{\partial u_{j}} \left( \tilde{u}, T, \varepsilon \right) = \delta_{ij} + \varepsilon \frac{\partial q_{i}^{(1)}}{\partial u_{j}} (c, T)
\]

(5.5)

\[
- \varepsilon^{3/2} \frac{1}{k} \sum_{\sigma = 1}^{k} \sum_{\beta = -1}^{k} \sum_{k+1}^{n} \frac{\partial^2 q_{i}^{(1)}}{\partial u_{\tau} \partial u_{\nu}} \frac{\partial q_{i}^{(1)}}{\partial u_{\nu}} D_{\beta \delta} + o(\varepsilon^{3/2})
\]

\[
+ \varepsilon^{3/2} \sum_{\mu = -k+1}^{n} \frac{\partial^2 q_{i}^{(1)}}{\partial u_{j} \partial u_{\mu}} d_{\mu} + o(\varepsilon^{3/2})
\]

\((i, j = 1, 2, \ldots, n)\).

Let \( A \) be the matrix of the coefficients of the linear parts in the right members of (5.4). Then, by (5.5), \( A \) is of the form as follows:

\[
A = E + \varepsilon A_1 + \varepsilon^{3/2} A_2 + o(\varepsilon^{3/2})
\]

(5.6)

\[
= E + \varepsilon \begin{pmatrix} A_{11}^{(1)} & A_{12}^{(1)} \\ A_{21}^{(1)} & A_{22}^{(1)} \end{pmatrix} + \varepsilon^{3/2} \begin{pmatrix} A_{11}^{(2)} & A_{12}^{(2)} \\ A_{21}^{(2)} & A_{22}^{(2)} \end{pmatrix} + o(\varepsilon^{3/2}),
\]

where \( A_{ij}^{(1)} \) and \( A_{ij}^{(2)} \) are respectively \( k \times k \) and \((n-k) \times (n-k)\) matrices. As is seen from (5.4) and (5.5), the elements of \( A_{11}^{(1)}, A_{12}^{(1)}, \ldots \) are as follows:
\[
\begin{align*}
\left[ A_{11}^{(1)} \right]_{\alpha \beta} &= -\frac{\partial \phi_{\alpha}^{(1)}}{\partial u_{\beta}} (c, T) - \sum_{\mu = k+1}^{n} \xi_{\mu \beta} \frac{\partial \phi_{\mu}^{(1)}}{\partial u_{\mu}} (c, T), \\
\left[ A_{12}^{(1)} \right]_{\alpha \mu} &= -\frac{\partial \phi_{\alpha}^{(1)}}{\partial u_{\mu}} (c, T), \\
\left[ A_{21}^{(1)} \right]_{\nu \beta} &= 0, \\
\left[ A_{22}^{(1)} \right]_{\nu \mu} &= 0;
\end{align*}
\]
\[
\begin{align*}
\left[ A_{21}^{(2)} \right]_{\nu \beta} &= - \frac{1}{J_1} \sum_{\beta = 1}^{k} \sum_{\alpha = 1}^{k} \sum_{\gamma = 1}^{n} \frac{\partial \phi_{\gamma}^{(1)}}{\partial u_{\mu}} \frac{\partial \phi_{\mu}^{(1)}}{\partial u_{\nu}} D_{\nu \alpha} d_{\mu} + \sum_{\mu = k+1}^{n} \xi_{\mu \beta} \frac{\partial \phi_{\mu}^{(1)}}{\partial u_{\nu}} D_{\nu \alpha} d_{\mu} \\
&+ \frac{\sum_{\gamma = 1}^{n} \xi_{\mu \beta}}{J_1} \{ - \frac{1}{J_1} \sum_{\alpha = 1}^{k} \sum_{\gamma = 1}^{n} \frac{\partial \phi_{\gamma}^{(1)}}{\partial u_{\mu}} \frac{\partial \phi_{\mu}^{(1)}}{\partial u_{\nu}} D_{\nu \alpha} d_{\mu} + \sum_{\lambda = k+1}^{n} \xi_{\lambda \beta} \frac{\partial \phi_{\lambda}^{(1)}}{\partial u_{\nu}} \frac{\partial \phi_{\mu}^{(1)}}{\partial u_{\lambda}} D_{\nu \alpha} d_{\lambda} \\
&+ \sum_{\lambda = k+1}^{n} \xi_{\nu \lambda} \frac{\partial \phi_{\nu}^{(1)}}{\partial u_{\alpha}} \frac{\partial \phi_{\mu}^{(1)}}{\partial u_{\lambda}} d_{\lambda} \},
\end{align*}
\]
\[
\begin{align*}
\left[ A_{22}^{(2)} \right]_{\nu \mu} &= - \frac{1}{J_1} \sum_{\beta = 1}^{k} \sum_{\alpha = 1}^{k} \sum_{\gamma = 1}^{n} \frac{\partial \phi_{\gamma}^{(1)}}{\partial u_{\mu}} \frac{\partial \phi_{\mu}^{(1)}}{\partial u_{\nu}} D_{\nu \alpha} d_{\mu} + \sum_{\lambda = k+1}^{n} \xi_{\mu \beta} \frac{\partial \phi_{\mu}^{(1)}}{\partial u_{\nu}} D_{\nu \alpha} d_{\lambda} \\
&+ \frac{\sum_{\beta = 1}^{n} \xi_{\nu \beta}}{J_1} \{ - \frac{1}{J_1} \sum_{\alpha = 1}^{k} \sum_{\gamma = 1}^{n} \frac{\partial \phi_{\gamma}^{(1)}}{\partial u_{\mu}} \frac{\partial \phi_{\mu}^{(1)}}{\partial u_{\nu}} D_{\nu \alpha} d_{\lambda} + \sum_{\lambda = k+1}^{n} \xi_{\nu \lambda} \frac{\partial \phi_{\nu}^{(1)}}{\partial u_{\alpha}} \frac{\partial \phi_{\mu}^{(1)}}{\partial u_{\lambda}} d_{\lambda} \}.
\end{align*}
\]

Since \( A \) is of the form (5.6), \( A \) can be written in the exponential form
\[
A = \exp(\varepsilon B),
\]
where \( B \) is of the form as follows:
\[
(5.7) \quad B = A_1 + \varepsilon^{1/2} A_2 + o(\varepsilon^{1/2})
\]
\[
= \begin{pmatrix}
A_{11}^{(1)} + o(1) & A_{12}^{(1)} + o(1) \\
\varepsilon^{1/2} A_{21}^{(2)} + o(\varepsilon^{1/2}) & \varepsilon^{1/2} A_{22}^{(2)} + o(\varepsilon^{1/2})
\end{pmatrix}.
\]

If \( \det A_{11}^{(1)} \neq 0 \), the characteristic roots of \( B \) are given by Urabe's lemma [1] as follows:
\[
\mu_{\alpha} + o(1) \quad (\alpha = 1, 2, \ldots, k) \quad \text{and} \quad \varepsilon^{1/2} \left[ \lambda_{\nu} + o(1) \right] \quad (\nu = k+1, \ldots, n),
\]
where \( \mu_{\alpha} \) and \( \lambda_{\nu} \) are respectively the characteristic roots of the matrices
\[
(5.8) \quad A_{11}^{(1)} \quad \text{and} \quad A_{22}^{(2)} - A_{21}^{(2)} A_{11}^{(1)} A_{12}^{(1)} A_{12}^{(2)}.\]
But, as is well known, if $R_{\mu a} < 0$, $R_{\nu r} < 0$ ($\alpha = 1, 2, \ldots, k$; $\nu = k + 1, \ldots, n$), the periodic solution assured in Theorem 1 is stable since $\varepsilon > 0$ ($\S 3$).

Thus we have

**Theorem 3.** The periodic solution whose existence is guaranteed by Theorem 1 is stable if the real parts of the characteristic roots of the matrices (5.8) are all negative.

**Case II.** In this case, by (4.9), the partial derivatives $\frac{\partial q_i}{\partial u_j} (\bar{u}, T, \varepsilon)$ $(i, j = 1, 2, \ldots, n)$ can be written as follows:

\[
\frac{\partial q_i}{\partial u_j} (\bar{u}, T, \varepsilon) = \delta_{ij} + \varepsilon \frac{\partial q_i^{(1)}}{\partial u_j} (c, T) + \varepsilon^2 \left\{ - \sum_{a=1}^{k} \frac{\partial^2 q_i^{(1)}}{\partial u_j \partial u_a} \frac{1}{J_1} \sum_{\beta=1}^{k} D_{\beta a} \left( \sum_{\gamma = k+1}^{n} \frac{\partial q_i^{(1)}}{\partial u_{\gamma}} d_{\gamma} + q_i^{(2)} \right) \\
+ \sum_{\gamma = k+1}^{n} \frac{\partial^2 q_i^{(1)}}{\partial u_j \partial u_{\gamma}} d_{\gamma} + \frac{\partial q_i^{(2)}}{\partial u_j} \right\} + o(\varepsilon^2) \quad (i, j = 1, 2, \ldots, n).
\]

Then, substituting these into the right members of (5.4), we see that the matrix $A$ of the coefficients of their linear parts can be written as follows:

\[
A = E + \varepsilon A_1 + \varepsilon^2 A_2 + o(\varepsilon^2)
\]

\[
E + \varepsilon \begin{pmatrix} A_1^{(1)} & A_1^{(2)} \\ A_2^{(1)} & A_2^{(2)} \end{pmatrix} + \varepsilon^2 \begin{pmatrix} A_1^{(2)} & A_1^{(3)} \\ A_2^{(2)} & A_2^{(3)} \end{pmatrix} + o(\varepsilon^2),
\]

where

\[
[A_1^{(1)}]_{\alpha \beta} = \frac{\partial q_{\alpha}^{(1)}}{\partial u_{\beta}} (c, T) - \sum_{\mu = k+1}^{n} \frac{\partial q_{\alpha}^{(1)}}{\partial u_{\mu}} (c, T),
\]

\[
[A_1^{(2)}]_{\alpha \mu} = \frac{\partial q_{\alpha}^{(1)}}{\partial u_{\mu}} (c, T),
\]

\[
[A_1^{(2)}]_{\gamma \beta} = 0,
\]

\[
[A_1^{(3)}]_{\gamma \mu} = 0;
\]

\[
[A_2^{(1)}]_{\gamma \beta} = - \sum_{a=1}^{k} \frac{\partial^2 q_{\gamma}^{(1)}}{\partial u_{\beta} \partial u_a} \frac{1}{J_1} \sum_{\gamma = k+1}^{n} D_{\gamma a} \left( \sum_{\delta = k+1}^{n} \frac{\partial q_{\gamma}^{(1)}}{\partial u_{\delta}} d_{\delta} + q_{\gamma}^{(2)} \right) \\
+ \sum_{\gamma = k+1}^{n} \frac{\partial^2 q_{\gamma}^{(1)}}{\partial u_j \partial u_{\gamma}} d_{\gamma} + \frac{\partial q_{\gamma}^{(2)}}{\partial u_{\gamma}} \left\{ - \sum_{a=1}^{k} \frac{\partial^2 q_{\gamma}^{(1)}}{\partial u_{\beta} \partial u_a} \frac{1}{J_1} \sum_{\delta = k+1}^{n} D_{\delta a} \left( \sum_{\mu = k+1}^{n} \frac{\partial q_{\gamma}^{(1)}}{\partial u_{\mu}} d_{\mu} + q_{\gamma}^{(2)} \right) \\
+ \sum_{\gamma = 1}^{k} \frac{\partial^2 q_{\gamma}^{(1)}}{\partial u_j \partial u_{\gamma}} d_{\gamma} + \frac{\partial q_{\gamma}^{(2)}}{\partial u_{\gamma}} \right\} + o(\varepsilon^2).
\]
Since $A$ is of the form (5.10), $A$ can be written in the exponential form

$$A = \exp(\varepsilon B),$$

where $B$ is of the form as follows:

$$B = A_1 + \varepsilon \left( A_2 - \frac{1}{2} A_1^2 \right) + o(\varepsilon)$$

$$= \begin{pmatrix} A_{11}^{(1)} + o(1) & A_{12}^{(1)} + o(1) \\ \varepsilon A_{21}^{(2)} + o(\varepsilon) & \varepsilon A_{22}^{(2)} + o(\varepsilon) \end{pmatrix}.$$

If $\det A_{11}^{(1)} \neq 0$, the characteristic roots of the matrix $B$ are given by Urabe’s lemma [1] as follows:

$$\mu_\alpha + o(1) \quad (\alpha = 1, 2, \ldots, k) \quad \text{and} \quad \varepsilon [\lambda_\nu + o(1)] \quad (\nu = k + 1, \ldots, n),$$

where $\mu_\alpha$ and $\lambda_\nu$ are respectively the characteristic roots of the matrices

$$A_{11}^{(1)} \quad \text{and} \quad A_{22}^{(2)} = A_{21}^{(2)} A_{11}^{(1)-1} A_{12}^{(1)}.$$

But, as is well known, if $R_{\mu_\alpha} < 0, R_{\lambda_\nu} < 0 \ (\alpha = 1, 2, \ldots, k; \nu = k + 1, \ldots, n)$, the periodic solution assured in Theorem 2, is stable since $\varepsilon > 0$ (§3).
Thus we have

**Theorem 4.** The periodic solution whose existence is guaranteed by Theorem 2 is stable, if the real parts of the characteristic roots of the matrices (5.12) are all negative.

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**Reference**


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