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Evaluation of Hausdorff Measures of Generalized Cantor Sets

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§1. Introduction

The problem how a Hausdorff measure of a product set $A \times B$ is related to Hausdorff measures of A and B is not completely solved. This problem was first investigated by F. Hausdorff himself [3] and later by A. S. Besicovitch and P. A. P. Moran [1], J. M. Marstrand [4] and others. Their works and investigations of similar problem for capacity (e.g. [6], [7]) show that evaluation of Hausdorff measures of generalized Cantor sets supplies many clues to this problem.

In this paper we first evaluate the α-Hausdorff measure of generalized Cantor sets in the Euclidean space R^n. As a concequence we see the existence of a compact set in R^n which has infinite α-Hausdorff measure but zero α-capacity ($0<\alpha<n$). Next we estimate Hausdorff measures of product sets of one-dimensional generalized Cantor sets and then give examples which show that in case the α-Hausdorff measure of E_1 is infinite and the β-Hausdorff measure of E_2 is zero, the $(\alpha+\beta)$-Hausdorff measure of $E_1 \times E_2$ may either be zero, positive finite or infinite. Also these examples answer M. Ohtsuka's question in [7] (p. 114) in the negative.

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§2. Definitions and Notation

Let $R^n (n \geqq 1)$ be the n-dimensional Euclidean space with points $x=(x_1, x_2, \cdots, x_n)$. By an n-dimensional open cube (closed cube resp.) in R^n, we mean the set of points $x=(x_1, x_2, \cdots, x_n)$ satisfying the inequalities:

$$a_i < x_i < a_i + d \ (a_i \leqq x_i \leqq a_i + d \text{ resp.}) \qquad \text{for} \quad i=1, 2, \cdots, n,$$

where a_i $(i=1, 2, \cdots, n)$ are any numbers and $d>0$. We call d the length of the side, or simply the side, of the open (or closed) cube.

Let $\mathfrak{A}$ be the family of non empty open sets in R^n which is determined by the following properties:

(i) any n-dimensional open cube belongs to $\mathfrak{A}$,

(ii) if ω_1 and ω_2 belong to $\mathfrak{A}$, then so does $\omega_1 \cup \omega_2$,

(iii) if ω is an element of $\mathfrak{A}$, then there exists a finite number of n-

dimensional open cubes I_ν ($\nu=1, 2, \cdots, N$) such that $\omega=\bigcup_{\nu=1}^{N} I_\nu$.

Let $h(r)$ be a continuous increasing function defined for $r\geqq 0$ such that $h(0)=0$. Let E be an arbitrary set in R^n and ρ be any positive number. We put $\Lambda_h^{(\rho)}(E)=\inf\{\sum_\nu h(d_\nu)\}$, where the infimum is taken over all coverings of E by at most a countable number of n-dimensional open cubes I_ν with the side $d_\nu\leqq\rho$. Since $\Lambda_h^{(\rho)}(E)$ increases as ρ decreases, the limit

$$\Lambda_h(E)=\lim_{\rho\to 0}\Lambda_h^{(\rho)}(E) \qquad (\leqq\infty)$$

exists. As is easily seen, $\Lambda_h(E)$ is a Carathéodory's outer measure. Hence any Borel set is measurable with respect to Λ_h. For a measurable set E we call $\Lambda_h(E)$ the h-Hausdorff measure of E.

If $h(r)=r^\alpha$ ($\alpha>0$), then we use the notation Λ_α instead of Λ_h and call it the α-Hausdorff measure.

Let μ be a positive (Radon) measure in R^n with support S_μ and α be a positive number such that $0<\alpha<n$. The α-capacity $C_\alpha(F)$ of a compact set F is defined by

$$C_\alpha(F)=\{\inf\iint\frac{1}{|x-y|^\alpha}d\mu(x)d\mu(y)\}^{-1},$$

where the infimum is taken over the class of all positive measures μ with unit mass and $S_\mu\subset F$.

We shall define an n-dimensional generalized Cantor set. Let l be a positive number, q_0 be a positive integer, $\{k_q\}_{q=1}^\infty$ be a sequence of integers and $\{\lambda_q\}_{q=q_0}^\infty$ be a sequence of positive numbers. Suppose a system $[l, \{k_q\}_{q=1}^\infty, \{\lambda_q\}_{q=q_0}^\infty]$ satisfies the following condition (*):

$$(*):\qquad k_q>1\ (q\geqq 1),\ k_{q+1}\lambda_{q+1}<\lambda_q\ (q\geqq q_0)\ \text{and}\ k_1k_2\cdots k_{q_0}\lambda_{q_0}<l.$$

Let I be a one-dimensional closed interval with the length l.

In the first step, we remove from I $(k_1k_2\cdots k_{q_0}-1)$ open intevals each of the same length so that $k_1k_2\cdots k_{q_0}$ closed intervals $I_i^{(q_0)}$ ($i=1, 2, \cdots, k_1k_2\cdots k_{q_0}$) each of length λ_{q_0} remain. Set $E^{(q_0)}=\bigcup_{i=1}^{k_1k_2\cdots k_{q_0}} I_i^{(q_0)}$. Next in the second step, we remove from each $I_i^{(q_0)}(k_{q_0+1}-1)$ open intervals each of the same length so that k_{q_0+1} closed intervals $I_{i,j}^{(q_0+1)}$ ($j=1, 2, \cdots, k_{q_0+1}$) each of length λ_{q_0+1} remain. We set $E^{(q_0+1)}=\bigcup_{i=1}^{k_1\cdots k_{q_0}}\bigcup_{j=1}^{k_{q_0+1}} I_{i,j}^{(q_0+1)}$.

We continue this process and obtain the sets $E^{(q)}$, $q=q_0, q_0+1, \cdots$. We define $E_{(1)}=\bigcap_{q=q_0}^\infty E^{(q)}$. Note that $E_{(1)}$ is a compact set in R^1. It is called the one-dimensional generalized Cantor set constructed by the system $[l, \{k_q\}_{q=1}^\infty, \{\lambda_q\}_{q=q_0}^\infty]$. We call the product set $E_{(n)}=E_{(1)}\times E_{(1)}\times\cdots\times E_{(1)}$ of n ($n\geqq 2$) one-dimensional generalized Cantor set $E_{(1)}$ the n-dimensional symmetric generalized Cantor set constructed by the system $[l, \{k_q\}_{q=1}^{\infty}, \{\lambda_q\}_{q=q_0}^{\infty}]$. Evidently $E_{(n)}$ is a compact set in R^n. We can see that $E_{(n)}=\bigcap_{q=q_0}^{\infty} E^{(q)}\times E^{(q)}\times\cdots\times E^{(q)}$, where $E^{(q)}\times E^{(q)}\times\cdots\times E^{(q)}$ is a product set in R^n and consists of $(k_1k_2\cdots k_q)^n$ n-dimensional closed cubes with the side λ_q. We call $E^{(q)}\times\cdots\times E^{(q)}$ the qth approximation of $E_{(n)}$ $(n\geqq 1)$.

§3. Main theorem

LEMMA 1. (P. A. P. Moran [5]) *Let F be a compact set in R^n and let $\mathfrak{A}$ be the family defined in* §2. *Assume that there exists a set function Φ on $\mathfrak{A}$ satisfying the following conditions*:

(1) *$\Phi(\omega)\geqq 0$ for every set $\omega\in\mathfrak{A}$,*

(2) *if $\omega=\bigcup_{i=1}^{N}\omega_i$, $\omega_i\in\mathfrak{A}$ $(i=1, 2, \dots, N)$, then $\Phi(\omega)\leqq\sum_{i=1}^{N}\Phi(\omega_i)$,*

(3) *if $\omega\in\mathfrak{A}$ contains F, then $\Phi(\omega)\geqq b$, where b is some positive constant,*

(4) *there exist positive constants a and d_0 such that if I is any n-dimensional open cube with the side $d\leqq d_0$, then $\Phi(I)\leqq ah(d)$.*

Then $\Lambda_h(F)\geqq b/a$.

LEMMA 2. (M. Ohtsuka [6]) *Let α be a positive number such that $0<\alpha<n$ and let $E_{(n)}$ be the one-dimensional generalized Cantor set $(n=1)$ or the n-dimensional symmetric generalized Cantor set $(n\geqq 2)$ constructed by the system $[l, \{k_q\}_{q=1}^{\infty}, \{\lambda_q\}_{q=q_0}^{\infty}]$ which satisfies condition* $(*)$.

Then $C_\alpha(E_{(n)})=0$ if and only if $\sum_{q=q_0}^{\infty}(k_1k_2\cdots k_q)^{-n}\lambda_q^{-\alpha}=\infty$.

Using Lemma 1 we shall prove the following theorem.

THEOREM. *Let $E_{(n)}$ be the one-dimensional generalized Cantor set $(n=1)$ or the n-dimensional symmetric generalized Cantor set $(n\geqq 2)$ constructed by the system $[l, \{k_q\}_{q=1}^{\infty}, \{\lambda_q\}_{q=q_0}^{\infty}]$ which satisfies condition* $(*)$. *We assume $k_q\leqq M_1$ $(q=1, 2, \cdots)$ (M_1: a constant). Then*

(a) *$\Lambda_h(E_{(n)})=0$ if and only if $\varliminf_{q\to\infty}(k_1k_2\cdots k_q)^n h(\lambda_q)=0$,*

(b) *$0<\Lambda_h(E_{(n)})<\infty$ if and only if $0<\varliminf_{q\to\infty}(k_1k_2\cdots k_q)^n h(\lambda_q)<\infty$,*

(c) *$\Lambda_h(E_{(n)})=\infty$ if and only if $\lim_{q\to\infty}(k_1k_2\cdots k_q)^n h(\lambda_q)=\infty$.*

PROOF. If all the "if"-parts are proved, then all the "only if"-parts are immediately derived. Hence we shall prove the "if"-parts.

From the definition of the Hausdorff measure we can see that $\varliminf_{q\to\infty}(k_1k_2\cdots k_q)^n h(\lambda_q)=0$ ($<\infty$ resp.) implies $\Lambda_h(E_{(n)})=0$ ($<\infty$ resp.). Therefore we shall prove that $\varliminf_{q\to\infty}(k_1k_2\cdots k_q)^n h(\lambda_q)>0$ ($=\infty$ resp.) implies $\Lambda_h(E_{(n)})>0$ ($=\infty$ resp.).

We put $\varliminf_{q\to\infty}(k_1k_2\cdots k_q)^n h(\lambda_q)=A>0$. Let B be an arbitrary positive number such that $0<B<A$. Then there exists q_1 $(\geqq q_0)$ such that $(k_1k_2\cdots k_q)^n h(\lambda_q)>B$ for $q\geqq q_1$. We choose a sequence $\{\lambda'_q\}_{q=q_1}^{\infty}$ such that $(k_1k_2\cdots k_q)^n h(\lambda'_q)=B$. Evidently $0<\lambda'_q<\lambda_q$ and $k_{q+1}^n h(\lambda'_{q+1})=h(\lambda'_q)$ for $q\geqq q_1$.

We show that $\lim_{q\to\infty} N_q(\omega)h(\lambda'_q)$ exists for every $\omega\in\mathfrak{A}$, where $N_q(\omega)$ is the number of n-dimensional closed cubes in the qth approximation of $E_{(n)}$ which meet ω. By the construction of $E_{(n)}$, we see that

$$N_{q+1}(\omega)h(\lambda'_{q+1})\leqq N_q(\omega)k_{q+1}^n h(\lambda'_{q+1})=N_q(\omega)h(\lambda'_q) \qquad \text{for} \quad q\geqq q_1.$$

Thus $N_q(\omega)h(\lambda'_q)$ decreases as q increases. Now we define a set function Φ on $\mathfrak{A}$ by $\Phi(\omega)=\lim_{q\to\infty} N_q(\omega)h(\lambda'_q)$. Take $E_{(n)}$ as F in Lemma 1. We shall show that Φ satisfies conditions (1)-(4) in Lemma 1.

It is easy to see that Φ satisfies (1), (2) and (3) with $b=B$. We set $a=(2M_1)^n$ and $d_0=\lambda_{q_1}$. Let I be any open cube with the side $d\leqq d_0$. Then there is a uniquely determined positive integer $q(\geqq q_1)$ such that $\lambda_{q+1}<d\leqq\lambda_q$. Since $E_{(n)}$ is symmetric, we have $N_q(I)\leqq 2^n$, so that $N_{q+1}(I)\leqq k_{q+1}^n N_q(I)\leqq(2k_{q+1})^n\leqq(2M_1)^n=a$. Hence $\Phi(I)\leqq N_{q+1}(I)h(\lambda'_{q+1})\leqq ah(\lambda_{q+1})\leqq ah(d)$. Therefore Φ satisfies condition (4) in Lemma 1.

By Lemma 1, we obtain $\Lambda_h(E_{(n)})\geqq B/a$, where a is independent of the choice of B. Since B is an arbitrary number such that $0<B<A$, we have $\Lambda_h(E_{(n)})\geqq A/a=\dfrac{1}{a}\varliminf_{q\to\infty}(k_1k_2\cdots k_q)^n h(\lambda_q)$. By this inequality, we see that $\varliminf_{q\to\infty}(k_1k_2\cdots k_q)^n h(\lambda_q)>0$ $(=\infty$ resp.) implies $\Lambda_h(E_{(n)})>0$ $(=\infty$ resp.).

REMARK 1. We can easily see that $\Lambda_\alpha(E_{(n)})=0$ $(0<\Lambda_\alpha(E_{(n)})<\infty$, $\Lambda_\alpha(E_{(n)})=\infty$ resp.) is equivalent to $\Lambda_{\alpha/n}(E_{(1)})=0$ $(0<\Lambda_{\alpha/n}(E_{(1)})<\infty$, $\Lambda_{\alpha/n}(E_{(1)})=\infty$ resp.). In the case of capacity, however, the analogous relations are not always true. For instance, when $n\geqq 2$ and $0<\alpha<n$, we put $l=1$, $k_q=2$ $(q=1, 2, \cdots)$ and $\lambda_q=(q^2 2^{-nq})^{1/\alpha}$ for $q\geqq q_0$, where q_0 is a positive integer such that $2\lambda_{q+1}<\lambda_q$ for $q\geqq q_0$ and $2^{q_0}\lambda_{q_0}<1$. Let $E_{(1)}$ be the one-dimensional generalized Cantor set constructed by the system $[l, \{k_q\}_{q=1}^{\infty}, \{\lambda_q\}_{q=q_0}^{\infty}]$ and let $E_{(n)}=E_{(1)}\times\cdots\times E_{(1)}$, i.e., an n-dimensional symmetric generalized Cantor set. Then by Lemma 2, we can see that $C_\alpha(E_{(n)})>0$ but $C_{\alpha/n}(E_{(1)})=0$.

REMARK 2. Let α be a positive number and q_0 be a positive integer >1. We assume that a system $[l, \{k_q\}_{q=1}^{\infty}, \{\lambda_q\}_{q=1}^{\infty}]$ satisfies condition (*) and $k_q\leqq M_1<\infty$ (M_1: a constant). Let $E_{(n)}$ ($E'_{(n)}$ resp.) be the one-dimensional generalized Cantor set $(n=1)$ or the n-dimensional symmetric generalized Cantor set $(n\geqq 2)$ constructed by the system $[l, \{k_q\}_{q=1}^{\infty}, \{\lambda_q\}_{q=1}^{\infty}]$ ($[l, \{k_q\}_{q=1}^{\infty}, \{\lambda_q\}_{q=q_0}^{\infty}]$ resp.).

Then in general $E_{(n)}\neq E'_{(n)}$, but $C_\alpha(E_{(n)})$ and $C_\alpha(E'_{(n)})$ are zero simultaneously. Furthermore $\Lambda_\alpha(E_{(n)})$ and $\Lambda_\alpha(E'_{(n)})$ are zero (positive finite, infinite resp.) simultaneously.

REMARK 3. It is a well known result that if F is a compact set of positive α-capacity, then $\Lambda_\alpha(F)=\infty$, provided that $0<\alpha<n$ (cf. L. Carleson [2]). We show that the converse is not always true.

Let α be a positive number such that $0<\alpha<n$. We choose $l=1$, $k_q=2$ $(q=1, 2, \cdots)$ and $\lambda_q=(q2^{-nq})^{1/\alpha}$ for $q\geqq q_0$, where q_0 is any positive integer such that $2\lambda_{q+1}<\lambda_q$ for $q\geqq q_0$ and $2^{q_0}\lambda_{q_0}<1$. Let F be the one-dimensional generalized Cantor set $(n=1)$ or the n-dimensional symmetric generalized Cantor set $(n\geqq 2)$ constructed by the system $[l, \{k_q\}_{q=1}^\infty, \{\lambda_q\}_{q=q_0}^\infty]$. By Lemma 2 and the theorem, we can easily see that $C_\alpha(F)=0$ but $\Lambda_\alpha(F)=\infty$.

§4. Lemmas

We shall introduce an auxiliary α-Hausdorff measure Λ_α^*. Let ρ be any positive number. We put $\Lambda_\alpha^{(\rho)*}(E)=\inf\{\sum_\nu r_\nu^\alpha\}$ for an arbitrary set E in R^n, where the infimum is taken over all coverings of E by at most a countable number of closed convex sets with diameters $r_\nu\leqq\rho$. Since $\Lambda_\alpha^{(\rho)*}(E)$ increases as ρ decreases, the limit

$$\Lambda_\alpha^*(E)=\lim_{\rho\to 0}\Lambda_\alpha^{(\rho)*}(E) \qquad (\leqq\infty)$$

exists.

There exists a positive constant M_2, depending only on the dimension n, such that $(1/M_2)\Lambda_\alpha(E)\leqq\Lambda_\alpha^*(E)\leqq M_2\Lambda_\alpha(E)$ for every set E in R^n.

We shall deal with sets in R^2 in what follows.

LEMMA 3. *Let α, β, γ and δ be positive numbers such that $\alpha<1$ and $\beta<1$. Put $l=1$, $k_q=2$ $(q=1, 2, \cdots)$, $\lambda_q=q^\gamma 2^{-q/\alpha}$ for $q\geqq q_0$ and $\mu_q=q^{-\delta}2^{-q/\beta}$ for $q\geqq q_0$, where q_0 is any positive integer such that $2\lambda_{q+1}<\lambda_q$ for $q\geqq q_0$ and $2^{q_0}\lambda_{q_0}<1$. Let E_1 (E_2* resp.) *be the one-dimensional generalized Cantor set constructed by the system $[l, \{k_q\}_{q=1}^\infty, \{\lambda_q\}_{q=q_0}^\infty]$ ($[l, \{k_q\}_{q=1}^\infty, \{\mu_q\}_{q=q_0}^\infty]$* resp.). *Then*

$$\Lambda_{\alpha+\beta}^*(E_1\times E_2)\leqq M_3\varliminf_{q\to\infty}(2^q\lambda_q^\alpha)(2^q\mu_q^\beta), \quad \textit{where} \quad M_3=\sqrt{10}\max\left(1, \left(\frac{2\alpha}{\beta}\right)^{\beta\delta}\right).$$

PROOF. The case $\alpha<\beta$. There exists a positive integer q_1 $(\geqq q_0)$ such that $\lambda_q<\mu_q$ for $q\geqq q_1$. Let ρ be any positive number which satisfies $\rho<\lambda_{q_1}$. We can choose a positive integer q_2 $(\geqq q_1)$ such that $\mu_q<\rho$ for $q\geqq q_2$. For each $q\geqq q_2$, there is a uniquely determined positive integer $p=p(q)$ such that $\lambda_{p+1}<\mu_q\leqq\lambda_p$. We can see that $p<q$.

Now we assume $q\geqq q_2$. Then $E_1^{(p+1)}\times E_2^{(q)}$ $(\supset E_1\times E_2)$ consists of 2^{p+q+1} mutually congruent closed rectangles, where $E_1^{(q)}$ ($E_2^{(q)}$ resp.) is the qth approximation of E_1 (E_2 resp.). Let $r_{p+1,q}$ be the diameter of each rectangle. Then

$$r_{p+1,q}=\sqrt{\lambda_{p+1}^2+\mu_q^2}<\sqrt{2}\,\mu_q<2\rho,$$

$$r_{p+1,q}\leqq\sqrt{\lambda_{p+1}^2+\lambda_p^2}<\sqrt{\frac{5}{2}}\,\lambda_p.$$

By the definition of Λ_α^*,

$$\Lambda_{\alpha+\beta}^{(2\rho)*}(E_1\times E_2)\leqq\Lambda_{\alpha+\beta}^{(2\rho)*}(E_1^{(p+1)}\times E_2^{(q)})\leqq 2^{p+q+1}r_{p+1,q}^{\alpha+\beta}$$
$$=(2^{p+1}r_{p+1,q}^{\alpha})(2^q r_{p+1,q}^{\beta})<\sqrt{10}(2^p\lambda_p^\alpha)(2^q\mu_q^\beta).$$

Since $2^q\lambda_q^\alpha$ increases with q and $p<q$, we have $2^p\lambda_p^\alpha<2^q\lambda_q^\alpha$. Hence

$$\Lambda_{\alpha+\beta}^{(2\rho)*}(E_1\times E_2)\leqq\sqrt{10}\,\underline{\lim}_{q\to\infty}(2^q\lambda_q^\alpha)(2^q\mu_q^\beta).$$

Therefore we have

$$\Lambda_{\alpha+\beta}^{*}(E_1\times E_2)=\lim_{\rho\to 0}\Lambda_{\alpha+\beta}^{(2\rho)*}(E_1\times E_2)\leqq\sqrt{10}\,\underline{\lim}_{q\to\infty}(2^q\lambda_q^\alpha)(2^q\mu_q^\beta).$$

The case $\beta\leqq\alpha$. Interchanging λ_q and μ_q in the above proof for the case $\alpha<\beta$, we observe that there is q_2 such that

$$\Lambda_{\alpha+\beta}^{(2\rho)*}(E_1\times E_2)\leqq\sqrt{10}(2^q\lambda_q^\alpha)(2^p\mu_p^\beta)\qquad\text{for}\quad q\geqq q_2,$$

where $p=p(q)$ is determined so that $\mu_{p+1}<\lambda_q\leqq\mu_p$ and $p<q$. Obviously $2^p\mu_p^\beta=2^q\mu_q^\beta(q/p)^{\beta\delta}$. We shall prove $\overline{\lim}_{q\to\infty}q/p<2\alpha/\beta$. Suppose this is not true. Then there exist sequences $\{q(m)\}_{m=1}^\infty$ and $\{p(m)\}_{m=1}^\infty$ such that $q(m)/p(m)>3\alpha/2\beta$ $(m=1, 2, \dots)$. By $\mu_{p+1}<\lambda_q$, we see

$$2^{\frac{q(m)}{\alpha}}<2^{\frac{p(m)+1}{\beta}}(p(m)+1)^\delta q(m)^\gamma<2^{\frac{1}{\beta}\left(\frac{2\beta}{3\alpha}q(m)+1\right)}\left(\frac{2\beta}{3\alpha}q(m)+1\right)^\delta q(m)^\gamma.$$

Hence

$$2^{\frac{\beta q(m)}{3\alpha}}<2\left(\frac{2\beta}{3\alpha}q(m)+1\right)^{\beta\delta}q(m)^{\beta\gamma}\qquad\text{for}\quad m\geqq 1.$$

For sufficiently large m, it is contradictory. Thus we have $\overline{\lim}_{q\to\infty}q/p<2\alpha/\beta$. Hence

$$\Lambda_{\alpha+\beta}^{*}(E_1\times E_2)=\lim_{\rho\to 0}\Lambda_{\alpha+\beta}^{(2\rho)*}(E_1\times E_2)\leqq\sqrt{10}\left(\frac{2\alpha}{\beta}\right)^{\beta\delta}\underline{\lim}_{q\to\infty}(2^q\lambda_q^\alpha)(2^q\mu_q^\beta).$$

Therefore we have the required inequality in any case.

REMARK. This lemma is essentially due to F. Hausdorff [3].

We shall prove the following lemma by a method similar to the proof of the theorem.

LEMMA 4. *Under the same assumptions as in Lemma* 3,

$$\Lambda_{\alpha+\beta}(E_1\times E_2)\geqq(1/M_4)\varliminf_{q\to\infty}(2^q\lambda_q^\alpha)(2^q\mu_q^\beta),\quad \textit{where}\quad M_4=2^4\max\Big(1,\Big(\frac{2\beta}{\alpha}\Big)^{\alpha\gamma}\Big).$$

PROOF. If $\varliminf_{q\to\infty}(2^q\lambda_q^\alpha)(2^q\mu_q^\beta)=0$, then the conclusion is obvious. Hence assume $A=\varliminf_{q\to\infty}(2^q\lambda_q^\alpha)(2^q\mu_q^\beta)>0$.

Let B be an arbitrary positive number which satisfies $0<B<A$. Then we can choose a positive integer q_1 $(\geqq q_0)$ such that $(2^q\lambda_q^\alpha)(2^q\mu_q^\beta)>B$ for $q\geqq q_1$. Let $\{\mu_q'\}_{q=q_1}^\infty$ be a sequence defined by $(2^q\lambda_q^\alpha)(2^q\mu_q'^\beta)=B$. Then $0<\mu_q'<\mu_q$ and $2^2\lambda_{q+1}^\alpha\mu_{q+1}'^\beta=\lambda_q^\alpha\mu_q'^\beta$ for $q\geqq q_1$.

We show that $\lim_{q\to\infty}N_q(\omega)\lambda_q^\alpha\mu_q'^\beta$ exists for every $\omega\in\mathfrak{A}$, where $N_q(\omega)$ is the number of closed rectangles of the form $I_1^{(q)}\times I_2^{(q)}$ which meet ω. Here we denote by $I_1^{(q)}$ ($I_2^{(q)}$ resp.) any one of the closed intervals in the qth approximation of E_1 (E_2 resp.). By the construction of $E_1\times E_2$, we see that $N_{q+1}(\omega)\leqq 2^2N_q(\omega)$ for $q\geqq q_0$. It follows that

$$N_{q+1}(\omega)\lambda_{q+1}^\alpha\mu_{q+1}'^\beta\leqq N_q(\omega)2^2\lambda_{q+1}^\alpha\mu_{q+1}'^\beta=N_q(\omega)\lambda_q^\alpha\mu_q'^\beta\qquad\text{for}\quad q\geqq q_1.$$

Thus $N_q(\omega)\lambda_q^\alpha\mu_q'^\beta$ decreases as q increases. Now we define a set function Φ on $\mathfrak{A}$ by

$$\Phi(\omega)=\lim_{q\to\infty}N_q(\omega)\lambda_q^\alpha\mu_q'^\beta.$$

Take $E_1\times E_2$ as F in Lemma 1. We shall show that our Φ satisfies the conditions in Lemma 1. It is easy to see that Φ satisfies conditions (1), (2) and (3) with $b=B$. Hence it is enough to show that Φ satisfies (4).

The case $\beta\leqq\alpha$. There exists a positive integer q_2 $(\geqq q_1)$ such that $\mu_q<\lambda_{q+1}$ for $q\geqq q_2$. Put $d_0=\mu_{q_2}$. Let I be any 2-dimensional open cube with the side $d\leqq d_0$. Then there exist uniquely determined positive integers p and q such that $\lambda_{p+1}<d\leqq\lambda_p$ and $\mu_{q+1}<d\leqq\mu_q$. Since $\lambda_{p+1}<\mu_q<\lambda_{q+1}$ for $q\geqq q_2$, we have $q<p$. The open cube I meets at most 2^2 rectangles of the form $I_1^{(p)}\times I_2^{(q)}$ and so meets at most 2^4 rectangles of the form $I_1^{(p+1)}\times I_2^{(q+1)}$. It follows from $p>q$ that $N_{p+1}(I)\leqq 2^42^{p-q}$. Moreover $2^{p-q}\mu_{p+1}^\beta<\mu_{q+1}^\beta$, since $2^q\mu_q^\beta$ decreases as q increases. Hence we have

$$\Phi(I)\leqq N_{p+1}(I)\lambda_{p+1}^\alpha\mu_{p+1}'^\beta<2^{4+p-q}\lambda_{p+1}^\alpha\mu_{p+1}^\beta\leqq 2^4\lambda_{p+1}^\alpha\mu_{q+1}^\beta<2^4d^{\alpha+\beta}.$$

Therefore $\Phi(I)<2^4d^{\alpha+\beta}$.

The case $\alpha<\beta$. There exists a positive integer q_2 $(\geqq q_1)$ such that $\lambda_q<\mu_{q+1}$ for $q\geqq q_2$. For any positive number d which satisfies $0<d<\lambda_{q_2}$, there exist uniquely determined positive integers $p=p(d)$ and $q=q(d)$ such that $\lambda_{p+1}<d\leqq\lambda_p$ and $\mu_{q+1}<d\leqq\mu_q$. Since $\lambda_q<\mu_{q+1}<\lambda_p$, it follows that $p<q$.

We can prove $\overline{\lim_{d\to 0}}\, q/p<2\beta/\alpha$ as we did in the proof of Lemma 3. Accordingly we can choose a positive integer $q_3(\geqq q_2)$ such that $q/p<2\beta/\alpha$ for $q\geqq q_3$.

Put $d_0=\lambda_{q_3}$. Let I be any 2-dimensional open cube with the side d $(\leqq d_0)$. We can choose p and q as above for this d. The open cube I meets at most 2^2 rectangles of the form $I_1^{(p)}\times I_2^{(q)}$ and so meets at most 2^4 rectangles of the form $I_1^{(p+1)}\times I_2^{(q+1)}$. Hence $N_{q+1}(I)\leqq 2^{q-p}2^4$ and

$$\frac{2^{q+1}\lambda_{q+1}^{\alpha}}{2^{p+1}\lambda_{p+1}^{\alpha}}=\left(\frac{q+1}{p+1}\right)^{\alpha\gamma}<\left(\frac{2\beta}{\alpha}\right)^{\alpha\gamma}.$$

Then we have

$$\begin{aligned}\Phi(I)&\leqq N_{q+1}(I)\lambda_{q+1}^{\alpha}\mu_{q+1}'^{\beta}<2^{4+q-p}\lambda_{q+1}^{\alpha}\mu_{q+1}^{\beta}\\ &=2^4\lambda_{p+1}^{\alpha}\mu_{q+1}^{\beta}\frac{2^{q+1}\lambda_{q+1}^{\alpha}}{2^{p+1}\lambda_{p+1}^{\alpha}}<2^4\left(\frac{2\beta}{\alpha}\right)^{\alpha\gamma}\lambda_{p+1}^{\alpha}\mu_{q+1}^{\beta}<2^4\left(\frac{2\beta}{\alpha}\right)^{\alpha\gamma}d^{\alpha+\beta}.\end{aligned}$$

Therefore

$$\Phi(I)<2^4\left(\frac{2\beta}{\alpha}\right)^{\alpha\gamma}d^{\alpha+\beta}.$$

Thus Φ satisfies conditions (1), (2), (3) and (4) in Lemma 1. It follows from Lemma 1 that $\Lambda_{\alpha+\beta}(E_1\times E_2)\geqq B/M_4$, where $M_4=2^4\max\left(1,\left(\frac{2\beta}{\alpha}\right)^{\alpha\gamma}\right)$. Since B is an arbitrary number such that $0<B<A$, we have $\Lambda_{\alpha+\beta}(E_1\times E_2)\geqq (1/M_4)\underline{\lim}_{q\to\infty}(2^q\lambda_q^{\alpha})(2^q\mu_q^{\beta})$.

By Lemmas 3 and 4, we obtain

COROLLARY. *Under the same assumptions as in Lemma* 3, $\Lambda_{\alpha+\beta}(E_1\times E_2)$ *is zero, positive finite or infinite if and only if* $\underline{\lim}_{q\to\infty}(2^q\lambda_q^{\alpha})(2^q\mu_q^{\beta})$ *is zero, positive finite or infinite, respectively.*

§5. Examples

In this section we denote by $z=(x,y)$ a point of R^2. Let α and β be positive numbers such that $\alpha\leqq 1$ and $\beta\leqq 1$. Let Z be a set in R^2 and X be a set in the x-axis. Denote by Z_x the intersection of Z with the line parallel to the y-axis passing through $z=(x,0)$. J. M. Marstrand [4] proved that if M is a positive number such that $\Lambda_\beta(Z_x)\geqq M$ for all $x\in X$, then there exists a positive constant c such that

$$\Lambda_{\alpha+\beta}(Z)\geqq cM\Lambda_\alpha(X)\qquad \text{for all}\quad \alpha>0.$$

From this relation we derive immediately

$$\Lambda_{\alpha+\beta}(X\times Y)\geqq c\Lambda_\alpha(X)\Lambda_\beta(Y).$$

If $\alpha<1$ and $\beta<1$, then we shall show by examples that there exist compact sets E_1 and E_2 satisfying the following conditions:

1) $\Lambda_\alpha(E_1)=\infty$ and $\Lambda_{\alpha'}(E_1)=0$ for all $\alpha'>\alpha$,

2) $\Lambda_\beta(E_2)=0$ and $\Lambda_{\beta'}(E_2)=\infty$ for all $\beta'<\beta$,

3) $\Lambda_{\alpha+\beta}(E_1\times E_2)=0$ or 3′) $0<\Lambda_{\alpha+\beta}(E_1\times E_2)<\infty$ or 3″) $\Lambda_{\alpha+\beta}(E_1\times E_2)=\infty$.

Before constructing examples we observe that if

1′) $C_\alpha(E_1)>0$ and $C_{\alpha'}(E_1)=0$ for all $\alpha'>\alpha$

is true, then 1) is true. In fact, $C_\alpha(E_1)>0$ implies $\Lambda_\alpha(E_1)=\infty$ and $\Lambda_{\alpha'}(E_1)=0$ is true for all $\alpha'>\alpha$ if $C_{\alpha'}(E_1)=0$ for all $\alpha'>\alpha$ (cf. [2]).

We shall construct examples which satisfy 1′), 2) and 3) or 3′) or 3″).

Examples. Let $0<\alpha$, $\beta<1$. Put $l=1$, $k_q=2$, $\lambda_q=(q^2 2^{-q})^{1/\alpha}$ and $\mu_q^{(j)}=(q^{-j}2^{-q})^{1/\beta}$ ($j=1, 2, 3$) for $q=1, 2, \cdots$. Note that $2\mu_{q+1}^{(j)}<\mu_q^{(j)}$ is always true. Choose a positive integer q_0 such that $2\lambda_{q+1}<\lambda_q$ for $q\geqq q_0$ and $2^{q_0}\lambda_{q_0}<1$. Let E_1 ($E_2^{(j)}$ resp.) be the one-dimensional generalized Cantor set constructed by the system $[l, \{k_q\}_{q=1}^\infty, \{\lambda_q\}_{q=q_0}^\infty]$ ($[l, \{k_q\}_{q=1}^\infty, \{\mu_q^{(j)}\}_{q=q_0}^\infty]$ resp.).

First we show that 1′) and 2) are satisfied. By Lemma 2, we see that $C_\alpha(E_1)>0$ and $C_{\alpha'}(E_1)=0$ for all $\alpha'>\alpha$. Using the theorem for each j we infer that $\Lambda_\beta(E_2^{(j)})=0$ and $\Lambda_{\beta'}(E_2^{(j)})=\infty$ for all $\beta'<\beta$. Finally it follows from the corollary of Lemma 4 that $\Lambda_{\alpha+\beta}(E_1\times E_2^{(j)})$ is infinite, positive finite, zero according as $j=1, 2, 3$ respectively.

Remark. Let α, β be positive numbers such that $0<\alpha<n$, $0<\beta<n$. M. Ohtsuka raised the following question in [7]: Let E_1 and E_2 be compact sets in R^n. Suppose that $C_\alpha(E_1)>0$ and $C_{\beta'}(E_2)>0$ for all $\beta'<\beta$. Then is $C_{\alpha+\beta}(E_1\times E_2)$ always positive? Now it is easy to see that our E_1 and $E_2^{(2)}$ (or $E_2^{(3)}$) answer this question in the negative in the 2-dimensional case.

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