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Relations between Capacities and Maximum Principle

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§ 1. Introduction and problem setting

There are various definitions of capacities in potential theory. These capacities are usually used to determine potential theoretic exceptional sets. The aim of this paper is to study some properties of these capacities as set functions.

More precisely, let Ω be a locally compact Hausdorff space and Φ be a kernel, i. e., a lower semicontinuous function on $\Omega\times\Omega$ which takes values in $(0, +\infty]$. The adjoint kernel $\check{\Phi}$ is defined by $\check{\Phi}(x, y)=\Phi(y, x)$. Φ is called symmetric if $\check{\Phi}=\Phi$. A measure μ will always be a non-negative Radon measure with compact support $S\mu$. The Φ-potential of μ is defined by

$$\Phi(x, \mu)=\int\Phi(x, y)d\mu(y)$$

and the mutual energy of μ and ν is defined by

$$(\nu, \mu)=\int\Phi(x, \mu)d\nu(x).$$

We call (μ, μ) the energy of μ and denote by $\mathscr{E}$ the class of all measures of finite energy.

For a compact set K, we set

$$\begin{aligned}
\mathscr{E}_K &=\{\mu;\ S\mu\subset K \quad \text{and} \quad \mu\in\mathscr{E}\},\\
\mathscr{M}_K &=\{\mu;\ S\mu\subset K \quad \text{and} \quad \Phi(x, \mu)\leqq 1 \quad \text{in} \quad \Omega\},\\
\mathscr{L}_K &=\{\mu;\ S\mu\subset K \quad \text{and} \quad \Phi(x, \mu)\leqq 1 \quad \text{on} \quad K\},\\
\mathscr{N}_K &=\{\mu;\ S\mu\subset K \quad \text{and} \quad \Phi(x, \mu)\leqq 1 \quad \text{on} \quad S\mu\}.
\end{aligned}$$

We define

$$M(K)=\sup\{\mu(K);\ \mu\in M_K\} \qquad \text{if} \quad \mathscr{M}_K\neq\phi,$$

and

$$M(K)=0 \qquad \text{if} \quad \mathscr{M}_K=\phi,$$

where ϕ denotes the empty set. We define $L(K)$ for $\mathscr{L}_K$ and $N(K)$ for $\mathscr{N}_K$ in the same way. These set functions were utilized to determine exceptional sets for instance in [1], [3] and [6] and they may be regarded as capacities.

It is clear that

$$0 \leqq M(K) \leqq L(K) \leqq N(K) < \infty$$

for every compact set K.

We shall be concerned with the following problems:

Problem I. Does any one of the equalities $L=N$ and $L=M$ as set functions defined on the class of compact sets imply the other equality?

Problem II. Does any one of the equalities among M, L and N hold as set functions defined on the class of all compact sets if and only if Φ satisfies the maximum principle defined in §3?

In case Φ is a finite-valued continuous kernel, we shall study in §6

Problem III. Does any one of the equalities among M, L and N as set functions defined on the class of all finite sets imply that Φ satisfies the maximum principle?

§2. Study of Problem I

We say that a set function c defined on the class of all compact sets has a monotone property if $c(K_1) \leqq c(K_2)$ provided that $K_1 \subset K_2$. Note that M and N have a monotone property but L does not have a monotone property in general.

We shall prove

THEOREM 1. *If $L(K)=M(K)$ holds for every compact set K, then $L(K)=N(K)$ holds for every compact set K.*

PROOF. Let K be any compact set. There exists a measure $\mu \in \mathcal{N}_K$ such that $\mu(K)=N(K)$[1]. Since $S_\mu \subset K$, we have $L(S_\mu) \leqq N(S_\mu) \leqq N(K)$. From the fact that $\mu \in \mathcal{L}_{S_\mu}$, it follows that $L(S_\mu) \geqq \mu(S_\mu)=N(K)$, and hence $L(S_\mu)=N(K)$. By our assumption that $L=M$, we have

$$N(K)=L(S_\mu)=M(S_\mu) \leqq M(K)=L(K) \leqq N(K),$$

and hence $N(K)=L(K)=M(K)$. This completes the proof.

In the above proof, we have shown

COROLLARY 1. *If L has a monotone property, then $L(K)=N(K)$ holds for every compact set K.*

COROLLARY 2. *It is valid that $N(K)=\sup\{L(F); F$ is compact and $F \subset K\}$.*

The converse of Theorem 1 is not necessarily true in case Φ is not symmetric. This is shown by

1) This follows from Lemma 1 in [7], p. 46.

EXAMPLE 1. Let Ω be $\{x_1, x_2\}$ and Φ be given by the matrix:

$$\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix},$$

where $\Phi(x_i, x_j)$ is the (i, j)-element of the matrix. It is easily verified that $L=N$. On the other hand, we have $M(\{x_1\})=1/2<1=L(\{x_1\})$.

We shall establish in §5 that $L=N$ implies $L=M$ in case Φ is symmetric.

§ 3. Principles

We say that a property holds *nearly everywhere*, or *n. e.*, on a set A if it holds except on a subset A' of A such that $N(K)=0$ for every compact set $K\subset A'$.

We recall two principles:

Maximum principle. If $\Phi(x, \mu)\leqq 1$ on $S\mu$, then the same inequality holds everywhere in Ω.

Equilibrium principle. For every compact set K, there exists a measure μ supported by K such that

$$\Phi(x, \mu)=1 \qquad n.\,e. \quad on \quad K,$$

$$\Phi(x, \mu)\leqq 1 \qquad\qquad in \quad \Omega.$$

This measure μ is called an equilibrium measure of K.

Note that an equilibrium measure of K belongs to $\mathscr{E}_K$. It is easily seen that the total mass of every equilibrium measure of K is equal to $M(K)$ in case Φ is symmetric.

We shall give an answer to "if" part of Problem II:

THEOREM 2. *If Φ satisfies the maximum principle, then $M(K)=L(K)=N(K)$ holds for every compact set K.*

PROOF. Assume that Φ satisfies the maximum principle and let K be any compact set. Then we see easily that $\mathscr{M}_K=\mathscr{N}_K$, and hence $M(K)=N(K)$.

We show by an example that the converse of Theorem 2 is not always valid in case Φ is not symmetric.

EXAMPLE 2. Let Ω be $\{x_1, x_2, x_3\}$ and Φ be given by the matrix

$$\begin{bmatrix} 3 & 1 & 1 \\ 2 & 2 & 1 \\ 3 & 2 & 3 \end{bmatrix}.$$

Then we can easily verify that $M=N$. On the other hand, Φ does not satisfy the maximum principle. In fact, for $\mu=(\varepsilon_{x_1}+\varepsilon_{x_2})/4$ (ε_x denotes the unit point measure at x), we have

$$\Phi(x_1, \mu)=\Phi(x_2, \mu)=1<5/4=\Phi(x_3, \mu).$$

Similarly, we see that $\check{\Phi}$ does not satisfy the maximum principle.

§ 4. Lemmas and known results

Hereafter we shall always assume that Φ is symmetric. We introduce two set functions. For a non-empty compact set K, let $\mathcal{U}_K$ be the totality of unit measures supported by K and let $V(\mu)=\sup\{\Phi(x, \mu); x \in S\mu\}$ for $\mu\neq 0$. We define

$$\begin{aligned} V(K) &= \inf\{V(\mu); \mu \in \mathcal{U}_K\} && \text{if} \quad \mathcal{U}_K\neq\phi, \\ W(K) &= \inf\{(\mu, \mu); \mu \in \mathcal{U}_K\} && \text{if} \quad \mathcal{U}_K\neq\phi, \\ V(\phi) &= W(\phi)=\infty. \end{aligned}$$

It is not difficult to show that $N(K)=1/V(K)$ holds for every compact set K. By this fact and a result in [6][2], we see that $N(K)=0$ if and only if $\mu(K)=0$ for all $\mu \in \mathcal{E}$.

The following lemmas are well-known:

LEMMA 1.[3] *Let K be a compact set and f be a strictly positive finite-valued continuous function on K. There exists a measure $\mu \in \mathcal{E}_K$ such that*

$$\begin{aligned} \Phi(x, \mu)&\geqq f(x) && n.\,e. \quad on \quad K, \\ \Phi(x, \mu)&\leqq f(x) && on \quad S\mu. \end{aligned}$$

LEMMA 2.[4] *If μ is a measure in $\mathcal{U}_K$ such that $(\mu, \mu)= W(K)$, then we have $\Phi(x, \mu)\geqq W(K)$ n. e. on K.*

LEMMA 3.[5] *It is valid that $V(K)= W(K)$ for every compact set K.*

The following important result was proved in [5] for a continuous kernel[6]. The present proof essentially follows the argument in [4].

PROPOSITION 1. *The following three statements are equivalent.*

(a) *Φ satisfies the maximum principle.*

(b) *Φ satisfies the equilibrium principle.*

(c) *Φ has property*

[P]. *For any $\nu \in \mathcal{E}$ and $z \notin S\nu$, $\Phi(x, \nu)\leqq\Phi(x, z)$ on $S\nu$ implies $\nu(\Omega)\leqq 1$.*

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2) [6], p. 139 and p. 222.

3) [3], Theorems 2.3 and 2.4 and p. 185.

4) [3], Theorem 2.4.

5) [3], [6] and [8].

PROOF. First we prove that (a) implies (b). Let K be any compact set. There exists by Lemma 1 a measure $\mu \in \mathscr{E}_K$ such that

$$\Phi(x, \mu) \geqq 1 \qquad \text{n. e.} \quad \text{on} \quad K,$$

$$\Phi(x, \mu) \leqq 1 \qquad \text{on} \quad S\mu.$$

It follows from (a) that $\Phi(x, \mu) \leqq 1$ in Ω.

Next we prove that (b) implies (c). Assume that ν and z satisfy the hypotheses of [P], i. e., $\nu \in \mathscr{E}$, $z \notin S\nu$ and $\Phi(x, \nu) \leqq \Phi(x, z)$ on $S\nu$. By our assumption (b), there is a measure $\mu \in \mathscr{E}_{S\nu}$ such that $\Phi(x, \mu)=1$ n. e. on $S\nu$ and $\Phi(x, \mu) \leqq 1$ in Ω. Then we have

$$\nu(\Omega)=\int \Phi(x, \mu) d\nu(x)=\int \Phi(\nu, y) d\mu(y)$$

$$\leqq \int \Phi(z, y) d\mu(y)=\Phi(z, \mu) \leqq 1.$$

Finally we prove that (c) implies (a). Let μ be a measure whose potential satisfies $\Phi(x, \mu) \leqq 1$ on $S\mu$. Let z be an arbitrarily fixed point such that $z \notin S\mu$. We shall prove $\Phi(z, \mu) \leqq 1$ under assumption (c). We consider the directed set D of strictly positive finite-valued continuous functions f on $S\mu$ such that $f(y) \leqq \Phi(z, y)$ on $S\mu$. For any $f \in D$, there exists by Lemma 1 a measure $\nu_f \in \mathscr{E}_{S\mu}$ such that $\Phi(\nu_f, y) \geqq f(y)$ n. e. on $S\mu$ and $\Phi(\nu_f, y) \leqq f(y)$ on $S\nu_f$. Since $z \notin S\nu_f$, $\nu_f \in \mathscr{E}$ and

$$\Phi(\nu_f, y) \leqq f(y) \leqq \Phi(z, y) \qquad \text{on} \qquad S\nu_f,$$

we obtain by our assumption (c) that $\nu_f(\Omega) \leqq 1$. Thus we have

$$\int f d\mu \leqq \int \Phi(\nu_f, y) d\mu(y)=\int \Phi(x, \mu) d\nu_f(x) \leqq \nu_f(S\mu) \leqq 1$$

for any $f \in D$, and hence $\Phi(z, \mu) \leqq 1$.

§ 5. Study of Problem II

We shall establish

THEOREM 3. *If $M(K)=N(K)$ holds for every compact set K, then Φ satisfies the maximum principle.*

This theorem follows from Proposition 1 and the following

LEMMA 4. *Let K be a compact set such that $N(K)=M(K)$. Then there exists a measure $\mu \in \mathscr{E}_K$ such that*

6) Φ is called a continuous kernel if it is continuous in the extended sense and $\Phi(x, y)$ is finite whenever $x \neq y$.

$$\Phi(x, \mu)=1 \qquad n.\,e. \quad on \quad K,$$
$$\Phi(x, \mu)\leqq 1 \qquad in \quad \Omega,$$
$$\mu(K)=M(K).$$

PROOF. We may assume that $N(K)>0$. Since $\mathscr{M}_K$ is vaguely compact, there exists a measure $\mu \in \mathscr{M}_K$ such that $\mu(K)=M(K)$. It suffices to show that $\Phi(x, \mu)\geqq 1$ n. e. on K. Since $(\mu, \mu)\leqq\mu(K)$ and $\mu_0=\mu/M(K) \in \mathscr{U}_K$, we have

$$W(K)\leqq(\mu_0, \mu_0)=(\mu, \mu)/M(K)^2\leqq 1/M(K)=1/N(K)=W(K)$$

by our assumption and Lemma 3. Consequently $(\mu_0, \mu_0)=W(K)$. By means of Lemma 2, we have $\Phi(x, \mu)\geqq 1$ n. e. on K. Namely μ is an equilibrium measure of K.

Combining Theorem 1 with Theorem 3, we have

THEOREM 4. *If $M(K)=L(K)$ holds for every compact set K, then Φ satisfies the maximum principle.*

Next we shall prove

THEOREM 5. *If $L(K)=N(K)$ holds for every compact set K, then Φ satisfies the maximum principle.*

We shall prepare

LEMMA 5. *Let K be a compact set such that $L(K)=N(K)$. Then there exists a measure $\mu \in \mathscr{E}_K$ such that*

$$\Phi(x, \mu)=1 \qquad n.\,e. \quad on \quad K,$$
$$\Phi(x, \mu)\leqq 1 \qquad on \quad K,$$
$$\mu(K)=L(K).$$

This measure μ is called a weak equilibrium measure of K.

PROOF. We may suppose that $N(K)>0$. There exists a measure $\mu \in \mathscr{L}_K$ such that $\mu(K)=L(K)$. By the same argument as in the proof of Lemma 4, we see that $\Phi(x, \mu)\geqq 1$ n. e. on K.

LEMMA 6. *Assume that $L(K)=N(K)$ holds for every compact set K. Let K be a compact set and z be a point such that $z \notin K$. Then there exists a measure $\mu \in \mathscr{E}_K$ such that*

$$\Phi(x, \mu)=1 \qquad n.\,e. \quad on \quad K,$$
$$\Phi(z, \mu)\leqq 1.$$

PROOF. Let μ be a weak equilibrium measure of K. If $\Phi(z, \mu)\leqq 1$, then μ is the required one. We consider the case where $\Phi(z, \mu)>1$. Let μ_0 be a weak equilibrium measure of $K\cup\{z\}$, i.e., $\Phi(x, \mu_0)=1$ n.e. on $K\cup\{z\}$, $\Phi(x, \mu_0)\leqq 1$ on $K\cup\{z\}$ and $\mu_0(K\cup\{z\})=L(K\cup\{z\})$. We have

$$L(K)=\mu(K)=\int\Phi(x, \mu_0)d\mu(x)=\int\Phi(\mu, y)\, d\mu_0(y)$$

$$=\mu_0(K)+\Phi(\mu, z)\mu_0(\{z\}).$$

If $\mu_0(\{z\})>0$, then

$$L(K)>\mu_0(K)+\mu_0(\{z\})=\mu_0(K\cup\{z\})=L(K\cup\{z\})$$

$$=N(K\cup\{z\})\geqq N(K)=L(K),$$

which is a contradiction. Therefore $S_{\mu_0}\subset K$ and μ_0 satisfies the required relations.

Proof of Theorem 5: On account of Proposition 1, it is enough to show that Φ has property [P]. Let $\nu\in\mathscr{E}$ and $z\notin S_\nu$ and assume that $\Phi(x, \nu)\leqq\Phi(x, z)$ on S_ν. There exists by Lemma 6 a measure $\mu\in\mathscr{E}_{S_\nu}$ such that $\Phi(x, \mu)=1$ n.e. on S_ν and $\Phi(z, \mu)\leqq 1$. We have

$$\nu(S_\nu)=\int\Phi(x, \mu)d\nu(x)=\int\Phi(\nu, y)d\mu(y)$$

$$\leqq\int\Phi(z, y)d\mu(y)=\Phi(z, \mu)\leqq 1.$$

This completes the proof.

Making use of Theorems 2 and 5, we obtain an answer to Problem I:

Theorem 6. *If $L(K)=N(K)$ holds for every compact set K, then $L(K)=M(K)$ holds for every compact set K.*

Thus Problems I and II are completely solved in the case where Φ is symmetric.

By means of Theorems 2 and 5 and Corollary 1 of Theorem 1, we can summarize Theorems 2, 3, 4 and 5 as follows:

Theorem 7. *Φ satisfies the maximum principle if and only if L has a monotone property.*

§ 6. Study of Problem III

Throughout this section, we always assume that Φ is a finite-valued continuous kernel. Note that in this case $N(\{x\})>0$ for every $x\in\Omega$.

The following proposition plays a fundamental role in the sequel:

Proposition 2[7]. *Assume that there exists an equilibrium measure of every*

7) The author first proved this proposition under the additional assumption that every compact set is separable. The present proof is due to Professor M. Kishi.

finite set. Then Φ satisfies the maximum principle.

PROOF. Suppose that $\Phi(x, \mu)\leqq 1$ on $S\mu$ and let z be an arbitrarily fixed point such that $z \notin S\mu$. Let us prove $\Phi(z, \mu)\leqq 1$. Since $f(x)=\Phi(x, z)$ is a strictly positive finite-valued continuous function on $S\mu$, there exists by Lemma 1 a measure ν such that $S\nu \subset S\mu$, $\Phi(x, \nu)\geqq\Phi(x, z)$ on $S\mu$ and $\Phi(x, \nu)\leqq\Phi(x, z)$ on $S\nu$. There is a net $\{\nu_\alpha; \alpha \in D\}$ which converges vaguely to ν such that each support $S\nu_\alpha$ consists of a finite number of points of $S\nu$. For every finite set $S\nu_\alpha$, there is an equilibrium measure λ_α of $S\nu_\alpha$ by our assumption. Since $\lambda_\alpha(S\mu)=M(S\nu_\alpha)\leqq M(S\mu)<\infty$, the total masses of λ_α are bounded. We choose a vaguely convergent subnet $\{\lambda_\alpha; \alpha \in D'\}$ and let λ be the limit. We see easily that $S\lambda \subset S\nu$, $\Phi(z, \lambda)\leqq 1$ and

$$\nu_\alpha(S\mu)=\int\Phi(x, \lambda_\alpha)d\nu_\alpha(x).$$

It follows that

$$\begin{aligned}\nu(S\mu)&=\lim_{D'}\nu_\alpha(S\mu)=\lim_{D'}\int\Phi(x, \lambda_\alpha)d\nu_\alpha(x)\\&=\int\Phi(x, \lambda)d\nu(x)=\int\Phi(\nu, y)d\lambda(y)\\&=\int\Phi(z, y)d\lambda(y)=\Phi(z, \lambda)\leqq 1.\end{aligned}$$

Consequently we have

$$\Phi(z, \mu)\leqq\int\Phi(\nu, y)d\mu(y)=\int\Phi(x, \mu)d\nu(x)\leqq\nu(S\mu)\leqq 1.$$

This completes the proof.

By the aid of Lemma 4 and Proposition 2, we have

THEOREM 8. *If $M(F)=N(F)$ holds for every finite set F, then Φ satisfies the maximum principle.*

By the same argument as in the proof of Theorem 1, we can prove

LEMMA 7. *If $M(F)=L(F)$ holds for every finite set F, then $M(F)=N(F)$ holds for every finite set F.*

Combining Lemma 7 with Theorem 8, we have

THEOREM 9. *If $M(F)=L(F)$ holds for every finite set F, then Φ satisfies the maximum principle.*

We shall establish

THEOREM 10. *If $L(F)=N(F)$ holds for every finite set F, then Φ satisfies the maximum principle.*

PROOF. On account of Proposition 2, it suffices to show that there exists an equilibrium measure of every finite set. This follows from Lemma 1 and the following

LEMMA 8. *Assume that $L(F)=N(F)$ holds for every finite set F. Let μ be a measure such that $S\mu$ is a finite set. If $\Phi(x, \mu)\leqq 1$ on $S\mu$, then $\Phi(x, \mu)\leqq 1$ everywhere in Ω.*

PROOF. Suppose that $\Phi(x, \mu)\leqq 1$ on $S\mu$ and let z be an arbitrarily fixed point such that $z \notin S\mu$. Let us prove $\Phi(z, \mu)\leqq 1$. There exists by Lemma 1 a measure ν such that $S\nu \subset S\mu$,

$$\Phi(\nu, y)\geqq\Phi(z, y) \qquad \text{on} \qquad S\mu,$$

$$\Phi(\nu, y)\leqq\Phi(z, y) \qquad \text{on} \qquad S\nu.$$

By the same argument as in the proof of Lemma 6, we can prove that there exists a measure λ such that $S\lambda\subset S\nu$,

$$\Phi(x, \lambda)=1 \qquad \text{on} \qquad S\nu,$$

$$\Phi(z, \lambda)\leqq 1.$$

In this proof, we use the assumption that $L(F)=N(F)$ for every finite set F. It follows that

$$\nu(S\nu)=\int\Phi(x, \lambda)d\nu(x)=\int\Phi(\nu, y)d\lambda(y)$$

$$\leqq\int\Phi(z, y)d\lambda(y)=\Phi(z, \lambda)\leqq 1.$$

Therefore we have

$$\Phi(z, \mu)\leqq\int\Phi(\nu, y)d\mu(y)=\int\Phi(x, \mu)d\nu(x)\leqq\nu(S\nu)\leqq 1.$$

We complete the proof.

§ 7. Elementary proof of Theorem 9

In the preceding sections, a solution of the minimizing problem of (μ, μ) for $\mu \in \mathscr{U}_K$ has played an important role for our problems. In this section, we shall give an elementary proof of Theorem 9 which is the same pattern as in [2].

By Lemma 1 and Proposition 2, it is enough to show the following

LEMMA 9. *Assume that $L(F)=M(F)$ holds for every finite set F and let μ be a measure such that $S\mu$ is a finite set. If $\Phi(x, \mu)\leqq 1$ on $S\mu$, then $\Phi(x, \mu)\leqq 1$*

everywhere in Ω.

PROOF. First we consider the case where $S\mu$ consists of one point x_0. Then $\mu=a\varepsilon_{x_0}(a>0)$ and $\Phi(x_0, \mu)=a\Phi(x_0, x_0)\leqq 1$. Since $1/\Phi(x_0, x_0)=L(\{x_0\})=M(\{x_0\})\leqq 1/\Phi(x, x_0)$ for every x, we have $\Phi(x, \mu)=a\Phi(x, x_0)\leqq 1$ in Ω. Next supposing that our assertion is true for every measure whose support consists of at most $k-1$ points, we show that $\Phi(x, \mu)\leqq 1$ on $S\mu$ implies that $\Phi(x, \mu)\leqq 1$ in Ω in the case where $S\mu$ consists of k points. We set $S\mu=X=\{x_1, \cdots, x_k\}$. Let G be the restriction of Φ onto $X\times X$. Then G satisfies the maximum principle and hence the equilibrium principle by Proposition 1. Thus there exists a measure μ_0 on X such that $G(x, \mu_0)=1$ on X. Since $\Phi(x, \mu_0)=G(x, \mu_0)=1$ on X, it is readily verified that $\mu_0(X)=L(X)$. In case $S\mu_0\neq X$, it follows from the assumption of induction that $\Phi(x, \mu_0)\leqq 1$ in Ω. In case $S\mu_0=X$, let ν be a measure on X such that $\nu(X)=M(X)$ and $\Phi(x, \nu)\leqq 1$ in Ω. Writing $\Phi(x_i, \nu)=p_i$ and $\mu_0(\{x_i\})=a_i$, we have

$$p_1a_1+\cdots+p_ka_k=\int\Phi(x, \nu)d\mu_0(x)=\int\Phi(\mu_0, y)d\nu(y)$$

$$=\nu(X)=M(X)=L(X)=\mu_0(X)=a_1+\cdots+a_k.$$

Since $p_i\leqq 1$ and $a_i>0$ for every i $(i=1, \cdots, k)$, we see that $p_i=1$ for every i, and hence $\Phi(x, \nu)=1$ on X. Consequently there exists a measure ν_0 supported by X such that

$$\Phi(x, \nu_0)=1 \qquad \text{on} \qquad S\mu=X,$$

$$\Phi(x, \nu_0)\leqq 1 \qquad \text{in} \qquad \Omega.$$

Now we prove that $\Phi(x, \mu)\leqq 1$ on $S\mu$ implies that $\Phi(x, \mu)\leqq 1$ in Ω. Set $a=\inf[\mu(\{x_i\})/\nu_0(\{x_i\});\ x_i\in S\nu_0]$. Then $0<a\leqq 1$ and we can write $\mu=a\nu_0+\mu'$, where μ' is a measure on X such that $S\mu'\neq X$. If $\mu'=0$, then $\Phi(x, \mu)=a\Phi(x, \nu_0)\leqq a\leqq 1$ in Ω. Otherwise, let $b=\max\{\Phi(x, \mu');\ x\in S\mu'\}$. Since $S\mu'\neq X$, we have $\Phi(x, \mu')\leqq b$ in Ω. Since there is a point $x_i\in S\mu'$ such that $\Phi(x_i, \mu')=b$, we have

$$a+b=a\Phi(x_i, \nu_0)+\Phi(x_i, \mu')=\Phi(x_i, \mu)\leqq 1,$$

and hence

$$\Phi(x, \mu)=a\Phi(x, \nu_0)+\Phi(x, \mu')\leqq a+b\leqq 1 \qquad \text{in} \quad \Omega.$$

This completes the proof.

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