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Oscillation Criteria for a Second Order Differential Equation with a Damping Term

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1. Introduction

In this paper we are concerned with the oscillatory behavior of the second order differential equation with a damping term

$$x''+q(t)x'+p(t)f(x)=0 \tag{1}$$

where the following assumptions are assumed to hold:

(a) $p, q \in C(R^+)$, $R^+=(0, \infty)$;

(b) $f \in C(R)$, $R=(-\infty, \infty)$, and $xf(x)>0$ for all $x \in R-\{0\}$;

(c) $f \in C^1(R-\{0\})$, and there is a constant $k>0$ such that $f'(x) \geqq k$ for all $x \in R-\{0\}$.

We restrict our attention to solutions $x(t)$ of (1) which exist on some half-line $[T_x, \infty)$ and are nontrivial for all large t. A solution $x(t)$ of (1) is said to be oscillatory if $x(t)$ has an unbounded set of zeros $\{t_k\}_{k=1}^{\infty}$ such that $\lim_{k\to\infty} t_k=\infty$; otherwise, a solution is said to be nonoscillatory. Equation (1) is called oscillatory (or nonoscillatory) if all solutions of (1) are oscillatory (or nonoscillatory).

As a special case of (1) we have

$$x''+p(t)f(x)=0, \tag{2}$$

which has been the subject of intensive investigations since the pioneering work of Atkinson [1]. For results regarding oscillation of (2) with the assumption $p(t) \geqq 0$ we refer in particular to Wong [14]. Oscillation criteria for (2) with no sign assumption on $p(t)$ have been given by Waltman [12], Bhatia [3], Kiguradze [9], Kamenev [7], Staikos and Sficas [11] and others. Recently an attempt has been made by Erbe [6] to extend to (1) some of the known results for (2).

It is the object of this paper to present oscillation criteria for equation (1) with no explicit sign assumptions on $p(t)$ and $q(t)$. Our results do not overlap with those of Erbe.

Results for (1) with nonlinear damping have been obtained by Bobisud ([4],

[5]) and Baker [2]. (We note that extensions of the results of Bobisud and Baker to higher order equations have been given by Kartsatos and Onose [8] and Naito [10].)

2. Oscillation theorems

THEOREM 1. *In addition to* (a), (b), (c) *assume that*

(3) $$tq(t) \leqq 1 \quad \textit{for all sufficiently large } t;$$

(4) $$\int^{\infty} tq^2(t)dt < \infty;$$

(5) $$\int_a^{\infty} \frac{dx}{f(x)} < \infty, \qquad \int_{-a}^{-\infty} \frac{dx}{f(x)} < \infty \quad \textit{for some } a>0;$$

(6) $$\int^{\infty} tp(t)dt = \infty .$$

Then equation (1) *is oscillatory.*

PROOF. Let $x(t)$ be a nonoscillatory solution of (1). There is no loss of generality in assuming that $x(t)>0$ for $t \geqq t_0$, since a similar argument holds when $x(t)<0$ for $t \geqq t_0$. Multiplying (1) by $t/f(x(t))$ and integrating on $[t_0, t]$ we obtain

(7) $$\begin{aligned} &\frac{tx'(t)}{f(x(t))} - \int_{t_0}^{t} \frac{x'(s)}{f(x(s))} ds + \int_{t_0}^{t} \frac{sf'(x(s))[x'(s)]^2}{[f(x(s))]^2} ds \\ &\qquad + \int_{t_0}^{t} \frac{sq(s)x'(s)}{f(x(s))} ds + \int_{t_0}^{t} sp(s)ds = \frac{t_0 x'(t_0)}{f(x(t_0))} . \end{aligned}$$

By Schwarz's inequality

(8) $$\left\{\int_{t_0}^{t} \frac{sq(s)x'(s)}{f(x(s))} ds\right\}^2 \leqq K^2 \int_{t_0}^{t} \frac{s[x'(s)]^2}{[f(x(s))]^2} ds,$$

where $K^2 = \int_{t_0}^{\infty} tq^2(t)dt$ $(K>0)$. Using (c) and (8) in (7), we get for $t \geqq t_0$

(9) $$\begin{aligned} &\frac{tx'(t)}{f(x(t))} - \int_{t_0}^{t} \frac{x'(s)}{f(x(s))} ds + k\int_{t_0}^{t} \frac{s[x'(s)]^2}{[f(x(s))]^2} ds \\ &\qquad - K\left\{\int_{t_0}^{t} \frac{s[x'(s)]^2}{[f(x(s))]^2} ds\right\}^{1/2} + \int_{t_0}^{t} sp(s)ds \leqq \frac{t_0 x'(t_0)}{f(x(t_0))} . \end{aligned}$$

Observing that

$$k\int_{t_0}^{t} \frac{s[x'(s)]^2}{[f(x(s))]^2} ds - K\left\{\int_{t_0}^{t} \frac{s[x'(s)]^2}{[f(x(s))]^2}\right\}^{1/2}$$

remains bounded below as $t\to\infty$, and taking (5), (6) into account, we see from (9) that

$$\frac{tx'(t)}{f(x(t))}\to -\infty \qquad \text{as} \quad t\to\infty\,.$$

Hence, there exists $t_1 \geqq t_0$ such that

$$x'(t)<0 \qquad \text{for} \quad t\geqq t_1. \tag{10}$$

Rewriting (7) as

$$\begin{aligned}&\frac{tx'(t)}{f(x(t))}+\int_{t_1}^{t}\frac{sf'(x(s))[x'(s)]^2}{[f(x(s))]^2}ds\\ &=\frac{t_0x'(t_0)}{f(x(t_0))}+\int_{t_0}^{t_1}\frac{(1-sq(s))x'(s)}{f(x(s))}ds+\int_{t_1}^{t}\frac{(1-sq(s))x'(s)}{f(x(s))}ds\\ &-\int_{t_0}^{t_1}\frac{sf'(x(s))[x'(s)]^2}{[f(x(s))]^2}ds-\int_{t_0}^{t}sp(s)ds\,,\end{aligned}$$

and using (3), (6) and (10), we find a $t_2\geqq t_1$ such that

$$\frac{tx'(t)}{f(x(t))}+\int_{t_1}^{t}\frac{sf'(x(s))[x'(s)]^2}{[f(x(s))]^2}ds\leqq -1 \qquad \text{for} \quad t\geqq t_2\,,$$

and consequently

$$-\frac{tx'(t)}{f(x(t))}\geqq 1+\int_{t_2}^{t}\frac{sf'(x(s))[x'(s)]^2}{[f(x(s))]^2}ds \qquad \text{for} \quad t\geqq t_2\,. \tag{11}$$

The rest of the proof is based on the method due to Kiguradze [9] and Kamenev [7]. Multiplying (11) by

$$-\frac{f'(x(t))x'(t)}{f(x(t))}\left\{1+\int_{t_2}^{t}\frac{sf'(x(s))[x'(s)]^2}{[f(x(s))]^2}ds\right\}^{-1}>0$$

and integrating from t_2 to t, we obtain

$$\log\left\{1+\int_{t_2}^{t}\frac{sf'(x(s))[x'(s)]^2}{[f(x(s))]^2}ds\right\}\geqq \log\frac{f(x(t_2))}{f(x(t))} \qquad \text{for} \quad t\geqq t_2\,.$$

This combined with (11) yields

$$x'(t)\leqq -\frac{f(x(t_2))}{t} \qquad \text{for} \quad t\geqq t_2\,. \tag{12}$$

Integrating (12) from t_2 to t and taking the limit as $t\to\infty$, we conclude that $\lim_{t\to\infty} x(t) = -\infty$, which contradicts the hypothesis that $x(t)>0$ for all large t. This completes the proof of the theorem.

From the proof of Theorem 1 it is easy to see that when $q(t)\equiv 0$ the assumption (c) can be replaced by the following weaker one:

(c′) $\quad f\in C^1(R-\{0\})$, and $f'(x)\geqq 0$ for all $x\in R-\{0\}$.

We state this fact in the following

COROLLARY 1. (Kamenev [7]) *Under the assumptions* (a), (b), (c′), (5) *and* (6), *equation* (2) *is oscillatory.*

EXAMPLE 1. The following equation is oscillatory:

$$x''+\frac{\sin t}{t^2}x'+\frac{\sin t}{t(2-\sin t)}(x+x^3)=0.$$

A close look at the proof of Theorem 1 enables us to obtain the following theorem.

THEOREM 2. *In addition to* (a), (b), (c) *and* (3) *assume that*

$$\int^{\infty}\frac{(sq(s)-1)^2}{s}ds<\infty. \tag{13}$$

If (6) *holds, then equation* (1) *is oscillatory.*

PROOF. As in the proof of Theorem 1, if there exists a nonoscillatory solution $x(t)$ of (1), then $x(t)$ satisfies (7). Combining the first and the third integrals in the left hand side of (7) and using Schwarz's inequality, we obtain

$$\left\{\int_{t_0}^{t}\frac{(sq(s)-1)x'(s)}{f(x(s))}ds\right\}^2\leqq K^2\int_{t_0}^{t}\frac{s[x'(s)]^2}{[f(x(s))]^2}ds$$

where $K^2=\int_{t_0}^{\infty}(sq(s)-1)^2/s\,ds$. Hence the inequality (9) without the first integral holds for $t\geqq t_0$. The rest of the proof proceeds exactly as in that of Theorem 1.

EXAMPLE 2. Willett [13] has shown that the equation

$$x''+\frac{\lambda+\mu t\sin\nu t}{t^2}x=0 \qquad (\lambda,\ \mu,\ \nu\neq 0 \text{ constants})$$

is oscillatory if

$$\lambda>\frac{1}{4}-\frac{1}{2}\left(\frac{\mu}{\nu}\right)^2,$$

and nonoscillatory if

$$\lambda < \frac{1}{4} - \frac{1}{2}\left(\frac{\mu}{\nu}\right)^2 .$$

Theorem 2 implies that the equation

$$x'' + \frac{1}{t}x' + \frac{\lambda + \mu t \sin \nu t}{t^2}x = 0$$

is oscillatory for any $\lambda > 0$. This shows that the oscillatory behavior of an original equation may or may not be affected by adding a damping term.

THEOREM 3. *In addition to* (a), (b), (c) *assume that there is a constant* α, $0 \leqq \alpha < 1$, such that

$$tq(t) \leqq \alpha \quad \textit{for all sufficiently large } t; \tag{14}$$

$$\int^{\infty} t^{\alpha} q^2(t)dt < \infty ; \tag{15}$$

$$\int^{\infty} t^{\alpha} p(t)dt = \infty . \tag{16}$$

Then equation (1) *is oscillatory.*

PROOF. Let $x(t)$ be a nonoscillatory solution of (1). We may assume without loss of generality that $x(t) > 0$ for $t \geqq t_0$. Multiplying (1) by $t^{\alpha}/f(x(t))$ and integrating from t_0 to t, we have

$$\begin{aligned} &\frac{t^{\alpha}x'(t)}{f(x(t))} - \alpha\int_{t_0}^{t} \frac{s^{\alpha-1}x'(s)}{f(x(s))}ds + \int_{t_0}^{t} \frac{s^{\alpha}f'(x(s))[x'(s)]^2}{[f(x(s))]^2}ds \\ &\qquad + \int_{t_0}^{t} \frac{s^{\alpha}q(s)x'(s)}{f(x(s))}ds + \int_{t_0}^{t} s^{\alpha}p(s)ds = \frac{t_0^{\alpha}x'(t_0)}{f(x(t_0))} . \end{aligned} \tag{17}$$

With the use of Schwarz's inequality we obtain

$$\left\{\int_{t_0}^{t} \frac{s^{\alpha-1}x'(s)}{f(x(s))}ds\right\}^2 \leqq K_1^2 \int_{t_0}^{t} \frac{s^{\alpha}[x'(s)]^2}{[f(x(s))]^2}ds ,$$

$$\left\{\int_{t_0}^{t} \frac{s^{\alpha}q(s)x'(s)}{f(x(s))}ds\right\}^2 \leqq K_2^2 \int_{t_0}^{t} \frac{s^{\alpha}[x'(s)]^2}{[f(x(s))]^2}ds ,$$

where $K_1^2 = \int_{t_0}^{\infty} s^{\alpha-2}ds$, $K_2^2 = \int_{t_0}^{\infty} s^{\alpha}q^2(s)ds$ $(K_1 > 0,\ K_2 > 0)$. Using these inequalities in (17), we obtain

$$\begin{aligned} &\frac{t^{\alpha}x'(t)}{f(x(t))} - (K_1\alpha + K_2)\left\{\int_{t_0}^{t} \frac{s^{\alpha}[x'(s)]^2}{[f(x(s))]^2}ds\right\}^{1/2} \\ &\qquad + k\int_{t_0}^{t} \frac{s^{\alpha}[x'(s)]^2}{[f(x(s))]^2}ds + \int_{t_0}^{t} s^{\alpha}p(s)ds \leqq \frac{t_0^{\alpha}x'(t_0)}{f(x(t_0))} . \end{aligned} \tag{18}$$

Proceeding as in the proof of Theorem 1 we can see that $x'(t)$ is eventually negative and that there exists $t_2 \geqq t_1$ such that

$$(19) \qquad -\frac{t^\alpha x'(t)}{f(x(t))} \geqq 1+\int_{t_2}^{t} \frac{s^\alpha f'(x(s))[x'(s)]^2}{[f(x(s))]^2} ds \qquad \text{for} \quad t \geqq t_2 .$$

It is easy to derive from (19) the desired contradiction $\lim_{t\to\infty} x(t) = -\infty$. The proof is thus complete.

The following corollaries follow immediately from Theorem 3.

COROLLARY 2. *Let* (a), (b) *and* (c) *be satisfied. If*

$$\int^{\infty} t^\alpha p(t) dt = \infty \qquad \text{for some } \alpha,\ 0 \leqq \alpha < 1 ,$$

then equation (2) *is oscillatory.*

COROLLARY 3. (Bhatia [3]) *Let* (a), (b) *and* (c′) *be satisfied. If*

$$\int^{\infty} p(t) dt = \infty ,$$

then equation (2) *is oscillatory.*

EXAMPLE 3. The following equation is oscillatory:

$$x'' + \frac{\alpha \sin t}{t} x' + \frac{\sin t}{t^\alpha (2 - \sin t)} x [\log (2 + |x|)]^\beta = 0 ,$$

where $0 \leqq \alpha < 1$, $\beta \geqq 0$.

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